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Uniformly persistence of a predator-prey model with Holling III type functional response

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Abstract—In this paper, a predator-prey model with Holling III type functional response

$$\begin{cases} \dot{x}(t) = x(t)(r_1(t) - b_1(t)x(t)) - \dfrac{c_1(t)x^2(t)}{kx^2(t)+1}y^m(t) \\ \dot{y}(t) = y(t)(-r_2(t) - b_2(t)y(t)) + \dfrac{c_2(t)x^2(t)}{kx^2(t)+1}y^m(t) \end{cases}$$

is studied. It is interesting that the system is always uniformly persistent, which yields the existence of at least one positive periodic solutions for the corresponding periodic system. The result improves the corresponding ones in [11]. Moreover, an example is illustrated to verify the results by simulation.

Keywords—Predator-prey model, Uniformly persistence, Comparison theorem, Holling III type functional response.

I. Introduction

THE first differential equation of predator-prey model was introduced by A.J. Lotka (1925) and V. Volterra (1926), respectively. After that many more complicated but realistic predator-prey model have been formulated by ecologists and mathematicians. The dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Recently predator-prey models with the mutual interference between the predators and preys have been extensively studied(see [1-7]), which was introduced by Hassell in 1971. From the observation Hassell introduced the concept of mutual interference constant $m(0 < m \le 1)$ and established a Volterra model with mutual interference as follows (see [8-10])

$$\begin{cases} \dot{x} = xg(x) - \varphi(x)y^m, \\ \dot{y} = y\left(-d + k\varphi(x)y^{m-1} - q(y)\right). \end{cases}$$

In [11] the authors discussed a Lotka-Volterra model with mutual interference and Holling III type functional response as follows

$$\begin{cases} \dot{x}(t) = x(t)(r_1(t) - b_1(t)x(t)) - \dfrac{c_1(t)x^2(t)}{x^2(t)+k^2}y^m(t), \\ \dot{y}(t) = y(t)(-r_2(t) - b_2(t)y(t)) + \dfrac{c_2(t)x^2(t)}{x^2(t)+k^2}y^m(t), \end{cases} \tag{1}$$

where x is the size of the prey population, and y is the size of the predator population; $k > 0$ is a constant; r_1, b_1, r_2, b_2, c_1 and c_2 are positive functions. In [11] some sufficient conditions are obtained for the existence, uniqueness and global attractivity of positive periodic solution of the model. But the authors do not discuss the uniformly persistence of the model. As far as we know, the existence of periodic solution is the special persistence. So the discussion of the uniformly persistence of the model is very important and significative.

Motivated by the above reason, in this paper by using some new analysis techniques and comparison theorem we investigate the permanence of system (2) as follows

$$\begin{cases} \dot{x}(t) = x(t)(r_1(t) - b_1(t)x(t)) - \dfrac{c_1(t)x^2(t)}{kx^2(t)+1}y^m(t), \\ \dot{y}(t) = y(t)(-r_2(t) - b_2(t)y(t)) + \dfrac{c_2(t)x^2(t)}{kx^2(t)+1}y^m(t). \end{cases} \tag{2}$$

Obviously, the models in [1-3] are the special cases of (2) with $m = 1$, i.e., there is no mutual interference between the predator and prey. Here, we only investigate system (2) in the case of $0 < m < 1$. It is interesting that the system is always uniformly persistent.

II. Definition and Lemmas

Definition 2.1 System (2) is said to be uniformly persistent if there exist positive constants $m_i,\ M_i,\ i = 1, 2$ and $T > 0$ such that

$$m_1 \le x(t) \le M_1; \quad m_2 \le y(t) \le M_2, \quad \text{for} \quad t \ge T,$$

for any positive solution $(x(t),\ y(t))$ of system (2).

Lemma 2.1 (*See [12]*) If $a > 0,\ b > 0$, and

$$z'(t) \ge (\le) b - a\, z(t), z(0) > 0,$$

then

$$z(t) \ge (\le) \frac{b}{a}\,[1 + (\frac{az(0)}{b} - 1)e^{-at}], \forall t \ge 0.$$

Lemma 2.2 If $a > 0,\ b > 0$, and

$$z'(t) \ge (\le) z(t)\,(b - a\, z(t)), z(0) > 0,$$

then

$$z(t) \ge (\le) \frac{b}{a}\,[1 + (\frac{b}{az(0)} - 1)e^{-bt}]^{-1}, \forall t \ge 0.$$

Proof. From

$$z'(t) \ge z(t)\,(b - a\, z(t)),$$

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we can easily obtain

$$\frac{d(z^{-1})}{dt} \le a - b\,z^{-1}.$$

By Lemma 2.1 we have

$$z^{-1}(t) \le \frac{a}{b}\,[1 + (\frac{bz^{-1}(0)}{a} - 1)e^{-bt}], \forall t \ge 0,$$

i.e.,

$$z(t) \ge \frac{b}{a}\,[1 + (\frac{b}{az(0)} - 1)e^{-bt}]^{-1}, \forall t \ge 0.$$

Similarly, we can prove if

$$z'(t) \le z(t)\,(b - a\,z(t)),$$

then

$$z(t) \le \frac{b}{a}\,[1 + (\frac{b}{az(0)} - 1)e^{-bt}]^{-1}, \forall t \ge 0.$$

Lemma 2.3 If $a > 0,\ b > 0$, and

$$z'(t) \ge (\le) z^m(t)\,(b - a\,z^{1-m}(t)), z(0) > 0,$$

then

$$z(t) \ge (\le)[\frac{b}{a} + (z^{1-m}(0) - \frac{b}{a})e^{-a(1-m)t}]^{\frac{1}{1-m}}, \forall t \ge 0.$$

Proof. From

$$z'(t) \ge z^m(t)\,(b - a\,z^{1-m}(t)),$$

we can easily obtain

$$\frac{d(z^{1-m})}{dt} \ge (1-m)(b - a\,z^{1-m}).$$

By Lemma 2.1 we have

$$z^{1-m}(t) \ge \frac{b}{a} + (z^{1-m}(0) - \frac{b}{a})e^{-a(1-m)t}], \forall t \ge 0,$$

i.e.,

$$z(t) \ge [\frac{b}{a} + (z^{1-m}(0) - \frac{b}{a})e^{-a(1-m)t}]^{\frac{1}{1-m}}, \forall t \ge 0.$$

Similarly, we can prove the other part of this Lemma.

Before the main results we give some useful notations as follows, for any continuous bounded function f defined on $[0, +\infty)$,

$$f^L := \inf_{t\in[0,+\infty)}\{f(t)\},\ f^U := \sup_{t\in[0,+\infty)}\{f(t)\}$$

and

$$K_1 := \frac{r_1^U}{b_1^L} + \varepsilon,\ K_2 := \frac{c_2^U}{2kr_2^L} + \varepsilon,$$

where ε is a positive constant.

III. Main results

Theorem 3.1 System (2) is uniformly persistent.

Proof. The first equation in system (2) leads to that

$$x'(t) \le x(t)\left[r_1^U - b_1^L x(t)\right], \forall t \ge 0,$$

which together with Lemma 2.2 yields that

$$x(t) \le \frac{r_1^U}{b_1^L}[1 + (\frac{r_1^U}{b_1^L x(0)} - 1)e^{-r_1^U t}]^{-1}, \forall t \ge 0. \quad (3)$$

Thus, $\forall \varepsilon > 0, \exists T_1 > 0$, such that

$$x(t) \le \frac{r_1^U}{b_1^L} + \varepsilon =: K_1, \text{for } t \ge T_1. \quad (4)$$

On the other hand, the second equation in system (2) implies

$$y'(t) \le y(t)\left[-r_2^L + \frac{c_2^U}{2k}y^{m-1}(t)\right]$$
$$\le y^m(t)\left[\frac{c_2^U}{2k} - r_2^L y^{1-m}(t)\right], \forall t \ge 0,$$

which together with Lemma 2.3 yields that $\forall t \ge 0$,

$$y(t) \le \left[\frac{c_2^U}{2kr_2^L} + \left(y^{1-m}(0) - \frac{c_2^U}{2kr_2^L}\right)e^{-r_2^L(1-m)t}\right]^{\frac{1}{1-m}}. \quad (5)$$

Therefore, for above $\varepsilon > 0, \exists T_2 > 0$, such that

$$y(t) \le \frac{c_2^U}{2kr_2^L} + \varepsilon =: K_2, \text{for } t \ge T_2. \quad (6)$$

Furthermore, from the first equation in system (2) we get

$$x'(t) \ge x(t)\left[r_1^L - b_1^U x(t) - c_1(t)x(t)y^m(t)\right]$$
$$\ge x(t)\left[r_1^L - (b_1^U + c_1^U K_2^m)x(t)\right].$$

It follows from Lemma 2.2 that there exists a constant $T_3 \in R_+$ such that

$$x(t) \ge \frac{r_1^L}{b_1^U + c_1^U K_2^m} - \varepsilon, \text{for } t \ge T_3.$$

Noticing that ε is an arbitrary small constant, we can let ε be so small that

$$\varepsilon < \frac{r_1^L}{2b_1^U + 2c_1^U K_2^m}.$$

So we get

$$x(t) \ge \frac{r_1^L}{2b_1^U + 2c_1^U K_2^m} =: K_3, \text{for } t \ge T_3. \quad (7)$$

Similarly, the second equation in system (2) yields

$$y'(t) \ge y(t)\left(-r_2^U - b_2^U K_2 + \frac{c_2^L K_3}{kK_1^2 + 1}y^{m-1}(t)\right]$$
$$= y^m(t)\left(\frac{c_2^L K_3}{kK_1^2 + 1} - (r_2^U + b_2^U K_2)y^{1-m}(t)\right].$$

it follows from Lemma 2.3 that for the above ε there exists $T_4 > 0$ such that

$$y^{1-m}(t) \ge \frac{c_2^L K_3}{(kK_1^2 + 1)(r_2^U + b_2^U K_2)} - \varepsilon, \text{for } t \ge T_4.$$

Let ε be so small that

$$\varepsilon < \frac{c_2^L K_3}{2(kK_1^2+1)(r_2^U+b_2^U K_2)},$$

so we get

$$y(t) \geq \left[\frac{c_2^L K_3}{2(kK_1^2+1)(r_2^U+b_2^U K_2)}\right]^{\frac{1}{1-m}} =: K_4, \text{for } t \geq T_4. \quad (8)$$

Let $T_0 = \max\{T_1, T_2, T_3, T_4\}$, by formula (4), (6), (7) and (8) we get

$$K_3 \leq x(t) \leq K_1 \text{ and } K_4 \leq y(t) \leq K_2, \text{ for } t > T_0.$$

Now we complete the proof of Theorem 3.1.

Remark If system (2) is a periodic system, then by Brouwer fixed point theorem we know System (2) is uniformly persistent, which implies that system (2) has at least one positive T-periodic solution. Thus the result of the existence of positive periodic solutions is improved, which is weaker than the ones in [11].

IV. Stimulation

As an application, we consider the following system:

$$\begin{cases} \dot{x}(t) = x(t)(r_1(t) - b_1(t)x(t)) - \dfrac{c_1(t)x^2(t)}{kx^2(t)+1}y^m(t), \\ \dot{y}(t) = y(t)(-r_2(t) - b_2(t)y(t)) + \dfrac{c_2(t)x^2(t)}{kx^2(t)+1}y^m(t), \end{cases} \quad (9)$$

where

$$r_1(t) = 5-0.3\sin t, b_1(t) = 4+0.5\cos t, c_1(t) = 1+0.6\sin t,$$

$$r_2(t) = 3+0.4\sin t, b_2(t) = 4+0.7\sin t, c_2(t) = 5+0.8\cos t,$$

$k = 0.5$ and $m = 2/3$.

We easily obtain system (9) is uniformly persistent.

Noticing that this system is a periodic system, system (9) has at least one positive 2π-periodic solution. In order to verify our conclusions further, we take the initial values by

$$(x(0), y(0)) = (0.3, 0.3), (x(0), y(0)) = (0.6, 0.7)$$

and

$$(x(0), y(0)) = (1.8, 0.8), (x(0), y(0)) = (2, 0.2),$$

respectively.

From the following figure, one can easily see that the positive solutions of system (9) are eventually tend to a periodic orbits, which yields that the predator and prey are uniformly persistent.

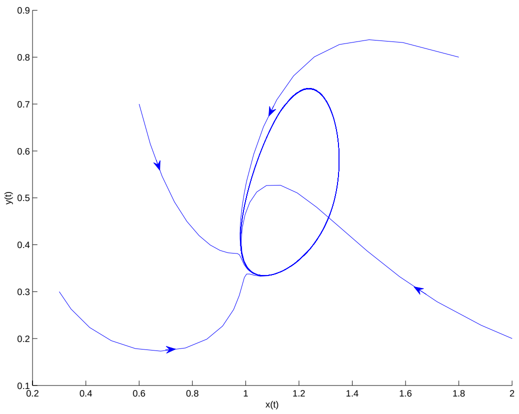

Fig. Evolution of the solutions of system (9).

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