International Journal of Architectural, Civil and Construction Sciences
ISSN: 2415-1734
Vol:13, No:6, 2019

Weak Instability in Direct Integration Methods for Structural Dynamics

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***Abstract*—**Three structure-dependent integration methods have been developed for solving equations of motion, which are second-order ordinary differential equations, for structural dynamics and earthquake engineering applications. Although they generally have the same numerical properties, such as explicit formulation, unconditional stability and second-order accuracy, a different performance is found in solving the free vibration response to either linear elastic or nonlinear systems with high frequency modes. The root cause of this different performance in the free vibration responses is analytically explored herein. As a result, it is verified that a weak instability is responsible for the different performance of the integration methods. In general, a weak instability will result in an inaccurate solution or even numerical instability in the free vibration responses of high frequency modes. As a result, a weak instability must be prohibited for time integration methods.

***Keywords*—**Dynamic analysis, high frequency, integration method, overshoot, weak instability.

I. Introduction

In a nonlinear dynamic analysis or a substructure pseudo-dynamic test, an integration method is generally required to carry out time integration. Although both explicit and implicit integration methods can be adopted for the calculations, either an explicit or implicit integration method will experience its own difficulty. A small step size must be adopted for an explicit integration method to meet stability and thus it will significantly increase the total number of time steps for nonlinear dynamic analysis or it is incapable of conducting pseudodynamic tests due to the presence of high frequency modes. On the other hand, an iteration procedure must be generally adopted for an implicit integration method. Hence, the computation details of each time step will become complex and is time consuming for nonlinear dynamic analysis. In addition, extra hard wares are also needed for implementing an implicit pseudodynamic algorithm. An unconditionally stable, explicit structure-dependent integration method was first developed by Chang [1] for overcoming the difficulty experienced in the pseudodynamic tests, where a test specimen with high frequency modes cannot be performed due to numerical instability for explicit pseudodynamic algorithm. This integration method is known as the Chang Explicit Method (CEM). Some of this structure-dependent type of integration methods were developed subsequently [2]-[8].

In addition to CEM, two similar integration methods were also developed for time integration. The method developed by Chen and Ricles [9] is referred as CRM, while that developed by Tang and Lou [10] is referred as TLM. Notice that CEM is a member of the first Chang family method [11] and CRM is a member of the second Chang family method [12]. In general, two difference equations are required for solving an equation of motion. One is a displacement difference equation and the other is a velocity difference equation. In general, CEM is considered as a semi-explicit integration method. This is because that it has an explicit displacement difference equation and an implicit velocity difference equation, whereas both CRM and TLM are considered as fully explicit integration methods. This is because that both difference equations are explicit for CRM and TLM. It will be shown that these three integration methods generally have the same numerical properties for linear elastic systems since they share the same characteristic equation for zero viscous damping for linear elastic systems. However, they have different performance in calculating a free vibration response to the systems with high frequency modes. This phenomenon will be numerically illustrated and the cause of this phenomenon will be analytically explored.

II. Formulations and Basic Numerical Properties

The general formulation of CEM, CRM and TLM for a single degree of freedom system can be simply written as:

$$\begin{aligned} & ma_{i+1} + cv_{i+1} + kd_{i+1} = f_{i+1} \\ & d_{i+1} = d_i + \beta_1 (\Delta t) v_i + \beta_2 (\Delta t)^2 a_i \\ & v_{i+1} = v_i + \gamma_1 (\Delta t) a_i + \gamma_2 (\Delta t) a_{i+1}. \end{aligned} \tag{1}$$

where d_i, v_i, a_i and f_i are the displacement, velocity, acceleration and external force at the end of the $i-th$ time step, respectively. The coefficients of β_1, β_2, γ_1 and γ_2 for CEM, CRM and TLM are found to be:

$$\begin{aligned} & \beta_1 = \frac{1}{D}(1+\xi\Omega_0) \;,\; \beta_2 = \frac{\frac{1}{2}}{D} \;,\; \gamma_1 = \tfrac{1}{2} \;,\; \gamma_2 = \tfrac{1}{2} && \text{CEM} \\ & \beta_1 = 1 \;,\; \beta_2 = \frac{1}{D} \;,\; \gamma_1 = \frac{1}{D} \;,\; \gamma_2 = 0 && \text{CRM} \\ & \beta_1 = \frac{1}{D} \;,\; \beta_2 = \frac{1}{D}\left[1 - \xi\Omega_0 - \xi^2\right] \;,\; \gamma_1 = 1 \;,\; \gamma_2 = \tfrac{1}{2} && \text{TLM} \end{aligned} \tag{2}$$

where $\Omega = \omega(\Delta t)$ and $\omega = \sqrt{k/m}$ is a natural frequency determined from the stiffness, where k is the initial stiffness; ξ is a viscous damping ratio. In addition, $D = 1 + \xi\Omega + \frac{1}{4}\Omega^2$ is defined. Apparently, the very different formulations are found for the three integration methods.

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The analysis of each integration method has been conducted in their original developments and thus it will not be elaborated. Alternatively, the numerical properties of each integration method are summarized for comparison. The characteristic equation for linear elastic systems is found to be:

$$\lambda\left[\lambda^2-2\left(\frac{1-\frac{1}{4}\Omega^2}{1+\frac{1}{4}\Omega^2}\right)\lambda+1\right]=0 \qquad (3)$$

for each integration method with zero viscous damping. In general, λ denotes an eigenvalue of the characteristic equation. It can be shown that (3) is the same as that of the constant average acceleration method (AAM). This implies that stability, period distortion and numerical damping properties for CEM, CRM and TLM are the same as those of AAM for linear elastic systems with zero viscous damping.

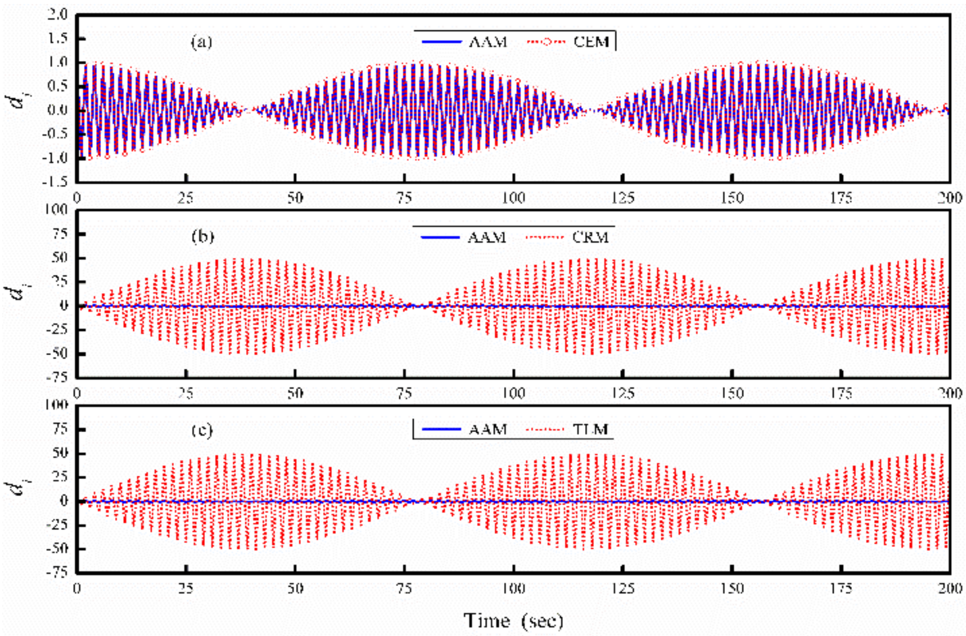

Fig. 1 Comparisons of displacement responses for Case 1 for CEM, CRM and TLM

III. Overshoot in Early Transient Response

An adverse overshoot in the early high frequency transient responses has been discovered [13] if the Wilson-θ method is applied to carry out time integration [14]. Notice that this overshoot cannot be detected by evaluating the characteristic equation. The root cause of this overshoot has been well explored [15] and a technique has been proposed to detect such an unusual overshoot behavior. A tendency to overshoot an exact solution for an integration method can be disclosed by examining the free vibration response to a single degree of freedom system subject to the initial conditions of the previous step data in the limit $\Omega\to\infty$. Hence, it is of great interest to investigate whether each of CEM, CRM and TLM has this adverse property and thus results in a different performance in solving a free vibration to the systems with high frequency modes. As a result, the results for CEM, CRM and TLM in the limit are found to be:

$$\begin{aligned}
&d_{i+1}\approx -d_i \quad , \quad v_{i+1}\approx -v_i && \text{CEM}\\
&d_{i+1}\approx -3d_i+(\Delta t)v_i \quad , \quad v_{i+1}\approx -4(d_i/\Delta t)+v_i && \text{CRM}\\
&d_{i+1}\approx -3d_i \quad , \quad v_{i+1}\approx (d_i/\Delta t)+v_i && \text{TLM}
\end{aligned} \qquad (4)$$

It is manifested from this equation that no overshoot both in displacement and velocity is expected for CEM, CRM and TLM since each term is independent of Ω.

To corroborate the analytical predictions for the overshooting behaviors of the three integration methods, both the displacement and velocity responses to a linear elastic single degree of freedom system as shown in the first line of (1) are computed by using CEM, CRM and TLM. Hence, $m=1$, $c=0$ and $k=10^6$, and are adopted. The natural frequency is found to be $\omega=10^3$ rad/sec. Two initial conditions are considered:

$$\begin{aligned}
&d_0=1 \quad , \quad v_0=0 && \text{Case 1}\\
&d_0=0 \quad , \quad v_0=\omega && \text{Case 2}
\end{aligned} \qquad (5)$$

In general, a free vibration response excited by a nonzero initial displacement is treated in Case 1 while for Case 2 a free vibration response excited by a nonzero initial velocity is considered. A time step of $\Delta t=0.1$sec is chosen for time integration and thus the value of is as large as 100. As a result, numerical solutions corresponding to Case 1 and Case 2 are plotted in Figs. 1 and 2. It is seen in Fig. 1 (a) that the results obtained from CEM coincide with those obtained from AAM and show no overshoot in displacement. Meanwhile, it is manifested from Figs. 1 (b) and (c) that the results calculated from CRM and TLM significantly overshoot the results obtained from AAM. Clearly, the results of CEM is in good agreement with the analytical prediction of no overshoot in displacement while the overshoot of CRM and TLM is totally inconsistent with the analytical prediction of no overshoot in displacement.

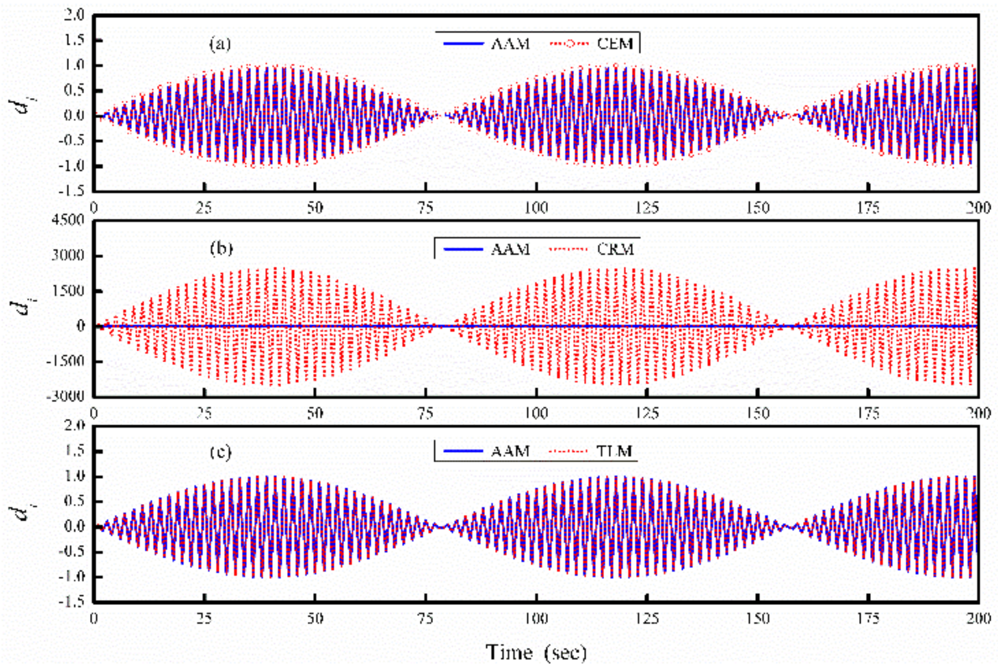

Fig. 2 Comparisons of displacement responses for Case 2 for CEM, CRM and TLM

On the other hand, the displacement responses obtained from CEM and TLM are the same as those obtained from AAM and no overshoot in displacement is found as shown in Figs. 2 (a) and (c). These phenomena are consistent with analytical predictions. Whereas, the results obtained from TLM clearly displays a very significant overshoot behavior as shown in Fig. 2 (b). As a result, this behavior is inconsistent with the analytical result. In summary, the numerical results for CEM are in good agreement with the analytical predictions both in displacement and velocity while for CRM and TLM inconsistency is found. Hence, the cause of this inconsistency between the calculated results and the analytical predictions as shown in (4) must be further explored.

IV. Cause of Overshoot

To explore the root cause of the overshoot in the calculated results obtained from CRM and TLM while there is no overshoot for the results obtained from CEM as shown in Figs. 1 and 2, an analytical scheme is applied to evaluate these overshoot behaviors. In general, the scheme is to analytically derive the numerical solution obtained from an integration method for a free vibration response of an undamped, linear elastic single degree of freedom system. On the other hand, an exact free vibration response is derived from the fundamental theory of structural dynamics. Hence, comparing the numerical solution to the exact solution, an overshoot phenomenon can be disclosed.

It is very straightforward to yield an exact free vibration response to a linear elastic single degree of freedom system subject to a combined initial condition of d_0 and v_0. As a result, an exact displacement response is found to be:

$$d_n = \cos(n\Omega) d_0 + \frac{\sin(n\Omega)}{\Omega} (\Delta t) v_0 \tag{6}$$

Clearly, d_n is a bounded oscillatory displacement response. Meanwhile, an integration method can be also applied to compute the free vibration response. In general, the application of an integration method to compute the complete free vibration response can be expressed in a recursive matrix form as:

$$\mathbf{X}_n = \mathbf{A}\mathbf{X}_{n-1} = \mathbf{A}^2\mathbf{X}_{n-2} = \cdots\cdots = \mathbf{A}^n\mathbf{X}_0 \tag{7}$$

where $\mathbf{X}_n = \left[d_n, \ (\Delta t) v_n, \ (\Delta t)^2 a_n \right]^T$ and a_0 is found from $(\Delta t)^2 a_0 = -2\xi\Omega_0 (\Delta t) v_0 - \Omega_0^2 d_0$ for the given initial condition of d_0 and v_0. Besides, $\mathbf{A}$ is known as an amplification matrix. It is well recognized that the matrix $\mathbf{A}$ is diagonalizable if it has three linearly independent eigenvectors. As a result, one can have:

$$\mathbf{X}_n = \mathbf{A}^n\mathbf{X}_0 = \mathbf{\Phi}\mathbf{\Lambda}^n\mathbf{\Phi}^{-1}\mathbf{X}_0 \tag{8}$$

where $\mathbf{\Lambda}$ is a diagonal matrix and its diagonal term λ_i is an eigenvalue of $\mathbf{A}$ for $i = 1 \sim 3$, and $\mathbf{\Phi}$ is an eigenvector matrix and each column ϕ_i is the eigenvector corresponding to λ_i. It can be found from (3) that the eigenvalues for the three integration methods are:

$$\lambda_{1,2} = \frac{1-\frac{1}{4}\Omega^2}{1+\frac{1}{4}\Omega^2} \pm i\frac{\Omega}{1+\frac{1}{4}\Omega^2} \quad , \quad \lambda_3 = 0 \tag{9}$$

This equation reveals that the three integration methods generally have three different eigenvalues for a general value of Ω and thus they have three linearly independent eigenvectors. This implies that the matrix $\mathbf{A}$ is generally diagonalizable and then (8) is applicable to calculate the free vibration responses without any difficulty. As a result, no overshoot in displacement and velocity is strongly indicated.

Since an overshoot phenomenon is found in the responses of the numerical example for a large value of Ω, it is necessary to examine the limiting case of $\Omega \to \infty$. Although the three eigenvalues of the three integration methods are generally different for a general value of Ω, the two principal eigenvalues will become identical in the limit $\Omega \to \infty$. In fact, the three eigenvalues are found to be:

$$\lambda_{1,2} = 1 \quad , \quad \lambda_3 = 0 \tag{10}$$

On the other hand, their corresponding eigenvector matrices for CEM, CRM and TLM are found to be:

$$\mathbf{\Phi}_{\mathrm{CEM}} = \begin{bmatrix} 1 & 1 & 0 \\ i\Omega_0 & -i\Omega_0 & -2 \\ -\Omega_0^2 & -\Omega_0^2 & -1 \end{bmatrix}$$
$$\mathbf{\Phi}_{\mathrm{CRM}} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ -\Omega_0^2 & -\Omega_0^2 & 1 \end{bmatrix} \quad , \quad \mathbf{\Phi}_{\mathrm{TLM}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix} \tag{11}$$

It is clear that CEM can still have three linearly independent eigenvectors although it possesses two identical principal eigenvalues in the limit $\Omega \to \infty$. This indicates that its amplification matrix can be still decomposed by $\mathbf{\Phi}_{\mathrm{CEM}}$ by using (8). Consequently, after substituting $\mathbf{\Phi}_{\mathrm{CEM}}$ as shown in (11) into (8), the numerical solution in a mathematical form for CEM in the limit $\Omega \to \infty$ can be derived and it is found to be:

$$d_n = \cos\left(n\bar{\Omega}\right) d_0 + \frac{\sin\left(n\bar{\Omega}\right)}{\Omega}\left(\Delta t\right) v_0 \tag{12}$$

where $\bar{\Omega} = \bar{\omega}(\Delta t)$ and $\bar{\omega}$ is a calculated natural frequency in a numerical procedure in contrast to a true natural frequency ω. In general, it can be found that $\bar{\Omega}$ is close to Ω as it is small while $\bar{\Omega}$ will be significantly different from Ω as it is large.

Unlike CEM, both CRM and TLM do not possess three linearly independent eigenvectors as shown in (11) and thus their amplification matrices are not diagonalizable. However, (7) can be alternatively expressed by using a Jordan canonical form. In fact, there exists a non-singular matrix to have $\mathbf{A} = \mathbf{\Psi J \Psi}^{-1}$. As a result, one can have:

$$\mathbf{X}_n = \mathbf{A}^n \mathbf{X}_0 = \mathbf{\Psi J}^n \mathbf{\Psi}^{-1} \mathbf{X}_0 \tag{13}$$

where

$$\mathbf{\Psi}_{\mathrm{CRM}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -\Omega_0^2 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{\Psi}_{\mathrm{TLM}} = \begin{bmatrix} 2 & 1 & 0 \\ \Omega_0^2 & \Omega_0^2 & 1 \\ -2\Omega_0^2 & -\Omega_0^2 & -1 \end{bmatrix}$$
$$\mathbf{J}^n = \begin{bmatrix} (-1)^n & n(-1)^{n-1} & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

Notice that both CRM and TLM share the same Jordan canonical form and thus $\mathbf{J}_{\mathrm{CRM}} = \mathbf{J}_{\mathrm{TLM}} = \mathbf{J}$. Thus, after substituting $\mathbf{\Psi}_{\mathrm{CRM}}$ and $\mathbf{J}^n$ as shown in (14) into (13), the displacement of d_n in mathematical form for CRM in the limit $\Omega \to \infty$ can be derived. Similarly, that for TLM can be also obtained. As a result, they are found to be:

$$d_n = (2n+1)(-1)^n d_0 - n(-1)^n (\Delta t) v_0 \qquad \text{CRM}$$
$$d_n = (2n+1)(-1)^n d_0 + \frac{\sin\left(n\bar{\Omega}\right)}{\Omega}(\Delta t) v_0 \qquad \text{TLM} \tag{15}$$

After obtaining the mathematically derived numerical solutions for the three integration methods, they can be compared to the exact solution for discussing the overshoot behaviors found in Figs. 1 and 2. For this purpose, the coefficients of d_0 and $(\Delta t)v_0$ for the solutions obtained from CEM, CRM and TLM are summarized in Table I for comparison.

TABLE I
RESPONSE COEFFICIENTS OF d_0 AND $(\Delta t)v_0$

Method	Coefficient of d_0	Coefficient of $(\Delta t)v_0$
exact	$\cos(n\Omega)$	$\frac{\sin(n\Omega)}{\Omega}$
CEM	$\cos(n\bar{\Omega})$	$\frac{\sin(n\bar{\Omega})}{\Omega}$
CRM	$(2n+1)(-1)^n$	$-n(-1)^n$
TLM	$(2n+1)(-1)^n$	$\frac{\sin(n\bar{\Omega})}{\Omega}$

After comparing the third row to the second row in Table I, the coefficients of d_0 and $(\Delta t)v_0$ for CEM are almost the same as those of the exact solution. In fact, the only difference is the true Ω is found for the exact solution while it is replaced by a calculated $\bar{\Omega}$ for the solution calculated from CEM. As a result, there is no overshoot in the limit $\Omega \to \infty$ for CEM. This explains why there is no overshoot in displacement as shown in Figs. 1 (a) and 2 (a) for CEM. On the contrary, it is found that the coefficients of d_0 and $(\Delta t)v_0$ of CRM is drastically different from those of the exact solution. In fact, both the coefficients of d_0 and $(\Delta t)v_0$ increase with the increase of the number of n. Notice that the coefficient of d_0 for the exact solution is $\cos(n\Omega)$, which generally varies from -1 to 1; and that for $(\Delta t)v_0$ is $\sin(n\Omega)/\Omega$, which will diminish to zero for a large Ω. Thus, the difference between (5) and the first line of (14) will become very significant for a large Ω. Since the coefficients of d_0 and $(\Delta t)v_0$ for CRM generally increases with increasing n for a large value of Ω, it has a weak instability. In fact, either a nonzero initial displacement or a nonzero initial velocity will cause an instability. Meanwhile, the coefficient of d_0 for TLM is the same as that of CRM, and thus, TLM also has a weak instability for nonzero initial displacement. Notice that the coefficient of $(\Delta t)v_0$ for TLM is the same as that of CEM and thus TLM a nonzero initial velocity will not result in a weak instability.

The results of this analytical study can be completely applied to explain the phenomena found in both Figs. 1 and 2. A weak instability is applicable to polynomial growth in n of arbitrary order. Consequently, either a nonzero d_0 or $(\Delta t)v_0$ for CRM will lead to a weak instability and a nonzero d_0 will result in a weak instability for TLM. The analytical predictions are consistent with the results found in Figs. 1 and 2.

V. Conclusion

Although an overshoot in high frequency early transient responses has been found by Goudreau and Taylor, there exists a different type of high frequency overshoot in transient responses. The former overshoot behavior can be detected by evaluating the displacement difference equation after removing acceleration in the limit $\Omega \to \infty$, whereas the latter overshoot can be disclosed by assessing the eigenvector matrix in the limit $\Omega \to \infty$. Since both CRM and TLM exhibit a weak instability, they will give inaccurate results or even instability in high frequency transient responses. Hence, their applications are very limited. On the other hand, CEM involves no such an adverse property, and thus, it is preferred over CRM and TLM in structural dynamics and earthquake engineering applications.

Acknowledgment

The author is grateful to acknowledge that this study is supported by the Ministry of Science and Technology, Taiwan, R.O.C., under Grant No. MOST-106-2221-E-027-021.

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