Young's Modulus Variability: Influence on Masonry Vault Behavior

A. Zanaz, S. Yotte, F. Fouchal, A. Chateauneuf

Abstract—This paper presents a methodology for probabilistic assessment of bearing capacity and prediction of failure mechanism of masonry vaults at the ultimate state with consideration of the natural variability of Young's modulus of stones. First, the computation model is explained. The failure mode corresponds to the four-hinge mechanism. Based on this consideration, the study of a vault composed of 16 segments is presented. The Young's modulus of the segments is considered as random variable defined by a mean value and a coefficient of variation. A relationship linking the vault bearing capacity to the voussoirs modulus variation is proposed. The most probable failure mechanisms, in addition to that observed in the deterministic case, are identified for each variability level as well as their probability of occurrence. The results show that the mechanism observed in the deterministic case has decreasing probability of occurrence in terms of variability, while the number of other mechanisms and their probability of occurrence increases with the coefficient of variation of Young's modulus. This means that if a significant change in the Young's modulus of the segments is proven, taking it into account in computations becomes mandatory, both for determining the vault bearing capacity and for predicting its failure mechanism.

Keywords—Masonry, mechanism, probability, variability, vault.

I. INTRODUCTION

OVER the last two decades, a large amount of researches on masonry structures or masonry structural elements behavior has been conducted. By introducing the concept of representative volume element (RVE) of the composite material (stone or brick and mortar), many authors have proposed homogenization models considering the masonry as a periodic structure [1]-[6] or non-periodic [7], [8] using the "test window" method coupled with a probabilistic convergence criterion [9] previously developed for the analysis of composite structures. The objective of these models is to statistically determine the most representative elementary volume. However, the homogenized characteristics of RVE are obtained by considering the Young's modulus as constant value for all solid blocks (deterministic approach). Nonetheless, this setting can considerably vary in a ratio of 1

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to 2 for stones extracted from the same massif [10]. It might be necessary to take this variation into account for the analysis. The aim of this study is to develop a methodology allowing to assess the bearing capacity and to predict the failure mechanism of masonry vaults with consideration of the natural Young's modulus variability of stones.

The masonry arches behavior assessment requires the use of several methods: MEXE (Military Engineering Experimental Establishment) [11], [12], the limit analysis method [13]-[16], REAM (Railway Empirical Assessment Method) [17]-[19], yield design method [20], [21], the finite element method [22]-[24] and finally the distinct element method [25], [26]. Most of these methods provide the basis for the development of several software packages and computer-based applications (Archie-M, Voute, Ring, Diana...). The vast majority of available assessment methods of masonry vaults are deterministic. They can predict the bearing capacity provided that all variables involved in the mechanical response are assumed as deterministic values, which are not really the case due to uncertainties involved in the geometry, materials, loads ... etc. The reliability-based assessment methods of structures have been developed to address these issues. These methods take into account the uncertainties on the involved variables, through Monte Carlo simulations and other reliability methods [27]. Several experiences have shown the large amount of money that can be saved by efficient and accurate assessment based on a probabilistic approach [28]. The latter often requires making a large number of mechanical response calculations. That is why an interesting compromise was found between the computation time and accuracy, through modeling the arch with 2D beam finite element.

In this study, the analysis of an arch composed of 16 segments is presented. The Young's modulus of segments is assumed as random variable defined by a mean value and a coefficient of variation *CV*. The arch is treated as a plane structure formed of beam elements loaded in bending and compression. The hinge appearance is conditioned by the central third theorem where the formation of four successive hinges is synonymous to failure mechanism. The calculation is implemented in a software application developed for this purpose (ArcProg_Z).

II. CALCULATION MODEL

The failure mode of a vault is global. Indeed, experimental tests carried out on bridges show that the failure of an arch is of a global nature more than due to the failure of one of its components [28]. The chosen model allows performing this global analysis of the vault while minimizing the computation

time. The homogenized characteristics of each voussoir are represented at its average fiber. The tensile strength is assumed nil, which satisfies the requirements of the central third theorem adopted in this study and hinge formation conditions. Backfill reaction is modeled by horizontal and vertical springs acting at each node (i) and affected by stiffness coefficients (k_{hi} and k_{vi}) taking into account the backfill modulus and contact surface corresponding to each node, as shown in Fig. 1. This spring stiffness is nil when the deformations have for effect to separate the structure from the backfill.

The principle of the probabilistic calculation is to assign random value to the segment Young's moduli (Fig. 2). ArcProg_Z calculates the critical load value and its position and determines the failure mechanism for each assigned combination of sampled values. The calculation is repeated for 10,000 Monte Carlo simulations leading to a total of 160,000 Young's modulus values randomly generated by the developed program following a truncated normal distribution. No negative value of the module was generated due to truncation (the chosen normal law parameters make the probability very low), no statistical bias is introduced into the method.
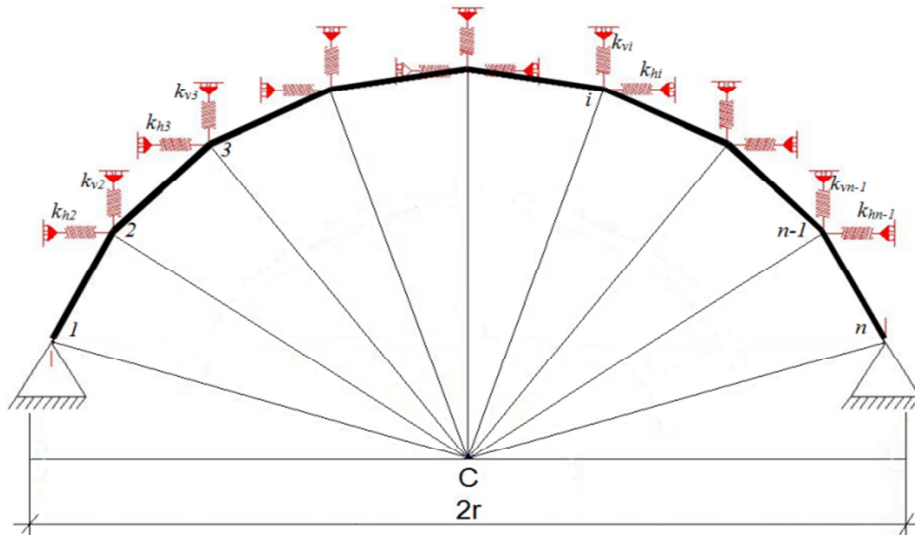

Fig. 1 Modeling of backfill reaction (ArcProg_Z)

Three values of the coefficient of variation (5%, 10% and 20%) of the Young's modulus were considered for this simulation, in order to determine the influence of this variability on the global vault behavior; the mean value being equal to 48GPa. The question marks in Fig. 2 identify the problem unknowns. For each affected combination of modulus values, the position of the load point is moved along the half-vault, with constant steps of 0.25m. Then, ArcProg_Z determines the critical load value for each position by increasing the point load by steps of 5 kN until the appearance of 4 hinges satisfying the failure criterion. The minimum value of the calculated critical load corresponding to the vault bearing capacity is then deduced, as well as the corresponding position X of this minimum load and the position of the 4 hinges forming the failure mechanism. The studied vault is composed of 16 segments of 0.58 m thick (Fig. 2), with a span of 6.18 m, a rise of 2.50 m, a filling depth above the crown of 0.85 m and a track thickness of 0.28 m. The physical and mechanical parameters are summarized in Table I.

The obtained results are analyzed from three perspectives: The first concerns the vault bearing capacity variation depending on the combinations of modulus values of the segments. The second is related to the position of the formed hinges and the third perspective studies the importance of taking account of this variability in the mechanical response calculation.
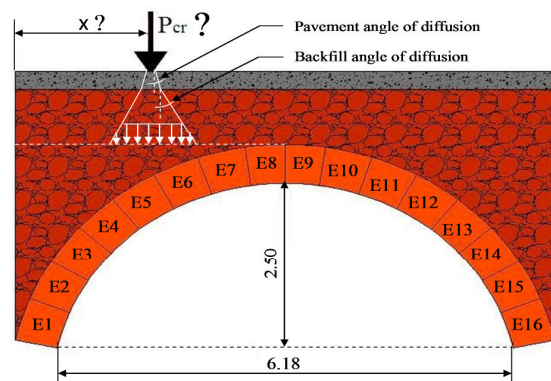

Fig. 2 Loading model

TABLE I
PHYSICAL AND MECHANICAL CHARACTERISTICS OF THE STUDIED VAULT

Designation	Unit	Value
Pavement unit weight	kN/m^3	21
Backfill unit weight	kN/m^3	18
Backfill cohesion	kN/m^2	0
Backfill angle of shearing resistance	rad	$\pi/6$
Pavement angle of diffusion	rad	$\pi/6$
Backfill angle of diffusion	rad	$\pi/6$
Tensile resistance of voussoirs	kN/m^2	0
Backfill Young modulus	kN/m^2	20.10^3

III. INFLUENCE OF YOUNG'S MODULUS VARIABILITY ON THE BEARING CAPACITY

The 10,000 critical load values resulting from Monte Carlo simulations are ordered in size, and grouped into size ranges of equal intervals of 5 kN. If the critical loads P_{cr1}, P_{cr2} ... P_{crk} appear a_1, a_2 ..., a_k times, with $P_{cr1} < P_{cr2} < ... < P_{crk}$, the probability of each of these loads is given by:

$$p(P_{cri}) = \frac{a_i}{\sum_1^k a_i} = \frac{a_i}{10000} \tag{1}$$

This probability estimate is an approximation of the probability density function of the load capacity $P(P_{cri})$. Fig. 3 gives the probability density of critical load and its distribution function for the coefficient of variation CV of 10%. The cumulative probability estimate is given by:

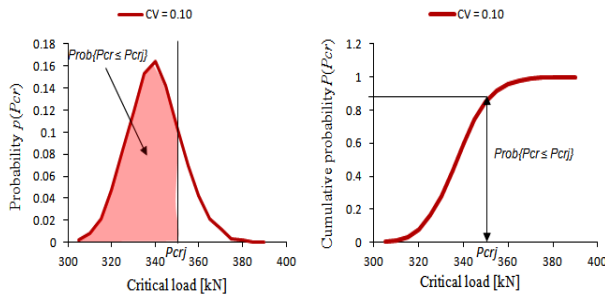$$p(P_{cri}) = \frac{\sum_1^j a_i}{\sum_1^{18} a_i} = \frac{\sum_1^j a_i}{10000} \tag{2}$$

Fig. 3 Probability density and cumulative functions of the load capacity ($CV = 10\%$)

The obtained distributions for the three CV values (5, 10 and 20%) seem to be symmetrical with mean value of 340 kN. Normality test has proved that they were normal distributions with a mean value of 340 kN and standard deviations of 6.44, 12.55 and 27.23 kN (fitting with 0.98%, 0.93% and 4.77% error respectively).

The output data domain (critical load) increases with the coefficient of variation. Thus the consideration of the mean critical load is inappropriate since the loss of stability will result as soon as an extreme variation of Yong modulus of the segments is realized. In this case, conservative value calculations are required in order to predict the critical limit of bearing capacity with sufficient confidence.

The proportion of the critical loads less than 305 kN (for example) can be read directly from the graph of cumulative probability in Fig. 4. Table II gives some characteristic values of the three dispersion levels.

A. Modeling of Critical Load Variability

In all three cases, limiting the vault bearing capacity to minimum value ensures its stability with a probability of 99.8% regardless of Young's modulus values of different segments. If the construction managers consider the lower confidence bound (95% for example), the vault bearing capacity is increased by 4 to 16%. In all cases, assuming that the mean value of critical load is the vault bearing capacity can overestimate the safety of the structure. In this context, it should be clarified that the mean value corresponds to the vault bearing capacity without taking into account the variability of Young's modulus (the deterministic case), and additional safety factors should be considered.

TABLE II
CHARACTERISTIC VALUES

| | Domain of P_{cr} | | | | Quantiles | | | | |
	Min	Max	μ	σ	1%	2%	3%	4%	5%
$P_{cr}(CV=5\%)$	315	370	340	6.44	324.38	326.14	327.25	328.09	328.77
$P_{cr}(CV=10\%)$	305	390	340	12.55	310.59	314.02	316.19	317.82	319.15
$P_{cr}(CV=20\%)$	250	515	340	27.23	277.93	285.35	290.35	293.60	296.48

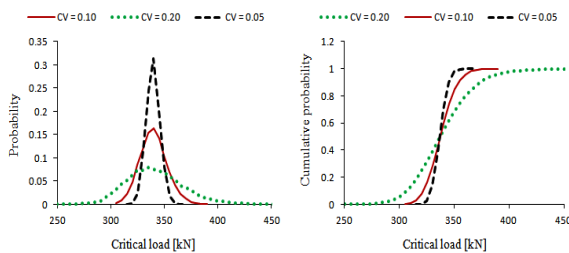μ = mean value, σ = standard deviation

Fig. 4 Probability density and distribution function of P_{cr} ($CV = 5\%$, 10% and 20%)

It is therefore important to find a relationship that links the critical load to Young's modulus variability of various segments, in order to define the influence of this variability on the vault bearing capacity. For that purpose, a linear regression using the least square method has been performed, leading to:

$$P_{cr} = 10^{-7} . [4.88\, E_1 + 3.85\, E_2 + 1.58\, E_3 + 0.44\, E_4 + 1.7\, E_5 + 4.92\, E_6 + 6.85\, E_7 + 4.96\, E_8 + 0.45\, E_9 - 4.74\, E_{10} - 8.77\, E_{11} - 10.82\, E_{12} - 10.46\, E_{13} - 7.28\, E_{14} - 0.17\, E_{15} + 12.34\, E_{16}] + 341 \tag{3}$$

This relationship can be written as:

$$P_{cr} = \sum_1^{16} a_i(E_i) + P_0 \tag{4}$$

The relationship (4) is composed of two parts. The first part is variable and presents the influence of Young's modulus variability on the critical load. The second part is constant, which is the mean value of the critical load. In the case of constant modulus E throughout the vault, the first part of the above relationship is nil, which requires:

$$\begin{cases} \sum_1^{16} a_i = 0 \\ P_{cr} = P_0 = \mu \end{cases} \tag{5}$$

The final relationship is written as:

$$P_{cr} = \sum_1^{16} a_i(E_i) + \mu \qquad (6)$$

The relationship (6) is valid for a point load applied on the left side of the vault. For a load applied on the right side, the relationship becomes:

$$P_{cr} = \sum_1^{16} a_i(E_{17-i}) + \mu \qquad (7)$$

If we put: $E_i = \bar{E} + \Delta E_i$, where $\bar{E}$ is the mean value of Young's modulus. The relationship (4) gives:

$$P_{cr} = \sum_1^{16} a_i(E_i) + \mu = \sum_1^{16} a_i(\bar{E} + \Delta E_i) + \mu \quad = \bar{E}\sum_1^{16} a_i + \sum_1^{16} a_i(\Delta E_i) + \mu \qquad (8)$$

$$\sum_1^{16} a_i = 0 \Rightarrow P_{cr} = \sum_1^{16} a_i(\Delta E_i) + \mu \qquad (9)$$

Similarly (7) takes the form:

$$P_{cr} = \sum_1^{16} a_i(\Delta E_{17-i}) + \mu \qquad (10)$$

Finally, (9) and (10) are respectively written as:

$$P_{cr} = 10^{-7}.[4.88\,\Delta E_1 + 3.85\,\Delta E_2 + 1.58\,\Delta E_3 + 0.44\,\Delta E_4 + 1.7\,\Delta E_5 + 4.92\,\Delta E_6 + 6.85\,\Delta E_7 + 4.96\,\Delta E_8 + 0.45\,\Delta E_9 - 4.74\,\Delta E_{10} - 8.77\,\Delta E_{11} - 10.82\,\Delta E_{12} - 10.46\,\Delta E_{13} - 7.28\,\Delta E_{14} - 0.17\,\Delta E_{15} + 12.34\,\Delta E_{16}] + 341 \quad (11)$$

$$P_{cr} = 10^{-7}.[4.88\,\Delta E_{16} + 3.85\,\Delta E_{15} + 1.58\,\Delta E_{14} + 0.44\,\Delta E_{13} + 1.7\,\Delta E_{12} + 4.92\,\Delta E_{11} + 6.85\,\Delta E_{10} + 4.96\,\Delta E_9 + 0.45\,\Delta E_8 - 4.74\,\Delta E_7 - 8.77\,\Delta E_6 - 10.82\,\Delta E_5 - 10.46\,\Delta E_4 - 7.28\,\Delta E_3 - 0.17\,\Delta E_2 + 12.34\,\Delta E_1] + 341 \quad (12)$$

B. Result Analysis

By observing (11) and (12), it can be seen that Young's modulus variations of segments number: 1, 8, 9 and 16 are weighted by positive coefficients regardless the point load position. It means that ensuring that these segments have E values greater than the mean value $\bar{E}$ will certainly lead to an increase of the vault bearing capacity. The variation of the modulus would have an adverse affect when the point load switches to the other side of the vault (Table III). Conversely, a decrease in modulus values of the above-mentioned segments gives negative variations ΔE, weighted by positive coefficients, leads to a decrease of the vault bearing capacity.

Table III shows that from the segments number 2 to 7 and from 10 to 15 the modulus variation must be low, i.e. the modulus of these segments must be close to the average value, especially for segments 3 to 6 and for segments 11 to 14 which are the vault haunches. A thickness loss at one of these segments substantially decreases the vault bearing capacity, and consequently increases the collapse risk of the structure. It is therefore recommended during inspection to start first by the springers of the vault then, the haunches and finally the keystone. The same priority order is maintained in the repair phase. This can be used to optimize the inspection and repair schedules, and consequently, to reduce the budget expenditure.

TABLE III
A_I VALUES ACCORDING TO THE POSITION OF THE POINT LOAD P

Number of Voussoir	P applied on :		Number of Voussoir	P applied on :	
	Left side [x 10⁻⁷]	Right side [x 10⁻⁷]		Left side [x 10⁻⁷]	Right side [x 10⁻⁷]
1	4.88	12.34	9	0.45	4.96
2	3.85	-0.17	10	-4.74	6.85
3	1.58	-7.28	11	-8.77	4.92
4	0.44	-10.46	12	-10.82	1.7
5	1.7	-10.82	13	-10.46	0.44
6	4.92	-8.77	14	-7.28	1.58
7	6.85	-4.74	15	-0.17	3.85
8	4.96	0.45	16	12.34	4.88

IV. INFLUENCE OF YOUNG'S MODULUS VARIABILITY ON THE FAILURE MECHANISM

ArcProg_Z determines the hinge positions of the mechanism at failure. The hinge position is then determined in the three treated cases. The search results are sorted by mechanism. The probability of appearance of each hinge is also calculated. In the following section, the obtained results are analyzed for each *CV* values.

A. Failure Mechanism with CV = 5%

Fig. 5 gives a graphic summary of the obtained results in both cases of point load position, on the left (Fig. 5 (a)), and on the right (Fig. 5 (b)) of the vault key. The analysis developed in this subsection concerns the case of Fig. 5 (a) relative to point load applied on the left side of the vault. The methodology is the same for the case of load applied on the right of the keystone (Fig. 5 (b)) and leads to the same results. Out of the 10,000 random simulations of the modulus distributions of the segments, the first hinge is observed at the joint number 2 with 100% of probability:

$$P(1) = 1 \qquad (13)$$

The second hinge occurres at joint 17 with the same probability:

$$P(17|1) = 1 \qquad (14)$$

Generally, the hinge position depends on those of the previous but the results have shown that the third hinge may occur at one of the two joints: either the joint number 6 with 13.7 probability of occurrence or the joint number 7 with a probability of 86.3%.

$$\begin{cases} P(6|1,17) = 0.137 \\ P(7|1,17) = 0.863 \end{cases} \qquad (15)$$

In this case, the first and the second hinge are observed at only one position, which means that the fourth hinge position depends solely on the position of the third one. The formation of the fourth hinge is synonymous to failure and expected to occur at two joints: 12 or 13. The result (16) gives the values of the probability of appearance of the fourth hinge at these joints conditioned by the previous hinge positions.

The results (16) reveal the probability of occurrence of three

mechanisms whose values strongly favor the mechanism (2-17-7-13) that corresponds to the same mechanism observed in the deterministic case where the variability of the Young's modulus is not taken into account. Nevertheless, the two other mechanisms are also observed, which should be taken even more seriously, namely: the mechanism (12-6-17-2) with 6.96% probability to be the failure mechanism and the

mechanism (13-6-17-2) with probability of occurrence of 6.73%.

$$\begin{cases} P(12|6,1,17) = 0.07 \\ P(13|6,1,17) = 0.067 \\ P(13|7,1,17) = 0.863 \end{cases} \qquad (16)$$
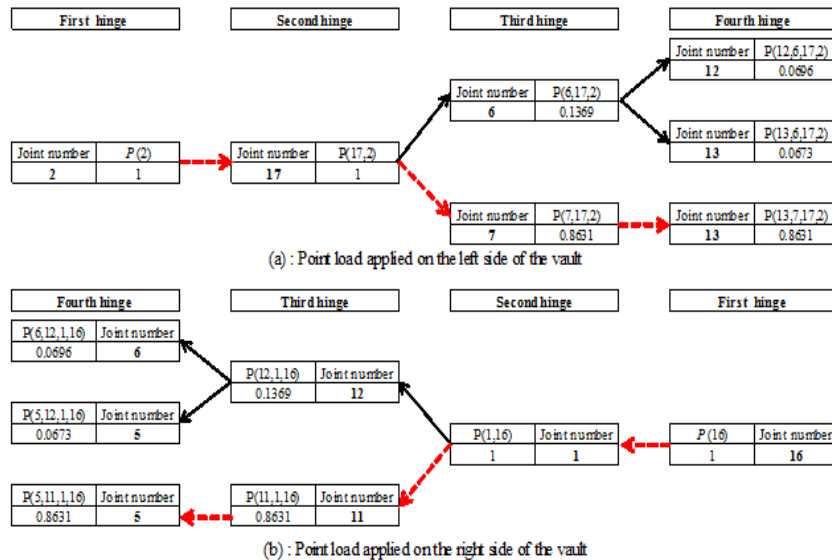

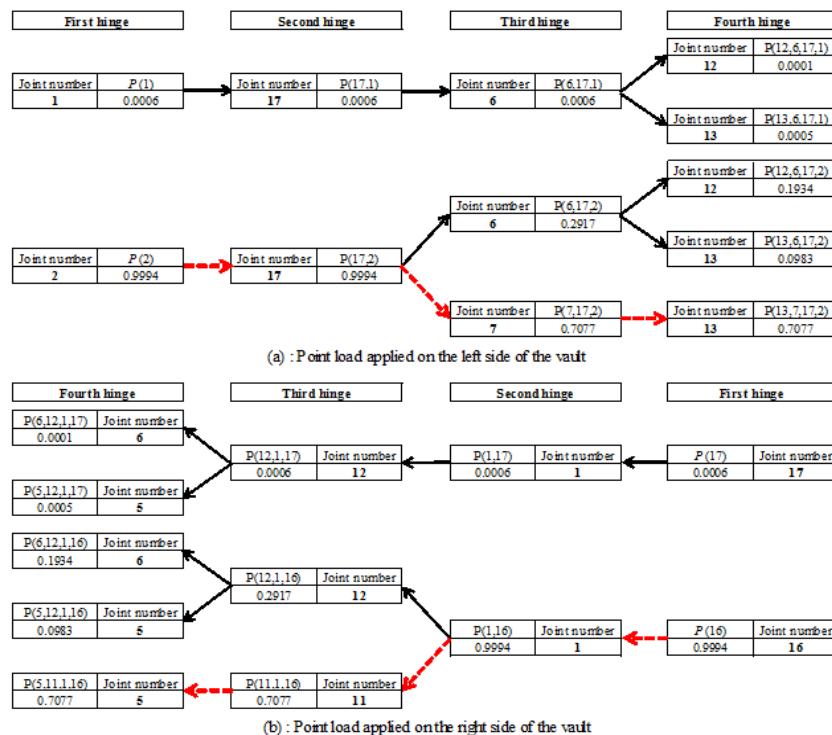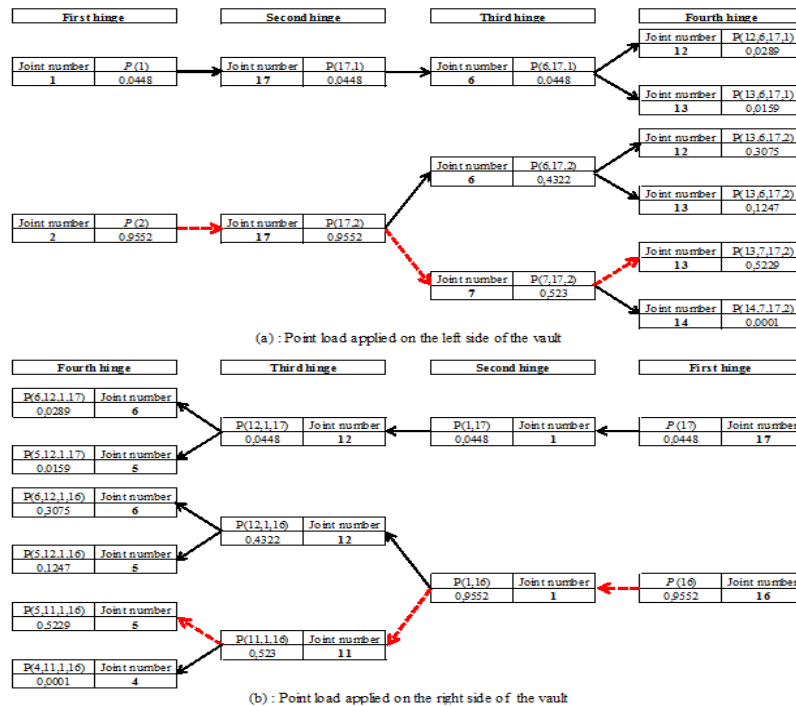Fig. 5 Probability of hinges appearance ($CV = 5\%$)

Fig. 6 Probability of hinges appearance ($CV = 10\%$)

(a) : Point load applied on the left side of the vault

(b) : Point load applied on the right side of the vault

Fig. 7 Probability of hinges appearance (*CV* = 20%)

B. Failure Mechanism with CV = 10%

In addition to the mechanism formed in the deterministic case, four other mechanisms are observed: (12-1-17-6), (13-1-17-6), (12-2-17-6) and (13-2-17-6) with probabilities of occurrence of 0.01%, 0.05%, 19.3% and 9.8% respectively. The most probable mechanism is still the mechanism (13-2-17-7) but with a lower probability: 70.8% rather than 86.3% in the previous case, as shown in Fig. 6. The probabilities of occurrence of the two observed mechanisms in the first case (*CV* = 5%), namely (2-17-6-12) and (2-17-6-13) have increased from 7% to 19.3% and from 6.7% to 9.8% respectively, which highlights even more the importance of taking into account the variability of Young's modulus in the vault behavior assessment.
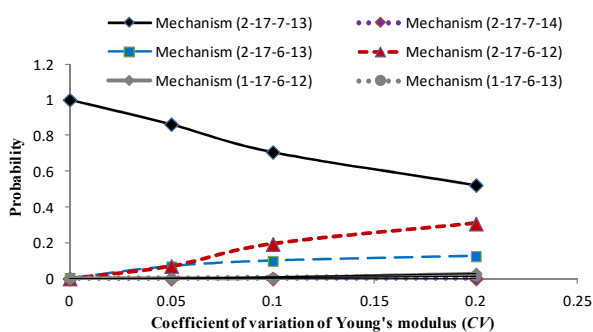
Fig. 8 Probability of occurrence of observed mechanisms depending to coefficient of variation CV

C. Failure Mechanism with CV = 20%

In this last case, five other mechanisms are observed with significant probabilities: (1-17-6-12), (1-17-6-13), (2-17-6-12), (2-17-6-13) and (2-17-7-14) with probabilities of occurrence of 2.9%, 1.6%, 30.8%, 12.5% and 0.01% respectively (Fig. 7).

The most probable mechanism is once again the mechanism (13-2-17-7) but with even lower probability of 52.29%. Fig. 8 clearly shows the variation of probability of various mechanisms depending on the coefficient of variation of the Young's modulus. This means that if a significant change in the Young's modulus of the segments is proven; taking into account in the computations becomes mandatory, both for determining the vault bearing capacity and for predicting its failure mechanism.

TABLE IV
PROBABILITY OF OCCURRENCE OF OBSERVED MECHANISMS

CV	Observed mechanisms					
	2-17-7-13	2-17-6-12	2-17-6-13	1-17-6-12	1-17-6-13	2-17-7-14
0%	100%	0%	0%	0%	0%	0%
5%	86,31%	6,96%	6,73%	0%	0%	0%
10%	70,77%	19,34%	9,83%	0,1%	0,5%	0%
20%	52,29%	30,75%	12,47%	2,89%	1,59%	0,1%

Table IV summarizes all the mechanisms identified for each considered *CV* value and their probability of occurrence.

V. CONCLUSION

A methodology for probabilistic assessment of bearing capacity and prediction of failure mechanisms of masonry vaults at the ultimate state is presented. This methodology involves the natural variability of Young's modulus of vault

segments in calculation. Considering the Young's modulus of the segments as random variable allows determining the distribution function of the critical load. The calculation has been performed for three values of the coefficient of variation CV (5%, 10% and 20%).

The obtained results enabled us to propose relationship linking the vault bearing capacity to the segment modulus variation. A relationship that allowed identifying the segments whose modulus decrease will mostly contribute to the decrease of the bearing capacity i.e. segments for which no defect is tolerated. This allows us to establish an order of priority, valid for both inspection and repair phases: firstly, the springing of the vault, secondly, the haunches, and thirdly the vault key. This classification may optimize the inspection and repair schedules, and consequently, minimizes the budget expenses.

In each case (deterministic, $CV = 5\%$, $CV = 10\%$ and $CV = 20\%$), the failure mechanisms are identified as well as their probability of occurrence. The results show that the observed mechanism in deterministic case corresponds to that with the highest occurrence probability, which decreases depending on the coefficient of variation CV, while the number of the other mechanisms as well as their occurrence probabilities increase. This means that if a significant change in the Young's modulus of the segments is proven, taking it into account in the computations becomes mandatory, for both determining the vault bearing capacity and for predicting its failure mechanism.

REFERENCES

[1] P. de Buhan and G. de Felice, "A homogenization approach to the ultimate strength of brick masonry," *J. Mech. Phys. Solids*, vol. 45, no. 7, pp. 1085–1104, Jul. 1997.

[2] P. Pegon and A. Anthoine, "Numerical strategies for solving continuum damage problems with softening: Application to the homogenization of Masonry," *Comput. Struct.*, vol. 64, no. 1–4, pp. 623–642, Jul. 1997.

[3] R. Luciano and E. Sacco, Homogenization technique and damage model for old masonry material," *Int. J. Solids Struct.*, vol. 34, no. 24, pp. 3191–3208, Aug. 1997.

[4] A. Zucchini and P. B. Lourenço, "A micro-mechanical model for the homogenisation of masonry," *Int. J. Solids Struct.*, vol. 39, no. 12, pp. 3233–3255, Jun. 2002.

[5] M. Mistler, A. Anthoine, and C. Butenweg, "In-plane and out-of-plane homogenisation of masonry," *Comput. Struct.*, vol. 85, no. 17–18, pp. 1321–1330, Sep. 2007.

[6] A. Zucchini and P. B. Lourenço, "A micro-mechanical homogenisation model for masonry: Application to shear walls," *Int. J. Solids Struct.*, vol. 46, no. 3–4, pp. 871–886, Feb. 2009.

[7] F. Cluni and V. Gusella, "Homogenization of non-periodic masonry structures," *Int. J. Solids Struct.*, vol. 41, no. 7, pp. 1911–1923, Apr. 2004.

[8] Gusella and Cluni, "Random field and homogenization for masonry with nonperiodic microstructure," *J. Mech. Mater. Struct.*, vol. 1, no. 2, p. 357e386, 2006.

[9] C. Huet, "Application of variational concepts to size effects in elastic heterogeneous bodies," *J. Mech. Phys. Solids*, vol. 38, no. 6, pp. 813–841, 1990.

[10] L. Binda, G. Baronio, C. Tiraboschi, and C. Tedeschi, "Experimental research for the choice of adequate materials for the reconstruction of the Cathedral of Noto," *Constr. Build. Mater.*, vol. 17, no. 8, pp. 629–639, Dec. 2003.

[11] PIPPARD A. J. S., "A study of the Voussoir arch," *His Majesty's stationery Office, National Building Studies, Research Paper n°11, 52p.*, 1951.

[12] Christchurch, "MEXE. Military Engineering experimental Establishment." «Military load classification of civil bridges by reconnaissance and correlation methods», 1963.

[13] Kooharian, "Limit Analysis of Voussoir (Segmental) and Concrete Arches," *Journal of the American Concrete Institute 317-328, V. 24, N° 4, Dec. 1952, Proceedings V. 49.*, 1953.

[14] G. A. Drosopoulos, G. E. Stavroulakis, and C. V. Massalas, "Limit analysis of a single span masonry bridge with unilateral frictional contact interfaces," *Eng. Struct.*, vol. 28, no. 13, pp. 1864–1873, Nov. 2006.

[15] A. Cavicchi and L. Gambarotta, "Lower bound limit analysis of masonry bridges including arch–fill interaction," *Eng. Struct.*, vol. 29, no. 11, pp. 3002–3014, Nov. 2007.

[16] E. Milani, G. Milani, and A. Tralli, "Limit analysis of masonry vaults by means of curved shell finite elements and homogenization," *Int. J. Solids Struct.*, vol. 45, no. 20, pp. 5258–5288, Oct. 2008.

[17] TRAUTWINE, J. C., *Civil engineer's pocket-book*. New York Wiley publisher, 1871, 770 p., 1871.

[18] Harvey, "Rule of thumb method for the assessment of arches." Rapport UIC, draft, 2007, pp. 22, 2007.

[19] ORBAN, Z., "Improving assessment, optimisation of maintenance and development of database for masonry arch bridges." International Union of Railways, UIC infrastructure Department, 13 p., 2008, 2008.

[20] Harvey WEJ, "Application of the mechanism analysis to masonry arches," *Struct Eng 1988*, vol. 66, no. 5, pp. 77–84, 1988.

[21] Buhan P., Mangiavacchi R., Nova R., Pellegrini G., and Salencon J., "Yield design of reinforced earth walls by a homogenization method," *J Geotech.*, vol. 39, no. 2, pp. 189–201, 1989.

[22] B. T. Rosson, K. Søyland, and T. E. Boothby, "Inelastic behavior of sand-lime mortar joint masonry arches," *Eng. Struct.*, vol. 20, no. 1–2, pp. 14–24, Jan. 1998.

[23] P. J. Fanning and T. E. Boothby, "Three-dimensional modelling and full-scale testing of stone arch bridges," *Comput. Struct.*, vol. 79, no. 29–30, pp. 2645–2662, Nov. 2001.

[24] E. Reccia, G. Milani, A. Cecchi, and A. Tralli, "Full 3D homogenization approach to investigate the behavior of masonry arch bridges: The Venice trans-lagoon railway bridge," *Constr. Build. Mater.*, vol. 66, pp. 567–586, Sep. 2014.

[25] A. R. Tóth, Z. Orbán, and K. Bagi, "Discrete element analysis of a stone masonry arch," *Mech. Res. Commun.*, vol. 36, no. 4, pp. 469–480, Jun. 2009.

[26] G. Milani and P. B. Lourenço, "3D non-linear behavior of masonry arch bridges," *Comput. Struct.*, vol. 110–111, pp. 133–150, Nov. 2012.

[27] K.-H. Ng and C. A. Fairfield, "Monte Carlo simulation for arch bridge assessment," *Constr. Build. Mater.*, vol. 16, no. 5, pp. 271–280, Jul. 2002.

[28] J. R. Casas, "Reliability-based assessment of masonry arch bridges," *Constr. Build. Mater.*, vol. 25, no. 4, pp. 1621–1631, Apr. 2011.