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Variational Iteration Method for the Solution of Boundary Value Problems

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Abstract**—**In this work, we present a reliable framework to solve boundary value problems with particular significance in solid mechanics. These problems are used as mathematical models in deformation of beams. The algorithm rests mainly on a relatively new technique, the Variational Iteration Method. Some examples are given to confirm the efficiency and the accuracy of the method.

Keywords**—**Variational iteration method, boundary value problems, convergence, restricted variation.

I. Introduction

THIS paper discussed the approximate solution of the equation of the form.

$$y^{(2n)} + f(x,y) = k, \quad n \geq 2 \tag{1}$$

Subject to the boundary condition

$$y(a) = \alpha_1, y^1(a) = \alpha_2, y(b) = \beta_1, y^1(b) = \beta_2 \tag{2}$$

Ma and Silva [20] adopted iterative solution for (1) representing beams on elastic foundation when $k = 0$.In the configuration of the deformed beam, the bending moment satisfies the relation $M = -EIU''$, where E is the Young modulus of elasticity and I is the inertial moment. Considering the deformation caused by a load $f = f(x)$ then $f = -v'$ and $v = M' = -EIU'''$,where v denotes the shear force. For u representing an elastic beam of length $L = 1$, which is clamped at its left side $x = 0$, and resting on an elastic bearing at its right side $x = 1$, and adding a load f along its length to cause deformation. $u = u(x)$. This lead to the boundary value problem:

$$U^{iv}(x) = f(x, u(x)), 0 < x < 1 \tag{3}$$

$$u(0) = u'(0) = 1, \quad u''(1) = 0, \quad u'''(1) = g(v(1))$$

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Solving (3) by means of iterative procedure, Ma and Silva obtained solution and argued that accuracy of result depends highly upon the integration method used in the iterative process.

Barari *et al* [3] solved the problem with variational iteration method, a case of k = 0.

In this work, it is aimed to apply the VIM proposed by He [12, 13, 14, 16] to different forms of (1) with boundary condition of physical significance.

II. Variational Iteration Method

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x) \tag{4}$$

Where L is a linear operator, N a nonlinear operator and g(x) is the forcing term. According to variational iteration method He [12, 13, 14, 17], Inokuti et al [15], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\left(Lu_n(\tau) + N\bar{u}_n(\tau) - g(\tau)\right)d\tau \tag{5}$$

Where λ is a Lagrange multiplier which can be identified optimally via variational iteration method. The subscripts n denotes the nth approximation, $\bar{u}_n$ is considered as a restricted variation, i.e, $\delta \bar{u}_n = 0$ equation (5) is called a correction functional. The solution of the linear problems can be solved in a single iteration steps due to exact identification of Lagrange multiplier [3, 16, 20].

III. Numerical Example

Example 1:

$$u^{iv}(x) - u(x) = 4e^x + k \tag{6}$$

$$u(0) = k + 1, u'(0) = 1, u(1) = 2e + k, \quad u'(1) = 3e + k$$

We construct a correction functional for (6), as follows:

$$U_{n+1}(t) = U_n(t) + \int_{t_0}^{t} \lambda\{LU_n(s) + N\underline{u}_n(s) - g(s)\}ds \tag{7}$$

The variational iteration formula corresponding to (6) is therefore

$$u_{n+1}(x) = u_n(x) + \int_0^x \left\{\frac{1}{6}(\tau - x)^3 u^4{}_n(\tau) + f(x, u(\tau)) - k\right\}\partial\tau \tag{8}$$

Let $u_0(x) = ax^3 + bx^2 + cx + d$ (9)

Then (7) becomes

$$u_1(x) = ax^3 + bx^2 + cx + d - \frac{1}{6}\int_0^x (\tau - x)^3 \left(a\tau^3 + b\tau^2 + c\tau + d + 4e\tau + k\right)d\tau$$

$$u_1(x) = ax^3 + bx^2 + cx + d - \frac{1}{6}\left\{\int_0^x \left(a\tau^6 + \tau^5(b - 3ax) + \tau^4\left(c + 3bx^2 + 3ax^2\right)\right.\right.$$

$$+\tau^3\left(d - 3cx + 3dx^2 - ax^3 + k\right) + \tau^2\left(3cx^2 - 3dx - 3kx - bx^3\right)$$

$$\left.+\tau\left(3dx^2 - 3kx^2 - cx^3\right) \begin{matrix} +4\tau^3 e^\tau - 12\tau^2 e^\tau x \\ +12\tau x e^\tau - 4x^3 e^\tau - dx^3 - x^3 k \end{matrix} \right\} d\tau \Big\} \qquad (10)$$

$$u_1(x) = \frac{ax^7}{840} + \frac{bx^6}{360} + \frac{cx^5}{120} + \frac{x^4}{24}(d + k) + x^3\left(a - \tfrac{2}{3}\right) + x^2(b - 2) + x(c - 4) + 4e^x + d - 4 \qquad (11)$$

Introducing the boundary condition, we have:

$$d = k + 1 \qquad (12)$$

$$c = 2 \qquad (13)$$

$$\frac{7a}{840} + \frac{b}{60} + \frac{cx^5}{120} + \frac{1}{24}(d + k) + \left(a - \tfrac{2}{3}\right) + (b - 2) + (c - 4) + 4e + d - 4 = 2e + k \qquad (14)$$

$$\frac{7a}{840} + \frac{b}{60} + \frac{1}{24} + \frac{(d + k)}{6} + 3\left(a - \tfrac{2}{3}\right) + 2(b - 2) + (c - 4) + 4e = 3e \qquad (15)$$

Solving equations (12)- (15), we have:

Case 1: k = 1, a = 0.500947014, b = 1.582497142, c = 2, d = 2
Case 2: k = 2, a = 0.334347197, b = 1.665730754, c = 2, d = 3.

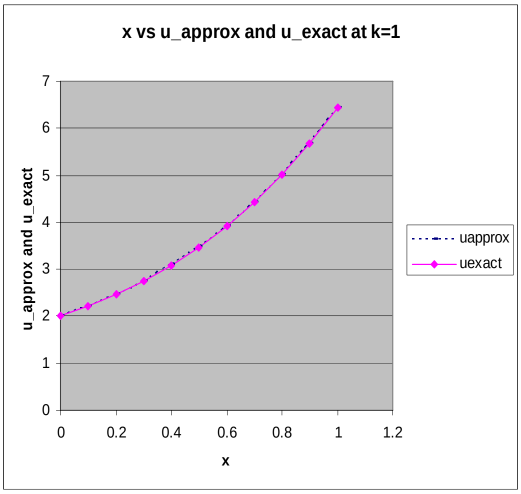

Fig. 1 (a)

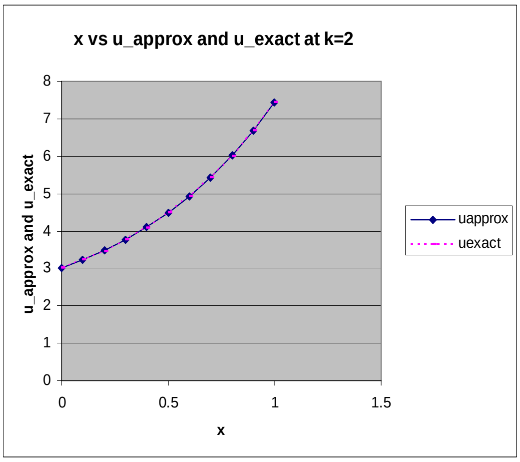

Fig. 1 (b)

Fig. 1 (a) and Fig. 1 (b) The graphs of exact solution (u_exact) and the approximate solution (u_approx.) against x when k=1 and k=2 respectively

Example 2:

Consider the following boundary value problem:

$$u^4(x) = u(x) + u''(x) + e^x(x - 3) + k, \quad 0 < x < 1 \qquad (16)$$

$$u(0) = 1 + k, u'(0) = 0, u(1) = k, u'(1) = -e \qquad (17)$$

The iteration formulation is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6}(\tau - x)^3\left(u_n^4(\tau) - u_n(\tau) - u_n''(\tau) - e^\tau(\tau - 3) - k\right)d\tau$$

$$u_1(x) = \frac{ax^7}{840} + \frac{bx^6}{360} + \left(\frac{a}{20} + \frac{c}{120}\right)x^5 + \left(\frac{b}{12} + \frac{d}{24} + \frac{k}{24}\right)x^4 + \left(a - \frac{2}{3}\right)x^2 + \left(b + \frac{5}{2}\right)x^2$$

$$+\left(c + 6 + e^x\right)x - 7e^x + d + 7 \qquad (18)$$

Incorporating the boundary into (18), we have

$$d = 1 + k \qquad (19)$$

$$c = 0 \qquad (20)$$

$$\frac{a}{840} + \frac{b}{360} + \left(\frac{a}{20} + \frac{c}{120}\right) + \left(\frac{b}{12} + \frac{d}{24} + \frac{k}{24}\right) + \left(a - \frac{2}{3}\right) + \left(b + \frac{5}{2}\right) + c + d + 13 - 6e = k \qquad (21)$$

$$\frac{a}{120} + \frac{b}{60} + \left(\frac{a}{4} + \frac{c}{24}\right) + \left(\frac{b}{3} + \frac{d}{6} + \frac{k}{6}\right) + (3a + 2)x^2 + (2b + 5) + c + 6 = 4e \qquad (22)$$

The solutions of equations (19)- (22) gives:

Case 1: k = 1, c=0, d=2, b=0.4101204308, a=-0.5104111543
Case 2: k = 2, c=0, d=3, b=-0.336315472, a=-0.6655431229

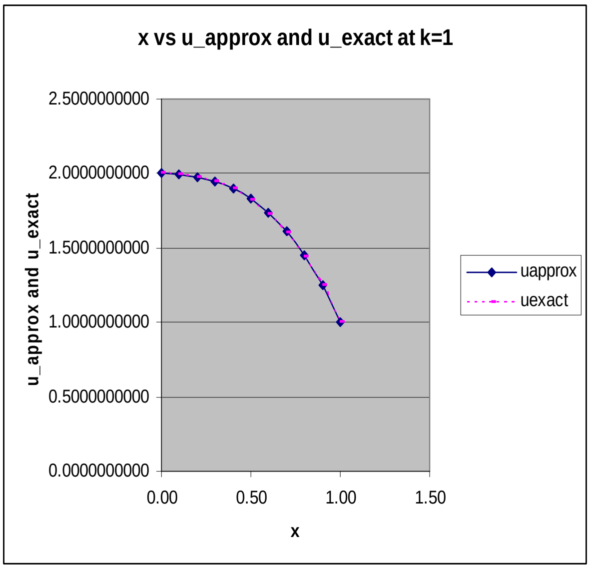

Fig. 2 (a)

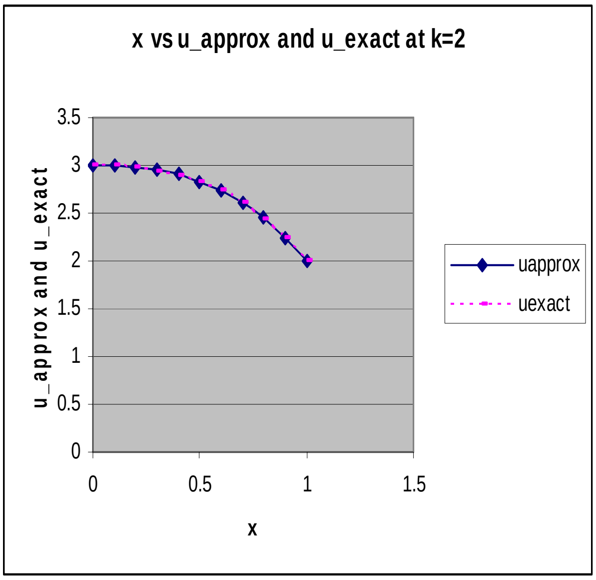

Fig. 2 (b)

Fig. 2 (a) and Fig. 2 (b) The graphs of exact solution (u_exact) and the approximate solution (u_approx.) against x when k=1 and k=2 respectively

Example 3:

Consider the following nonlinear boundary value problem:

$$u^{(4)}(x) = u^2(x) + g(x) + k \tag{23}$$

$$u(0)=k \quad u^1(0)=0 \quad u(1)=1+k \quad u^1(1)=1$$

$$g(x) = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48$$

$$u_{n+1} = u_n + \int_0^x \lambda\left(u_n^{(4)}(x) - u_n^2(\tau) - g(\tau) - k\right)d\tau$$

$$\begin{aligned} u_1(x) = & -\frac{1}{24024}x^{14} + \frac{1}{4290}x^{13} + \frac{1}{2970}x^{12} - \frac{1}{1980}x^{11} + \left(\frac{1}{5040}a^2 + \frac{1}{630}\right)x^{10} \\ & + \frac{1}{1512}abx^9 + \left(-\frac{1}{120} + \frac{b^2}{1680} + \frac{ac}{840}\right)x^8 + \left(\frac{bc}{420} + \frac{ad}{420} + \frac{ak}{420}\right)x^7 \\ & + \left(\frac{bd}{180} + \frac{bk}{180} + \frac{c^2}{360}\right)x^6 + \left(1 + \frac{ad}{60} + \frac{ck}{60}\right)x^5 \\ & + \left(\frac{d^2}{24} + \frac{k^2}{24} + 2\right)x^4 + ax^3 + bx^2 + cx + d \end{aligned} \tag{24}$$

Introducing boundary conditions, we have

$$d = 0 \tag{25}$$

$$c = 0 \tag{26}$$

$$\begin{aligned} & -\frac{1}{24024} + \frac{1}{4290} - \frac{1}{2970} - \frac{1}{1980} + \frac{a}{5040} + \frac{1}{630} + \frac{ab}{1512} + \frac{b^2}{1680} \\ & + \frac{ac}{840} - \frac{1}{420} + \frac{bc}{420} + \frac{bd}{420} + \frac{ak}{420} + \frac{bd}{180} + \frac{bk}{180} + \frac{c^2}{360} + 1 \\ & + \frac{cd}{60} + \frac{ck}{60} + \frac{d^2}{24} + \frac{k^2}{24} - 2 + a + b + c + d = 1 \end{aligned} \tag{27}$$

$$\begin{aligned} & -\frac{1}{1716} + \frac{1}{330} - \frac{12}{2970} - \frac{1}{180} + \frac{a^2}{504} + \frac{1}{63} + \frac{ab}{168} + \frac{b^2}{210} + \frac{ac}{105} - \frac{8}{420} \\ & + \frac{bc}{60} + \frac{ad}{60} + \frac{kd}{60} + \frac{bd}{30} + \frac{bk}{30} + \frac{c^2}{60} + 5 + \frac{cd}{12} + \frac{ck}{12} + \frac{d^2}{6} \\ & + \frac{k^2}{6} - 8 + 3a + 2b + c = 1 \end{aligned} \tag{28}$$

Solving (25)-(28), we have:

Case 1: k =1, d = 0, c = 0, a = -0.1363676737, b = 2.082180724
Case 2: k =2, d = 0, c = 0, a = -0.4395283713, b = 2.24694145

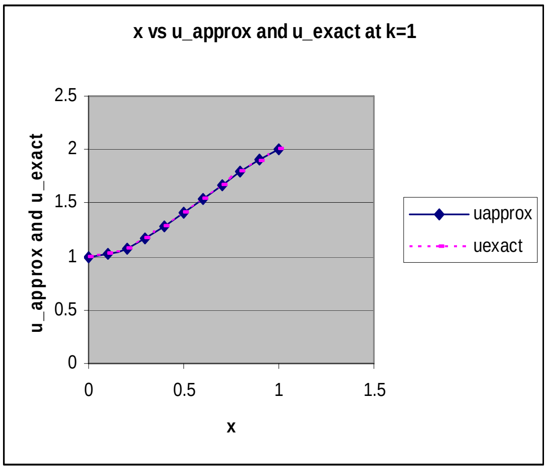

Fig. 3 (a)

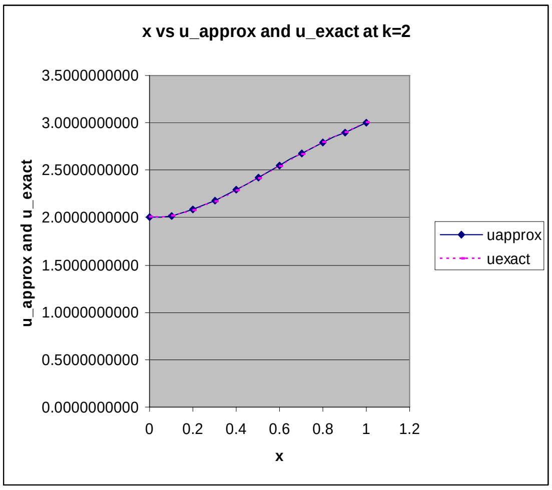

Fig. 3 (b)

Fig. 3 (a) and Fig. 3 (b) shows the graphs of exact solution (u_exact) and the approximate solution (u_approx.) against x when k=1 and k=2 respectively

IV. Conclusion

In this work, VIM has been successfully used to find the numerical solution of models which has fundamental importance in different field of engineering and applied sciences and can also be extended to those investigated in [1]-[11] and [18]-[31]. Many of the results attained in this work confirm the idea that VIM is powerful mathematical tool for solving different kinds of practical problems, having wide application in engineering.

Comparison between the approximate and exact solutions; Figs. 1a, 1b, 2a, 2b, 3a and 3b, shows that the one iteration of variational iteration method is enough. The methods also can be introduced to overcome the difficulties arising in calculating Adomian polynomials.

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