International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:4, 2013

Unscented Transformation for Estimating the Lyapunov Exponents of Chaotic Time Series Corrupted by Random Noise

K. Kamalanand and P. Mannar Jawahar

***Abstract*—**Many systems in the natural world exhibit chaos or non-linear behavior, the complexity of which is so great that they appear to be random. Identification of chaos in experimental data is essential for characterizing the system and for analyzing the predictability of the data under analysis. The Lyapunov exponents provide a quantitative measure of the sensitivity to initial conditions and are the most useful dynamical diagnostic for chaotic systems. However, it is difficult to accurately estimate the Lyapunov exponents of chaotic signals which are corrupted by a random noise. In this work, a method for estimation of Lyapunov exponents from noisy time series using unscented transformation is proposed. The proposed methodology was validated using time series obtained from known chaotic maps. In this paper, the objective of the work, the proposed methodology and validation results are discussed in detail.

***Keywords*—**Lyapunov exponents, unscented transformation, chaos theory, neural networks.

I. Introduction

THE discovery of chaotic behavior in deterministic dynamical systems has opened new perspectives for the design and analysis of time series [1], [2]. Methods for analyzing experimental or observational data for evidence of chaos have been applied to data in such diverse fields as physics, geology, astronomy, neurobiology, ecology and economics [3]. Deterministic chaos and fractal structure in dynamical systems are among the most important nonlinear paradigms [4].

The Lyapunov exponent, which measures the average rate of divergence or convergence of two nearby trajectories, is a useful measure of the stability of a dynamic system. Positivity of the Lyapunov exponent is an operational definition of chaos, Negative exponents indicate mean reverting behaviour, and the value zero is characteristic of cyclic behavior [5], [6].

The calculation of Lyapunov exponents for systems whose dynamical equations are known is straightforward. However, these methods cannot be applied directly to a set of measurement data [7]. The two common approaches for computing the Lyapunov exponents from output time series are geometrical and Jacobian approaches. In geometrical approach, Lyapunov exponents are calculated based on the long term evolution of an infinitesimal sphere of initial conditions [8]. On the other hand, in the Jacobian approach, local Jacobian matrices are estimated and the long term product of matrices is computed [7]-[9].

Eventhough there are several algorithms for estimation of Lyapunov exponents from experimental time series, most of them are usually unreliable except for long and noise-free time series [10]. The first algorithms for estimation of non-negative Lyapunov exponents from an experimental time series were presented by Wolf et al [11]. Rosenstein et al [10] have introduced a robust method for calculating the Lyapunov exponents of a time series using a simple measure of exponential divergence and by utilizing all the available data. Sano and Sawada [9] proposed a method to determine the spectrum of Lyapunov exponents from the observed time series of a single variable.

To obtain the Lyapunov exponent from observed data, Eckmann and Ruelle [12] and Eckmann et al [13] proposed a method based on nonparametric regression. Ataei et al [7] have estimated the Lyapunov exponents of a chaotic time series using a global polynomial model fitting to the given data followed by a Jacobian approach. McCaffrey et al [3] described procedures for estimating the Lyapunov exponents of time series data using different modeling approaches namely, thin-plate splines, radial basis functions and projection pursuit. While any nonparametric regression estimator can be employed in the Jacobian method, one of the most widely used approaches is the Lyapunov exponent estimator based on neural networks proposed by Nychka et al [14].

The estimation of Lyapunov exponents of a chaotic time series with a significant random noise is difficult. Most of the biological and economic systems are subjected to random perturbations and are observed over a limited period of time. In such time series, the dynamic information is limited by sample size and masked by noise. The Lyapunov exponents of stochastic systems were estimated by Nychka et. al. [14] and have concluded that to certain extent it is possible to identify chaotic dynamics in short noisy systems using thin plate splines.

In this paper, a method based on unscented transformation

K. Kamalanand is with the Department of Instrumentation Engineering, MIT Campus, Anna University, Chennai 600 044, India. (e-mail: kamalanand@mitindia.edu).
P. Mannar Jawahar is with the Centre for University Industry Collaboration, Anna University, Chennai 600 025, India. (e-mail: pmjmit@gmail.com).

and unscented Kalman filter based Artificial Neural Networks (ANN) is proposed for estimation of Lyapunov exponents of chaotic time series corrupted by a random noise. Further, the performance of the proposed method is analyzed using the time series obtained from well known chaotic maps.

II. Methodology

The proposed methodology for estimation of Lyapunov exponents from chaotic time series corrupted by random noise is described in Fig. 1. The first step of the proposed method is to develop a dynamic model from the measured time series. The generated model can be any nonlinear model and:

a. The generated model can be discontinuous

b. The generated model need not be differentiable

The next step is to apply the UKF algorithm to the measured time series using the developed model. Hence, a filtered estimate of the states is obtained. The estimated states are simultaneously used to develop a dynamic ANN model. Further, the Jacobian matrix of the developed ANN model is computed and the Lyapunov exponents of the noisy time series are obtained from the Jacobian matrix.

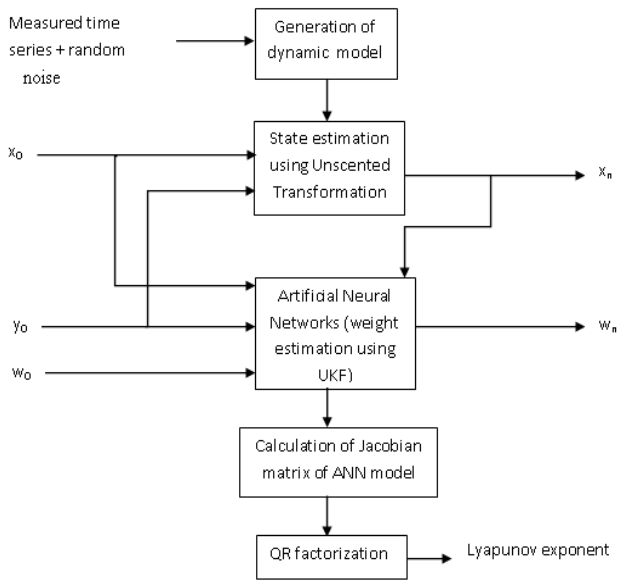


Fig. 1 Block diagram of the proposed methodology

A. The Unscented Kalman Filter

The unscented Kalman filter is founded on the intuition that it is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function [15]. The sigma points are chosen so that their mean and covariance to be exactly x_{k-1}^a and P_{k-1}. Each sigma point is then propagated through the nonlinearity yielding in the end a cloud of transformed points. The new estimated mean and covariance are then computed based on their statistics. This process is called unscented transformation. The unscented transformation is a method for calculating the statistics of a random variable which undergoes a nonlinear transformation [16].

Consider the following nonlinear system, described by the difference equation and the observation model with additive noise:

$$x_k = f\left(x_{k-1}\right) + w_{k-1} \tag{1}$$

$$z_k = h\left(x_k\right) + v_k \tag{2}$$

The initial state x_0 is a random vector with known mean $\mu_0 = E\left[x_0\right]$ and covariance $P_0 = E\left[\left(x_0 - \mu_0\right)\left(x_0 - \mu_0\right)^T\right]$.

1. Selection of Sigma Points

Let X_{k-1} be a set of $2n+1$ sigma points (where n is the dimension of the state space) and their associated weights:

$$X_{k-1} = \left\{\left(x_{k-1}^j, W^j\right) \middle| j = 0 \ldots 2n\right\} \tag{3}$$

Consider the following selection of sigma points, selection that incorporates higher order information in the selected points:

$$x_{k-1}^0 = x_{k-1}^a \tag{4}$$

$$-1 < W^0 < 1 \tag{5}$$

$$x_{k-1}^i = x_{k-1}^a + \left(\sqrt{\frac{n}{1-W^0}P_{k-1}}\right)_i \quad \text{for all } i = 1 \ldots n \tag{6}$$

$$x_{k-1}^{i+n} = x_{k-1}^a - \left(\sqrt{\frac{n}{1-W^0}P_{k-1}}\right)_i \quad \text{for all } i = 1 \ldots n \tag{7}$$

$$W^j = \frac{1-W^0}{2n} \quad \text{for all } j = 1 \ldots 2n \tag{8}$$

where the weights must obey the condition:

$$\sum_{j=0}^{2n} W^j = 1 \tag{9}$$

and $\left(\sqrt{\frac{n}{1-W^0}P_{k-1}}\right)_i$ is the row or column of the matrix square root of $\frac{n}{1-W^0}P_{k-1}$. W^0 controls the position of sigma points: $W^0 \geq 0$ points tend to move further from the origin, $W^0 \leq 0$ points tend to be closer to the origin.

2. Model Forecast Step

The Each sigma point is propagated through the nonlinear process model:

$$x_k^{f,j} = f\left(x_{k-1}^j\right) \tag{10}$$

The transformed points are used to compute the mean and covariance of the forecast value of x_k.

$$x_k^f = \sum_{j=0}^{2n} W^j x_k^{f,j} \tag{11}$$

$$P_k^f = \sum_{j=0}^{2n} W^j \left(x_k^{f,j} - x_k^f\right)\left(x_k^{f,j} - x_k^f\right)^T + Q_{k-1} \tag{12}$$

We propagate then the sigma points through the nonlinear observation model:

$$z_{k-1}^{f,j} = h\left(x_{k-1}^j\right) \tag{13}$$

With the resulted transformed observations, their mean and covariance (innovation covariance) is computed:

$$z_{k-1}^f = \sum_{j=0}^{2n} W^j z_{k-1}^{f,j} \tag{14}$$

$$Cov\left(\tilde{z}_{k-1}^f\right) = \sum_{j=0}^{2n} W^j \left(z_{k-1}^{f,j} - z_{k-1}^f\right)\left(z_{k-1}^{f,j} - z_{k-1}^f\right)^T + R_k \tag{15}$$

The cross covariance between $\tilde{x}_k^f$ and $\tilde{z}_{k-1}^f$ is:

$$Cov\left(\tilde{x}_k^f, \tilde{z}_{k-1}^f\right) = \sum_{j=0}^{2n} W^j \left(x_k^{f,j} - x_k^f\right)\left(z_{k-1}^{f,j} - z_{k-1}^f\right)^T \tag{16}$$

3. *Data Assimilation Step*

Assume that the estimate has the following form:

$$x_k^a = x_k^f + K_k\left(z_k - z_{k-1}^f\right) \tag{17}$$

The gain K_k is given by:

$$K_k = Cov\left(\tilde{x}_k^f, \tilde{z}_{k-1}^f\right) Cov^{-1}\left(\tilde{z}_{k-1}^f\right) \tag{18}$$

The posterior covariance is updated using

$$P_k = P_k^f - K_k Cov\left(\tilde{z}_{k-1}^f\right) K_k^T \tag{19}$$

B. Neural Network Training using UKF

The parameter estimation problem involves learning a nonlinear mapping $y_k=G(x_k,w)$, where w corresponds to a set of unknown parameters. In the considered methodology, $G(.)$ is the neural network and w is the set of weights to be estimated. The unscented Kalman filter can be used to estimate the weights of the network using the following set of equations [16], [17].

1. *Initialization:*

$$\hat{w}_0 = E\left[w\right] \tag{20}$$

$$Pw_0 = E\left[\left(w - \hat{w}_0\right)\left(w - \hat{w}_0\right)^T\right] \tag{21}$$

For $k \in \{1, \dots, \infty\}$,

2. *Time update and sigma point calculation:*

$$\hat{w}_k^- = \hat{w}_{k-1} \tag{22}$$

$$P_{wk}^- = P_{wk-1} + R_{k-1}^r \tag{23}$$

$$W_{k|k-1} = \left[\hat{w}_k^- \quad \hat{w}_k^- + \gamma\sqrt{P_{w_k}^-} \quad \hat{w}_k^- - \gamma\sqrt{P_{w_k}^-}\right] \tag{24}$$

$$D_{k|k-1} = G\left[x_k, W_{k|k-1}\right] \tag{25}$$

$$\hat{d}_k = G\left(x_k, \hat{w}_k^-\right) \tag{26}$$

3. *Measurement update equations:*

$$P_{\tilde{d}_k\tilde{d}_k} = \sum_{i=0}^{2L} W_i^{(c)}\left[D_{i,k|k-1} - \hat{d}_k\right]\left[D_{i,k|k-1} - \hat{d}_k\right]^T + R_k^e \tag{27}$$

$$P_{w_k d_k} = \sum_{i=0}^{2L} W_i^{(c)}\left[W_{i,k|k-1} - \hat{w}_k^-\right]\left[D_{i,k|k-1} - \hat{d}_k\right]^T \tag{28}$$

$$K_k = P_{w_k d_k} P_{\tilde{d}_k\tilde{d}_k}^{-1} \tag{29}$$

$$\hat{w}_k = \hat{w}_k^- + K_k\left(d_k - \hat{d}_k\right) \tag{30}$$

$$P_{w_k} = P_{w_k}^- = K_k P_{\tilde{d}_k\tilde{d}_k} K_k^T \tag{31}$$

where, $\gamma = \sqrt{(L+\lambda)}$, λ = composite scaling parameter, L =dimension of the state, R^r and R^e are noise covariances.

C. Calculation of Lyapunov Exponents

Further, the Lyapunov exponent of the considered time series is calculated using the Jacobian approach [7]. Consider the discrete dynamical system described in the following form:

$$x_k = F\left(x_{k-1}\right), k = 1, 2, 3... \tag{32}$$

where x_k is the state vector in the R^m space and $F(\cdot)$ is a continuously differentiable nonlinear function. In this case, $F(.)$ is the trained neural network. The linearized system for a small range around the operational trajectory in the phase space can be written as:

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:4, 2013

$$\delta x_k = J_{k-1}\delta x_{k-1}\text{, where } J_{k-1} = \left.\frac{\partial F}{\partial x}\right|_{x_{k-1}} \in R^{m\times m} \;\; k = 1,\ldots \tag{33}$$

where J_k is the Jacobian matrix in point k. Let $Y^k = J_{k-1}, J_{k-2}, \ldots, J_0$, then the following symmetric positive definite matrix exists:

$$A = \lim_{k\to x}\left(\left(Y^k\right)^T . Y^k\right)^{\frac{1}{2k}} \tag{34}$$

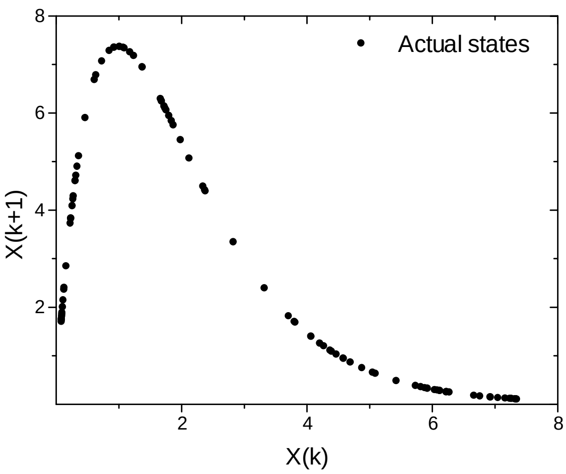
and the logarithms of its eigenvalues are called the Lyapunov exponents. For large value of k, the fundamental solution Y^k may go to very large values and actually, the calculation of A is not possible. Hence, the the QR factorisation algorithm is used for approximation of Lyapunov exponents [7].

D. Validation of Proposed Method

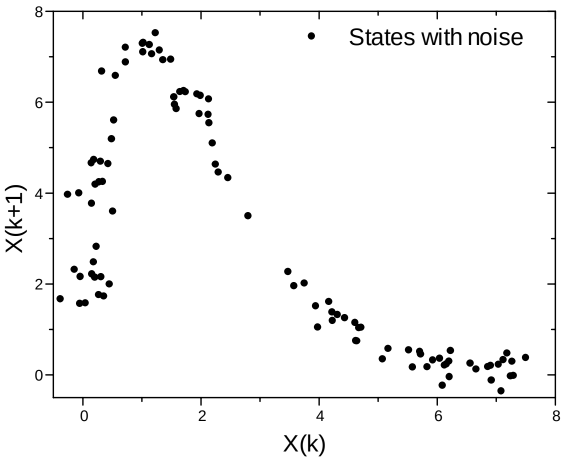
The validation of the proposed method was performed using chaotic time series obtained from well known chaotic maps such as the logistic map [10], tent map [19], sine map [20], and Ricker's map [18], [21]. Also, the Lyapunov exponents obtained using the proposed method, were compared with the actual values.

III. Results and Discussion

Fig. 2(a) shows the time series obtained from Ricker's population model. The obtained time series along with an additive random noise is shown in Fig. 2(b). The states estimated using the Extended Kalman Filter for the considered time series is shown in Fig. 3(a). It is seen that the UKF method can efficiently filter random noises from chaotic time series. Further, the states estimated using the neural network trained using unscented Kalman filter, is shown in Fig. 3(b). Also, it appears that the UKF method is efficient for developing dynamic ANN models of chaotic time series.



(a)

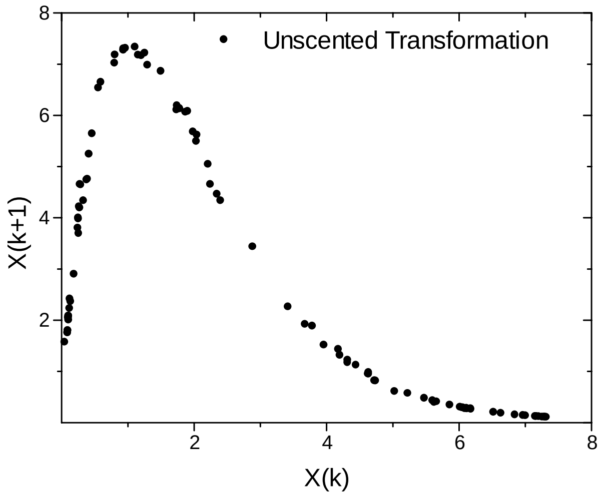


(b)

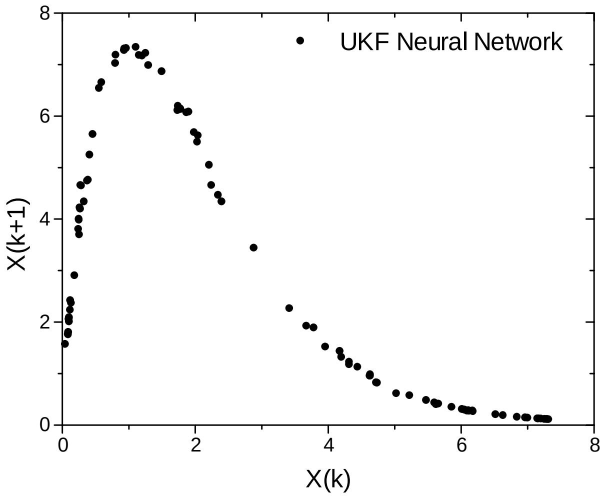
Fig. 2 (a) The actual states of Ricker's chaotic map, (b) the states of the Ricker's map with additive random noise

Fig. 4 shows the actual states and the states estimated using the UKF neural network, at each sampling instants. The error between the actual states and estimated states is found to be very less. Hence, the high capability of unscented Kalman filter for developing neural network models of chaotic time series is demonstrated by Fig. 4.

The actual values of Lyapunov exponents and the values obtained using the proposed method are compared in Table I, for four different time series obtained from known chaotic maps such as logistic map, sine map, tent map and Ricker's map. The values estimated using the proposed method was found to be close to the actual values of the Lyapunov exponents. The deviation from the actual value was found to be higher in the case of the tent map when compared to the other time series.



(a)



(b)

Fig 3 (a) The states of Ricker's map estimated using unscented transformation, (b) the states estimated using UKF neural network

The error in estimation of Lyapunov exponents using the proposed method is shown as a function of the standard deviation of random noise for four different time series in Fig. 5. It is seen that the estimation error increases with increase in the standard deviation of the additive noise. The estimation error was found to be higher for the time series obtained from tent map and logistic map when compared to the time series obtained from other chaotic maps. The error in estimation of Lyapunov exponents was found to be less and negligible for the chaotic time series obtained from Ricker's population model. Also, for the Ricker's time series, the estimation error was found to remain almost constant with increase in the standard deviation of the additive random noise.

The validation experiments demonstrate that the efficiency of the proposed method depends on the standard deviation of the random noise and also depends on the type of the chaotic time series.

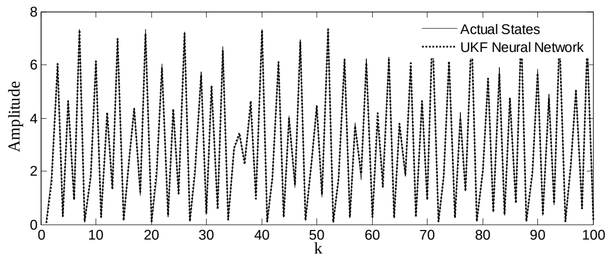


Fig. 4 The actual time series obtained from Ricker's chaotic map and the time series estimated using UKF neural network

TABLE I
THE EXPECTED VALUE OF LYAPUNOV EXPONENTS AND THE VALUES ESTIMATED USING THE PROPOSED METHOD

Time series	Lyapunov exponents		Deviation
	Expected value (Reference)	Proposed method	
Logistic map	0.67 (Rosenstein et al 1993)	0.66	0.01
Tent map	0.69 (Devaney 1989)	0.63	0.06
Sine map	0.68 (Strogatz 1994)	0.67	0.01
Ricker's map	0.39 (Ricker 1954)	0.38	0.01

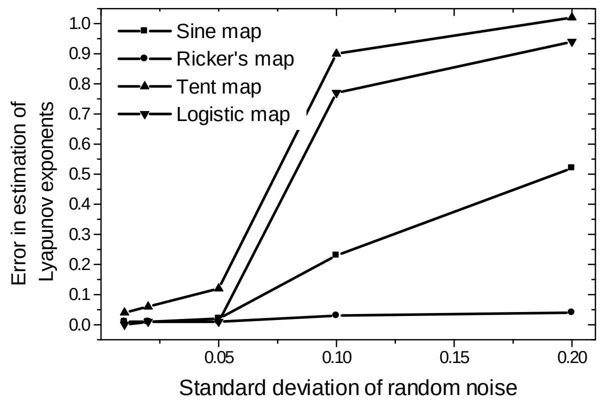


Fig 5 The error in estimation of Lyapunov exponents shown as a function of the standard deviation of additive random noise for four different chaotic time series

IV. CONCLUSION

Deterministic chaos appears in variety of fields like engineering, biomedical and life sciences, social sciences, and physical sciences and recognizing the chaotic behaviour of dynamical systems when only output data are available, is an important field of research [7]. Distinguishing deterministic chaos from random noise has become an important problem in many diverse fields such as physiology and economics [21]-[23].

The Lyapunov exponent is one of the basic quantities for characterizing the chaotic behavior of a system. [9]. However, the estimation of the Lyapunov exponents of chaotic time series corrupted by random noises is difficult [14] and only few methods for computing Lyapunov exponents of noisy time series are available. Also, most of the existing methods for estimating the Lyapunov exponents require relatively time series data of high precession but such high quality data cannot be obtained in many real-world situations [4].

In this work, a method using unscented transformation and artificial neural networks is proposed for estimation of the Lyapunov exponents of chaotic time series corrupted by a random noise. The proposed method was validated using time series obtained from well known chaotic maps such as sine map, tent map, logistic map and Ricker's population model. The estimated exponents were compared with the actual values of the Lyapunov exponents for the considered time series. Results demonstrate that the error in estimation of Lyapunov exponents increases with increase in the standard deviation of the additive noise. It was also found that the estimation error depends on the nature of the chaotic time series. Further, the UKF neural network seems to be efficient for development of dynamic models of chaotic time series. It appears that the proposed methodology can be efficiently used for estimation of Lyapunov exponents of chaotic time series measurements corrupted by random noises.

References

[1] N. L. Gerr, and J. C. Allen, “Stochastic versions of chaotic time series: generalized logistic and Henon time series models,” *Physics D*, vol. 68, 1993, pp. 232-249

[2] E. Takens, “Detecting nonlinearities in stationary time series,” *International Journal of Bifurcation and Chaos*, vol. 3, 1993, pp. 241-256

[3] D. F. McCaffrey, S. Ellner, A. R. Gallant and D. W. Nychka, “Estimating the Lyapunov exponent of a chaotic system with nonparametric regression,” *Journal of the Amer. Sta.l Assoc.*, vol. 87, 1992, pp. 682-695

[4] X. Zeng, R. Eykholt and R. A. Pielke, “Estimating the lyapunov-exponent spectrum from short time series of low precision,” *Phys. Rev. Let.*, vol. 66, 1991

[5] S. Mototsugu and L. Oliver, “Nonparametric neural network estimation of Lyapunov exponents and a direct test for chaos,” *Journal of Econometrics*, vol. 120, 2004, pp. 1–33

[6] A. N. Edmonds, “Time series prediction using supervised learning and tools from chaos theory,” Faculty of Science and Computing, University of Luton, Ph.D Thesis, 1996

[7] M. Ataei, A. Khaki-Sedigh, B. Lohmann and C. Lucas, “Estimating the lyapunov exponents of chaotic time series: a model based method,” *Proceedings of European Control Conference*, 2003

[8] H. Kantz, “A robust method to estimate the maximum Lyapunov exponent of a time series,” *Phys. Lett. A*, vol. 185, 1994, pp. 77-87

[9] M. Sano and Y. Sawada, “Measurement of the Lyapunov spectrum from a chaotic time series,” *Phys. Rev. Let.*, vol. 55, 1985, pp. 1082-1085

[10] M. T. Rosenstein, J. J. Collins and C. J. De Luca, “A practical method for calculating largest Lyapunov exponents from small data sets,” *Physica D*, vol. 65, 1993, pp. 117-134

[11] A. Wolf, J. B. Swift, H. L. Swinney and J. A. Vastano, “Determining Lyapunov exponents from a time series,” *Physica D*, vol. 16, 1985, pp. 285-317

[12] J.-P. Eckmann and D. Ruelle, “Ergodic theory of chaos and strange attractors,” *Reviews of Modern Physics*, vol. 57, 1985, pp. 617–656

[13] J.-P. Eckmann, S. O. Kamphorst, D. Ruelle and S. Ciliberto, “Liapunov exponents from time series,” *Physical Review A*, vol. 34, 1986, pp. 4971–4979

[14] D. Nychka, S. Ellner, A. R. Gallant and D. McCaHrey, “Finding chaos in noisy system,” *Journal of the Royal Statistical Society*, Series B, vol. 54, 1992, pp. 399–426

[15] S. J. Julier and J. K. Uhlmann, “Unscented filtering and nonlinear estimation,” *Proceedings of the IEEE*, vol. 92, 2004, pp. 401-422

[16] Wan, E. and R. Van der Merwe, “The unscented Kalman filter,” Wiley Publishing, 2001

[17] Li Hongli, J. Wang, Y. Che, H. Wang and Y. Chen, “On neural network training algorithm based on the unscented Kalman filter,” *Proceedings of 29th Chinese Control Conference*, 2010, pp. 1447-1450

[18] J. C. Sprott, “Chaos and time series analysis,” oxford university press, 2003

[19] R. L. Devaney, “An introduction to chaotic dynamical systems,” 2nd edition, Addison Wesley, Redwood City, CA, 1989

[20] S. H. Strogatz, “Nonlinear dynamics and chaos,” Perseus publishing, 1994

[21] W. E. Ricker, “Stock and recruitment,” *J. Fisheries Res. Board Can.*, vol. 11, 1954, pp. 559-623

[22] G. W. Frank, T. Lookman, M. A. H. Nerenberg, C. Essex, J. Lemieux and W. Blume, “Chaotic time series analysis of epileptic seizures,” *Physica D*, vol. 46, 1990, pp. 427

[23] P. Chen, “Empirical and theoretical evidence of economic chaos,” *Sys. Dyn. Rev.*, vol. 4, 1988, pp. 81