International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:8, No:9, 2014

Zero-Dissipative Explicit Runge-Kutta Method for Periodic Initial Value Problems

N. Senu, I. A. Kasim, F. Ismail, N. Bachok

***Abstract*—**In this paper zero-dissipative explicit Runge-Kutta method is derived for solving second-order ordinary differential equations with periodical solutions. The phase-lag and dissipation properties for Runge-Kutta (RK) method are also discussed. The new method has algebraic order three with dissipation of order infinity. The numerical results for the new method are compared with existing method when solving the second-order differential equations with periodic solutions using constant step size.

***Keywords*—**Dissipation, Oscillatory solutions, Phase-lag, Runge-Kutta methods.

I. Introduction

IN this paper, we are focused on initial value problems (IVP) related to second-order ODEs of the form:

$$y''(x) = f(x, y), y(x_0) = y_0, y'(x_0) = y'_0 \; x \in [a, b] \qquad (1)$$

where

$$y(x) = [y_1(x), y_2(x), \dots, y_s(x)]^T$$

$$f(x, y) = [f_1(x, y), f_2(x, y), \dots, f_s(x, y)]^T$$

and y_0 is a given vector of initial condition and their solution is oscillating. There are many procedures in order to develop efficient methods for the numerical solution (1) such as phase fitting, P-stability, and methods with minimal phase-lag. The results of these procedures are multistep methods (two-step or four-step) and hybrid multistep methods) (see [1]–[11]). From the above remark it is obvious that there is no efficient one-step method for the numerical solution (1). This is important since for the numerical solution of any problem using an one-step method, only the initial condition is required, while for the numerical solution of the same problem using a multistep method many initial conditions can be required. The first of them is the condition given by the problem. The rest are conditions that can produce errors that are much greater than the error of the numerical method. For this reason, it is important to investigate the production of efficient one-step methods and especially the well-known Runge-Kutta methods.

The term phase-lag was first introduced by Brusa and Nigro [12]. For the past three decades, several authors have developed RK or Runge-Kutta-Nystrom (RKN) methods based on the minimal phase-lag theory. See Van der Houwen and Sommeijer [13], Senu et al. [14], and Van de Vyver [15]. Simos [16] derived a Runge-Kutta-Fehlberg method based on the idea of phase-lag of order infinity.

In this paper, we will derive a new explicit RK method with three-stage third-order with dissipation of order infinity. In the next sections, we will discuss some basic theory of Runge-Kutta methods. Then, the construction of the new amplification-fitted Runge-Kutta method is described and its coefficients are displayed in the Butcher table. Finally, numerical tests are performed on first-order differential equation problems which are known have oscillatory solutions.

II. General Theory

RK methods for the numerical integration of the Initial Value Problem (IVP) is given by

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i f_i \qquad (2)$$

where

$$f_1 = f(x_n, y_n)$$

$$f_i = f\left(x_n + c_i h, y_n + h \sum_{j=1}^{i-1} a_{ij} f_j\right),$$

$$i = 1, \dots, s$$

The parameters a_{ij}, b_i and c_i are assumed to be real and $j \geq i$ then $a_{i,j} = 0$. m is the number of stages of the method. All the parameter can be tabulate in Butcher Tableau (see Table I) in the following form:

TABLE I
BUTCHER TABLEAU

C	A
	b^T

where

$$C = [c_1, c_2, \dots, c_m]^T, A = [a_{i,j}], b^T = [b_1, b_2, \dots, b_m]^T.$$

Consider the standard test problem of differential equation

$$y' = f(x, y) = \lambda y, \text{ and } y(x_n) = y_n \qquad (3)$$

which has true solution

$$y(x) = y_n e^{\lambda(x - x_n)}.$$

N. Senu is with the Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia (corresponding author: e-mail: norazak@upm.edu.my).

I.A. Kasim, F. Ismail, N. Bachok were with Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia (e-mail: fudziah@upm.edu.my, norfifah@upm.edu.my).

Applying (3) to the RK formula (2), by setting $v = \lambda h$ and factoring, we obtain

$$y_{n+1} = \{1 + vb_i(I - vA)^{-1}\}y_n. \quad (4)$$

$$y_{n+1} = R(v)y_n, \qquad \text{where } |R(v)| < 1, \quad (5)$$

and $R(v) = 1 + vb_i(I - vA)^{-1}$ is said to be stability polynomial of RK method.

Definition 1: The quantities of the stability equation in (5) corresponding to RK methods (2)

$$\phi(v) = v - \arctan\left(v\frac{B_m(v^2)}{A_m(v^2)}\right), \quad (6)$$

$$\alpha(v) = 1 - \sqrt{A_m^2(v^2) + v^2B_m^2(v^2)} \quad (7)$$

are called the dispersion (or phase lag or phase error) and the amplification error respectively. If $\phi(v) = O(v^{q+1})$ and $\alpha(v) = O(v^{r+1})$ then the method is said to be dispersive of order q and dissipative of order r.

Function $R(v)$ can be written as

$$R(v) = A_m(v^2) + ivB_m(v^2)\,,\ v = \lambda h \quad (8)$$

where

$$A_m(z) = 1 - \beta_2 z + \beta_4 z^2 + \cdots,$$

$$B_m(z) = 1 - \beta_3 z + \beta_5 z^2 + \cdots,$$

for $j > m\,, \beta_j = 0$ and $z = v^2$.

III. Construction of the New Method

In this section, we will derive a three-stage third order explicit RK method with phase-lag of order infinity. The derivation of the new RK method is based on the method derived as in [17] as given in Table II.

TABLE II
BUTCHER TABLEAU FOR THIRD-ORDER RK METHOD

0	0		
$\frac{1}{2}$	$\frac{1}{2}$	0	
$\frac{3}{4}$	0	$\frac{3}{4}$	0
	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$

Consider the stability polynomial in (5) for three-stage RK method, from the coefficients above then we have

$A_3 = 1 + \left(-\frac{1}{2} - \frac{4}{9}a_{31}\right)v^2$ and $B_3 = 1 - \frac{1}{6}v^2$ respectively.

As it has already been defined, in order to have amplification error of order infinity, (7) must hold. That is

$$1 - \sqrt{A_m^2(v^2) + v^2B_m^2(v^2)} = 0.$$

By applying $A_3(v^2)$ and $B_3(v^2)$ to the above formula, letting a_{31} as free parameter and solving for a_{31}, then we get

$$a_{31} = \frac{-3(3v^2 - 6 + \sqrt{12v^4 - 36v^2 + 36 - v^6})}{8v^2}.$$

The expansion Taylor series for a_{31}, which is given from the above formula is

$$a_{31} = -\frac{3}{32}v^2 - \frac{1}{64}v^4 - \frac{3}{512}v^6 - \frac{7}{3072}v^8 - \frac{31}{36864}v^{10} - \frac{7}{24576}v^{12} + O(v^{14})$$

The free parameter a_{31} is chosen in since it gives the smallest maximum global error than the other coefficients. This new method is denoted as RK3(D).

IV. Problems Tested and Numerical Results

In this section, we will apply the new method to some differential equation problems. The following explicit RK methods are selected for the numerical comparisons:

- RK3(D): The new derived third order RK method with dissipation of order infinity.
- RK3: The three stage third order RK method derived in [17].

Problem 1 [13]:

$$y'' = -64y, y(0) = 1, y'(0) = -2.$$

Theoretical solution:

$$y(x) = -\frac{1}{4}\sin(8x) + \cos(8x).$$

reduce to first order system:

$$y_1{}' = y_2, y_2{}' = -64y_1.$$

Problem 2 [14]:

$$y_1'' = -y_1 + 0.001\cos(x)\,, y_1(0) = 1, y_1{}'(0) = 0$$
$$y_2'' = -y_2 + 0.001\sin(x), y_2(0) = 0, y_2(0) = 0.9995.$$

Theoretical solutions:

$$y_1(x) = \cos(x) + 0.0005x\sin(x)$$
$$y_2(x) = \sin(x) - 0.0005x\cos(x).$$

reduce to first order system :

$$y_1' = y_2, y_2' = -y_1' \quad + 0.001\cos(x)$$
$$y_3' = y_4, y_4' = -y_3' \quad + 0.001\sin(x).$$

Problem 3 [18]:

$$y'' = -100y + 99\sin(x), y(x_0) = 1, y'(x_0) = 11.$$

Theoretical solution:

$$y(x) = \cos(10x) + \sin(10x) + \sin(x).$$

reduce to first order system:

$$y_1{}' = y_2, y_2' = -100y_1 + 99\sin(x).$$

From Tables III-V and Figs. 1 – 6, we can see that the RK3(D) method is always more accurate than the RK3 method.

TABLE III
COMPARISON MAXIMUM GLOBAL ERROR FOR RK3 AND RK3(D) FOR PROBLEM 1

h	Methods	End of Integration, *b*		
		100	1000	10000
0.003125	RK3	4.28976(-3)	4.28343(-2)	4.18466(-1)
	RK3(D)	2.14157(-5)	2.14380(-4)	2.14243(-3)
0.00625	RK3	3.42521(-2)	3.36504(-1)	2.81000(0)
	RK3(D)	3.42945(-4)	3.43943(-3)	3.44147(-2)
0.0125	RK3	2.69993(-1)	2.33820(0)	7.95937(0)
	RK3(D)	5.50679(-3)	5.52000(-2)	5.52251(-1)
0.025	RK3	1.93021(0)	7.70556(0)	8.24623(0)
	RK3(D)	8.91646(-2)	8.95096(-1)	8.53005(0)
0.05	RK3	7.38859(0)	8.25482(0)	8.25482(0)
	RK3(D)	1.50012(0)	1.31448(+1)	1.64920(1)

TABLE IV
COMPARISON MAXIMUM GLOBAL ERROR FOR RK3 AND RK3(D) FOR PROBLEM 2

h	Methods	End of Integration, *b*		
		100	1000	10000
0.003125	RK3	1.25751(-7)	1.30715(-6)	3.39464(-5)
	RK3(D)	7.21325(-11)	5.78939(-9)	1.00255(-6)
0.00625	RK3	1.00603(-6)	1.04649(-5)	2.73954(-4)
	RK3(D)	1.34269(-9)	1.53219(-8)	1.48067(-6)
0.0125	RK3	8.04855(-6)	8.37148(-5)	2.18995(-3)
	RK3(D)	2.05362(-8)	2.05006(-7)	5.31261(-6)
0.025	RK3	6.43903(-5)	6.69522(-4)	1.74834(-2)
	RK3(D)	3.23682(-7)	3.28764(-6)	8.41126(-5)
0.05	RK3	5.15065(-4)	5.34389(-3)	1.37612(-1)
	RK3(D)	5.16788(-6)	5.31213(-5)	1.37786(-3)

TABLE V
COMPARISON MAXIMUM GLOBAL ERROR FOR RK3 AND RK3(D) FOR PROBLEM 3

h	Methods	End of Integration, *b*		
		100	1000	10000
0.003125	RK3	1.79381(-2)	1.78660(-1)	1.68858(0)
	RK3(D)	1.13384(-4)	1.12408(-3)	1.12327(-2)
0.00625	RK3	1.42852(-1)	1.36760(0)	9.03167(0)
	RK3(D)	1.80905(-3)	1.80139(-2)	1.80205(-1)
0.0125	RK3	1.10279(0)	7.881169(0)	1.41391(1)
	RK3(D)	2.899448(-2)	2.897061(-1)	2.89349(0)
0.025	RK3	6.77466(0)	1.41350(+1)	1.41426(1)
	RK3(D)	4.73271(-1)	4.72114(0)	2.82838(1)
0.05	RK3	1.41931(+1)	1.41931(+1)	1.41931(1)
	RK3(D)	8.19326(0)	2.82805(+1)	2.82805(1)

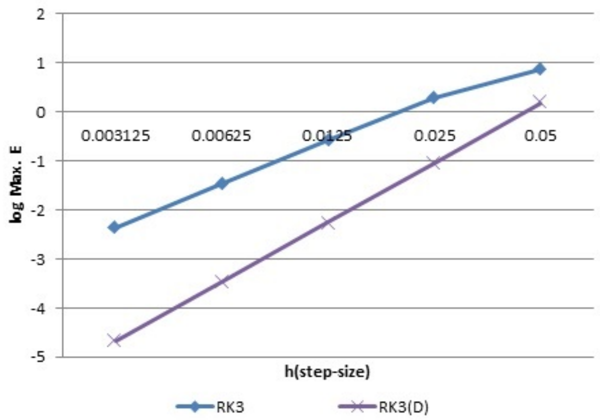

Fig. 1 Comparison for RK3 and RK3(D) methods for Problem 1 with *b*=100

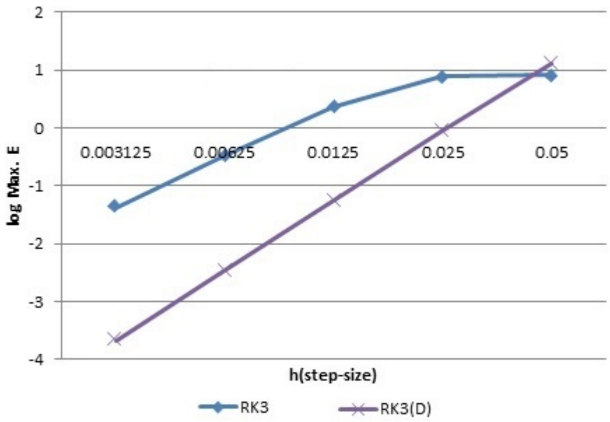

Fig. 2 Comparison for RK3 and RK3(D) methods for Problem 1 with *b*=1000

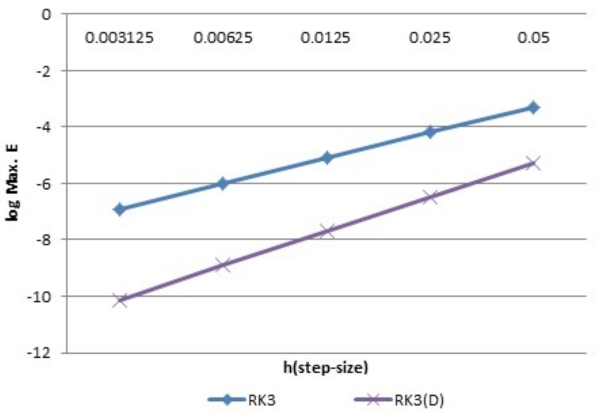

Fig. 3 Comparison for RK3 and RK3(D) methods for Problem 2 with *b*=100

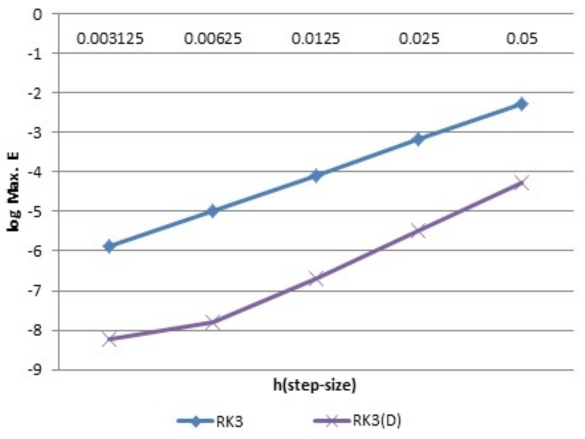

Fig. 4 Comparison for RK3 and RK3(D) methods for Problem 2 with *b*=1000

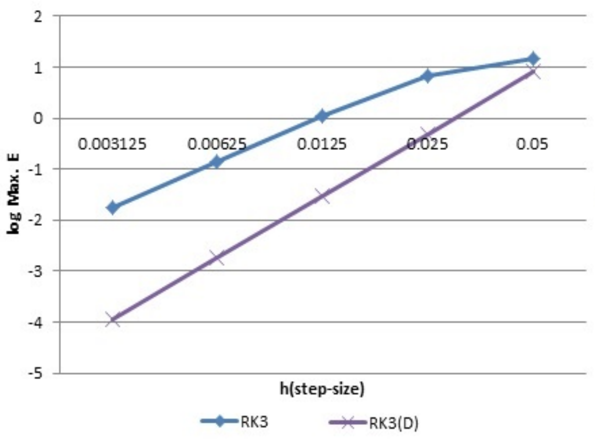

Fig. 5 Comparison for RK3 and RK3(D) methods for Problem 3 with *b*=100

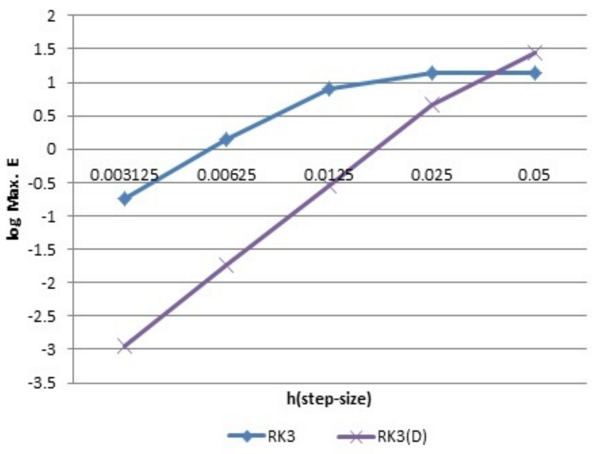

Fig. 6 Comparison for RK3 and RK3(D) methods for Problem 3 with *b*=1000

V. CONCLUSION

In this paper, we have derived a new third order zero-dissipative RK method. The new method is based on Dormand's third algebraic order RK method. Numerical results show that the new method is more accurate for solving second-order differential equations with oscillating solutions.

REFERENCES

[1] J.R. Cash, A.D. Raptis, T.E. Simos, "A sixth-order exponentially fitted method for the numerical solution of the radial Schrodinger equation,"*J. Comput. Phys.*,vol. 91, pp. 413–423, 1990.

[2] T.E. Simos, "A four-step method for the numerical solution of the Schrodinger equation," *J. Comput. Appl. Math.*,vol. 30,pp. 251–255, 1990.

[3] G. Avdelas, T.E. Simos, "Embedded methods for the numerical solution of the Schrodinger equation," *Comput. Math.Appl.*,vol. 31, pp. 85–102, 1996.

[4] G. Avdelas, T.E. Simos, "A generator of high-order embedded P-stable methods for the numerical solution of the Schrodinger equation,"*J. Comput. Appl. Math.*,vol. 72, pp. 345–358, 1996.

[5] T.E. Simos, "P-stable exponentially-fitted methods for the numerical integration of the Schrodinger equation," *J. Comput. Phys.*,vol. 148,no. 2, pp. 305 –321, 1999.

[6] S.Z. Ahmad, F. Ismail, N. Senu, & M. Suleiman, "Zero-Dissipative Phase-Fitted Hybrid Methods for Solving Oscillatory Second Order Ordinary Differential Equations," *Applied Mathematics and Computation*, vol. 219,pp. 10096–10104, 2013.

[7] S.Z. Ahmad, F. Ismail, N. Senu, & M. Suleiman, "Semi Implicit Hybrid Methods with Higher Order Dispersion for Solving Oscillatory Problems," *Abstract and Applied Analysis*, vol. 2013, Article ID 136961, 10 pages, 2013.

[8] N.Senu, M. Suleiman, F. Ismail, M. Othman, "A Singly Diagonally Implicit Runge-Kutta-Nystrom Method for Solving Oscillatory Problems," *IAENG International Journal of Applied Mathematics*, vol. 41, no. 2, pp.155–161, 2011.

[9] N.Senu, M. Suleiman, F. Ismail, M. Othman, "A New Diagonally Implicit Runge-Kutta-Nyström Method for Periodic IVPs,"*WSEAS Transactions On Mathematics*, vol 9, no. 9, pp. 679–688, 2010.

[10] N. Senu, M. Suleiman, F. Ismail, M. Othman, "A Zero-dissipative Runge-Kutta-Nystrom Method with Minimal Phase-lag, (2010),"*Mathematical Problems in Engineering*, vol. 2010 (2010), Article ID 591341, 15 pages, 2010.

[11] N.Senu, M. Suleiman, F. Ismail dan M. Othman, "KaedahPasangan 4(3) Runge-Kutta-Nystrom untuk Masalah Nilai Awal Berkala," *Sains Malaysiana*, vol. 39, no. 4, pp. 639–646, 2010.

[12] L. Brusa, L. Nigro, "A one-step method for direct integration of structural dynamic equations,"*Int. J. Numer. Methods Engin.*, vol. 15, pp. 685–699, 1980.

[13] P. J. van der Houwen and B. P. Sommeijer, "Explicit Runge-Kutta(-Nyström) methods with reduced phase errors for computing oscillating solutions," *SIAM J. Numer. Anal.*, vol. 24, no. 3, pp. 595–617, 1987.

[14] N. Senu, M. Suleiman, F. Ismail. "An embedded explicit Runge-Kutta-Nyström method for solving oscillatory problems,"*Phys. Scr.*,vol. 80, no. 1, pages 015005, 2009.

[15] H. Van der Vyver, "A symplecticRunge-Kutta-Nyström method with minimal phase-lag," *Physics Letters A*,vol. 367, pp. 16–24, 2007.

[16] T.E. Simos, "A Runge-Kutta-Fehlberg method with phase-lag of order infinity for initial-value problems with oscillating solution," *Comput. Math. Appl.*, vol. 25, pp. 95–101, 1993.

[17] J.R. Dormand, *Numerical Methods for Differential Equations*, CRC Press, Inc, Florida, 1996.

[18] G. Papadeorgiou, Ch. Tsitourus, and S.N Papakostas, "Runge-Kutta Pairs for Periodic Initial Value Problem,"*Computing*,vol. 1,pp. 151–163, 1993.