International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:3, 2011

Two-step Iterative Process For Common Fixed Points of Two Asymptotically Quasi-nonexpansive Mappings

Safeer Hussain Khan

Abstract—In this paper, we consider an iteration process for approximating common fixed points of two asymptotically quasi-nonexpansive mappings and we prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

Keywords—Asypmtotically quasi-nonexpansive mappings, Common fixed point, Strong and weak convergence, Iteration process.

I. Introduction

THROUGHOUT this paper, $\mathbb{N}$ will denote the set of all positive integers. Let C be a nonempty subset of a real Banach space E. Let $T : C \to C$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. A mapping $T : C \to C$ is said to be

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;

(ii) quasi-nonexpansive if $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$;

(iii) asymptotically nonexpansive if there exists a sequence $k_n \geq 1, \lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$. Note that $\{k_n\}$ is a nonincreasing bounded sequence, see [8].

(iv) asymptotically quasi-nonexpansive if there exists a sequence $k_n \geq 1, \lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - p\| \leq k_n \|x - p\|$$

for all $x \in C$, $p \in F(T)$ and $n \in \mathbb{N}$;

(v) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

From the above definitions, it is clear that each of a nonexpansive, a quasi-nonexpansive, an asymptotically nonexpansive is an asymptotically quasi-nonexpansive mapping. However, the converse of each of above statements may be not true. For the fact that an asymptotically quasi-nonexpansive mapping is not an asymptotically quasi-nonexpansive mapping; see, for example, [17].

Safeer Hussain Khan is with the Department of Mathematics, Statistics and Physics, Qatar University, Doha 2713, Qatar. e-mail: safeer@qu.edu.qa.

Manuscript received February 2, 2011; revised May 16, 2011.

In 1991, Schu [15] introduced the following Mann-type iterative process:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \in \mathbb{N} \end{cases} \tag{1}$$

where $T : C \to C$ is an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the condition $\delta \leq \alpha_n \leq 1 - \delta$ for all $n \in \mathbb{N}$ and for some $\delta > 0$. He concluded that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Since 1972, weak and strong convergence problems of iterative sequences (with errors) for asymptotically nonexpansive type mappings in a Hilbert space or a Banach space have been studied by many authors (see, for example, [5], [12], [13], [15]).

In 2007, Agarwal et al. [1] introduced the following iteration process:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \in \mathbb{N} \end{cases} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$. They showed that this process converges at a rate same as that of Picard iteration and faster than Mann for contractions.

The above process deals with one mapping only. The case of two mappings in iterative processes has also remained under study since Das and Debata [6] gave and studied a two mappings process. Also see, for example, [11] and [19]. The problem of approximating common fixed points of finitely many mappings plays an important role in applied mathematics, especially in the theory of evolution equations and the minimization problems; see [2], [3], [4], [18], for example.

Ishikawa-type iteration process

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \in \mathbb{N} \end{cases} \tag{3}$$

for two mappings has aslo been studied by many authors including [6], [11], [19].

Recently, Khan et al. [9] modified the iteration process (2) to the case of two mappings as follows.

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \in \mathbb{N} \end{cases} \tag{4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$.

Remarks. (i) Note that (4) reduces to (2) when $S = T$. Similarly, the process (4) reduces to (1) when $T = I$.

(ii) The process (2) does not reduce to (1) but (4) does. Thus (4) not only covers the results proved by (2) but also by (1) which are not covered by (2).

(iii) The process (4) is independent of (3): neither of them reduces to the other. Following the method of Agarwal et al. [1], it can be shown that (4) converges faster than (3) for contractions. For details, see [10].

In this paper, we prove some weak and strong convergence theorems for two asymptotically quasi-nonexpansive mappings using (4).

II. Preliminaries

For the sake of convenience, we restate the following concepts and results.

Let E be Banach space with its dimension greater than or equal to 2. The modulus of E is the function $\delta_E(\varepsilon) : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf \left\{ \begin{array}{c} 1 - \left\| \frac{1}{2}(x+y) \right\| : \|x\| = 1, \\ \|y\| = 1, \varepsilon = \|x-y\| \end{array} \right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

We recall the following. Let $S = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The space E has :

(i) *Gâteaux differentiable norm* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in S;

(ii) *Fréchet differentiable norm* (see e.g. [16]) if for each x in S, the above limit exists and is attained uniformly for y in S and in this case, it is also well-known that

$$\begin{aligned} \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 &\le \frac{1}{2}\|x+h\|^2 \\ &\le \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|) \quad (5) \end{aligned}$$

for all x, h in E, where J is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in X$, $\langle .,. \rangle$ is the pairing between E and E^*, and b is an increasing function defined on $[0,\infty)$ such that $\lim_{t\downarrow 0} \frac{b(t)}{t} = 0$;

(iii) *Opial condition* [14] if for any sequence $\{x_n\}$ in $E, x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \ne x$. Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces $l^p (1 < p < \infty)$. On the other hand, $L^p[0,2\pi]$ with $1 < p \ne 2$ fail to satisfy Opial condition.

A mapping $T : C \to C$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in C, the conditions x_n converges weakly to $x \in C$ and Tx_n converges strongly to 0 imply $Tx = 0$.

Two mappings $S, T : C \to C$, where C is a subset of a normed space E, are said to satisfy the Condition (A') [7] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Sx\| \ge f(d(x,F))$ or $\|x - Tx\| \ge f(d(x,F))$ for all $x \in C$ where $d(x,F) = \inf\{\|x-p\| : p \in F\}$.

We need the following useful known lemmas for the development of our results.

Lemma 1. [20] If $\{r_n\}, \{t_n\}$ and $\{s_n\}$ are sequences of nonnegative real numbers such that $r_{n+1} \le (1+t_n) r_n + s_n$, $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, then $\lim\limits_{n\to\infty} r_n$ exists.

Lemma 2. [15] Suppose that E is a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \le r$, $\limsup_{n\to\infty} \|y_n\| \le r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

III. Main Results

In this section, we prove some strong and weak convergence theorems of the iteration process (4) under some suitable conditions. Before proving our main results, we would like to remark as follows. Let $T, S : C \to C$ be two asymptotically quasi-nonexpansive mappings such that $\|T^n x - p\| \le k_{1n}\|x - p\|$ and $\|S^n x - p\| \le k_{2n}\|x - p\|$. Throughout this paper, we assume that $k_n = \max_{n\in\mathbb{N}}\{k_{1n}, k_{2n}\}$ and $F \ne \emptyset$ where $F = F(T) \cap F(S)$ is the set of common fixed points of the mappings T and S. We need the following lemma in order to prove our main theorems.

Lemma 3. Let C be a nonempty closed convex subset of a real Banach space E. Let T and S be two asymptotically quasi-nonexpansive self mappings of C with $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Let $\{x_n\}$ be defined by (4) and $F \ne \emptyset$. Then $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$.

Proof. Let $q \in F$. Then

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1-\alpha_n)T^n x_n + \alpha_n S^n y_n - q\| \\ &\le (1-\alpha_n)\|T^n x_n - q\| + \alpha_n \|S^n y_n - q\| \\ &\le (1-\alpha_n)(k_n \|x_n - q\|) + \alpha_n (k_n \|y_n - q\|) \\ &\le k_n \left[\begin{array}{c} (1-\alpha_n)\|x_n - q\| \\ +\alpha_n(1-\beta_n)\|x_n - q\| \\ +\alpha_n\beta_n \|T^n x_n - q\| \end{array} \right] \\ &\le k_n \left[\begin{array}{c} (1-\alpha_n)\|x_n - q\| \\ +\alpha_n(1-\beta_n)\|x_n - q\| \\ +\alpha_n\beta_n (k_n \|x_n - q\|) \end{array} \right] \\ &\le k_n \left[\left(\begin{array}{c} 1-\alpha_n + \alpha_n(1-\beta_n) \\ +k_n\alpha_n\beta_n \end{array} \right) \|x_n - q\| \right] \\ &= k_n[1 + (k_n - 1)\alpha_n\beta_n]\|x_n - q\| \\ &\le k_n[1 + k_n - 1]\|x_n - q\| \\ &= \left[1 + \left(k_n^2 - 1\right)\right]\|x_n - q\| \end{aligned}$$

Since $\{k_n\}$ (and hence $\{k_n + 1\}$) is a nonincreasing bounded sequence, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. It now follows from Lemma 1 that $\lim\limits_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$.

Theorem 1. Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T and S be two uniformly L_1 and L_2-Lipschitzian and asymptotically quasi-nonexpansive self mappings of C with $\sum_{n=1}^{\infty}(k_n-1)<\infty$ and $L=\max\{L_1,L_2\}$. Let $\{x_n\}$ be defined by (4) and $F\neq\emptyset$. Then$\lim_{n\to\infty}\|x_n-Tx_n\|=\lim_{n\to\infty}\|x_n-Sx_n\|=0$.

Proof. By Lemma 3, $\lim\limits_{n\to\infty}\|x_n-q\|$ exists. Assume that $\lim\limits_{n\to\infty}\|x_n-q\|=c$. If $c=0$, the conclusion is obvious. Suppose $c>0$.

Now

$$\begin{aligned}\|y_n-q\| &= \|(1-\beta_n)x_n+\beta_nT^nx_n-q\|\\ &\le (1-\beta_n)\|x_n-q\|+\beta_n(k_n\|x_n-q\|)\\ &\le (1-\beta_n)\|x_n-q\|+\beta_n(k_n\|x_n-q\|)\\ &\le (1+\beta_n(k_n-1))\|x_n-q\|\end{aligned}$$

implies that

$$\limsup_{n\to\infty}\|y_n-q\|\le c. \tag{6}$$

Since T is an asymptotically quasi-nonexpansive mappings, we have

$$\|T^nx_n-q\|\le k_n\|x_n-q\|$$

for all $n=1,2,\ldots$. Taking $\limsup$ on both sides, we obtain

$$\limsup_{n\to\infty}\|T^nx_n-q\|\le c. \tag{7}$$

In a similar way, we have

$$\|S^ny_n-q\|\le k_n\|y_n-q\|.$$

By using (6), we obtain

$$\limsup_{n\to\infty}\|S^ny_n-q\|\le c.$$

Also, it follows from $c=\lim\limits_{n\to\infty}\|x_{n+1}-q\|=\lim\limits_{n\to\infty}\|(1-\alpha_n)(T^nx_n-q)+\alpha_n(S^ny_n-q)\|$ and Lemma 2 that

$$\lim_{n\to\infty}\|T^nx_n-S^ny_n\|=0. \tag{8}$$

Now

$$\begin{aligned}\|x_{n+1}-q\| &= \|(1-\alpha_n)T^nx_n+\alpha_nS^ny_n-q\|\\ &= \|(T^nx_n-q)+\alpha_n(S^ny_n-T^nx_n)\|\\ &\le \|T^nx_n-q\|+\alpha_n\|T^nx_n-S^ny_n\|\end{aligned}$$

yields that

$$c\le\liminf_{n\to\infty}\|T^nx_n-q\|$$

so that (7) gives $\lim_{n\to\infty}\|T^nx_n-q\|=c$.
On the other hand,

$$\begin{aligned}\|T^nx_n-q\| &\le \|T^nx_n-S^ny_n\|+\|S^ny_n-q\|\\ &\le \|T^nx_n-S^ny_n\|+k_n\|y_n-q\|,\end{aligned}$$

so we have

$$c\le\liminf_{n\to\infty}\|y_n-q\|. \tag{9}$$

By using (6) and (9), we obtain

$$\lim_{n\to\infty}\|y_n-q\|=c. \tag{10}$$

Thus $c=\lim\limits_{n\to\infty}\|y_n-q\|=\lim\limits_{n\to\infty}\|(1-\beta_n)(x_n-q)+\beta_n(T^nx_n-q)\|$ gives by Lemma 2 that

$$\lim_{n\to\infty}\|T^nx_n-x_n\|=0. \tag{11}$$

Now

$$\|y_n-x_n\|=\beta_n\|T^nx_n-x_n\|.$$

Hence by (11),

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{12}$$

Also note that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|(1-\alpha_n)T^nx_n+\alpha_nS^ny_n-x_n\|\\ &\le \|T^nx_n-x_n\|+\alpha_n\|T^nx_n-S^ny_n\|\\ &\to 0 \text{ as } n\to\infty,\end{aligned} \tag{13}$$

so that

$$\begin{aligned}\|x_{n+1}-y_n\| &\le \|x_{n+1}-x_n\|+\|y_n-x_n\|\\ &\to 0 \text{ as } n\to\infty.\end{aligned} \tag{14}$$

Furthermore, from

$$\begin{aligned}\|x_{n+1}-S^ny_n\| \le\ & \|x_{n+1}-x_n\|+\|x_n-T^nx_n\|\\ &+\|T^nx_n-S^ny_n\|\end{aligned}$$

we find that

$$\lim_{n\to\infty}\|x_{n+1}-S^ny_n\|=0. \tag{15}$$

Then

$$\begin{aligned}\|x_{n+1}-Tx_{n+1}\| &\le \left\|x_{n+1}-T^{n+1}x_{n+1}\right\|\\ &\quad+\left\|T^{n+1}x_{n+1}-T^{n+1}x_n\right\|\\ &\quad+\left\|T^{n+1}x_n-Tx_{n+1}\right\|\\ &\le \left\|x_{n+1}-T^{n+1}x_{n+1}\right\|\\ &\quad+L\|x_{n+1}-x_n\|\\ &\quad+L\|T^nx_n-x_{n+1}\|\\ &= \left\|x_{n+1}-T^{n+1}x_{n+1}\right\|\\ &\quad+L\|x_{n+1}-x_n\|\\ &\quad+L\alpha_n\|T^nx_n-S^ny_n\|\end{aligned}$$

yields

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{16}$$

Now

$$\begin{aligned}\|x_n-S^nx_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-S^ny_n\|\\ &\quad+\|S^ny_n-S^nx_n\|\\ &\le \|x_n-x_{n+1}\|+\|x_{n+1}-S^ny_n\|\\ &\quad+L\|y_n-x_n\|\\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus

$$\begin{aligned}\|x_{n+1}-Sx_{n+1}\| &\leq \left\|x_{n+1}-S^{n+1}x_{n+1}\right\| \\ &\quad+\left\|S^{n+1}x_{n+1}-Sx_{n+1}\right\| \\ &\leq \left\|x_{n+1}-S^{n+1}x_{n+1}\right\| \\ &\quad+L\left\|S^{n}x_{n+1}-x_{n+1}\right\| \\ &\leq \left\|x_{n+1}-S^{n+1}x_{n+1}\right\| \\ &\quad+L\left(\begin{array}{c}\|S^{n}x_{n+1}-S^{n}y_{n}\| \\ +\|S^{n}y_{n}-x_{n+1}\|\end{array}\right) \\ &\leq \left\|x_{n+1}-S^{n+1}x_{n+1}\right\| \\ &\quad+L^{2}\left\|x_{n+1}-y_{n}\right\| \\ &\quad+L\left\|S^{n}y_{n}-x_{n+1}\right\|\end{aligned}$$

implies

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0.$$

Lemma 4. Assume that the conditions of Theorem 1 are satisfied. Then, for any $p_1, p_2 \in F$, $\lim_{n\to\infty}\langle x_n, J(p_1-p_2)\rangle$ exists; in particular, $\langle p-q, J(p_1-p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$.

Proof. Suppose that $x = p_1 - p_2$ with $p_1 \neq p_2$ and $h = t(x_n - p_1)$ in the inequality (5). Then, we get

$$\begin{aligned}& t\langle x_n-p_1, J(p_1-p_2)\rangle+\frac{1}{2}\|p_1-p_2\|^2 \\ \leq\ & \frac{1}{2}\|tx_n+(1-t)p_1-p_2\|^2 \\ \leq\ & t\langle x_n-p_1, J(p_1-p_2)\rangle+\frac{1}{2}\|p_1-p_2\|^2+ \\ & +b(t\|x_n-p_1\|).\end{aligned}$$

Since $\sup_{n\geq 1}\|x_n-p_1\| \leq M'$ for some $M'>0$, we have

$$\begin{aligned}& t\limsup_{n\to\infty}\langle x_n-p_1, J(p_1-p_2)\rangle+\frac{1}{2}\|p_1-p_2\|^2 \\ \leq\ & \frac{1}{2}\lim_{n\to\infty}\|tx_n+(1-t)p_1-p_2\|^2 \\ \leq\ & t\liminf_{n\to\infty}\langle x_n-p_1, J(p_1-p_2)\rangle \\ & +\frac{1}{2}\|p_1-p_2\|^2+b(tM').\end{aligned}$$

That is,

$$\begin{aligned}& \limsup_{n\to\infty}\langle x_n-p_1, J(p_1-p_2)\rangle \\ \leq\ & \liminf_{n\to\infty}\langle x_n-p_1, J(p_1-p_2)\rangle+\frac{b(tM')}{tM'}M'.\end{aligned}$$

If $t \to 0$, then $\lim_{n\to\infty}\langle x_n-p_1, J(p_1-p_2)\rangle$ exists for all $p_1, p_2 \in F$; in particular, we have $\langle p-q, J(p_1-p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$.

We now give our weak convergence theorem. In the sequel, $I : X \to X$ denotes the identity mapping.

Theorem 2. Let E be a uniformly convex Banach space satisfying Opial condition and let C, T, S and $\{x_n\}$ be taken as in Theorem 1. If the mappings $I-T$ and $I-S$ are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of T and S.

Proof. Let $q \in F$, then according to Lemma 3 the sequence $\{\|x_n-q\|\}$ is convergent and hence bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Thus there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q_1 \in C$. From Theorem 1, we have

$$\lim_{n\to\infty}\|x_{n_k}-Tx_{n_k}\|=0,\ \lim_{n\to\infty}\|x_{n_k}-Sx_{n_k}\|=0.$$

Since the mappings $I-T$ and $I-S$ are demiclosed at zero, therefore $Tq_1 = q_1$ and $Sq_1 = q_1$, which means $q_1 \in F$. Finally, let us prove that $\{x_n\}$ converges weakly to q_1. Suppose on contrary that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_2 \in C$ and $q_1 \neq q_2$. Then by the same method as given above, we can also prove that $q_2 \in F$. From Lemma 3, the limits $\lim_{n\to\infty}\|x_n-q_1\|$ and $\lim_{n\to\infty}\|x_n-q_2\|$ exist. By virtue of the Opial condition of E, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|x_n-q_1\| &= \lim_{n_i\to\infty}\|x_{n_i}-q_1\| \\ &< \lim_{n_i\to\infty}\|x_{n_i}-q_2\| \\ &= \lim_{n\to\infty}\|x_n-q_2\| \\ &= \lim_{n_j\to\infty}\|x_{n_j}-q_2\| \\ &< \lim_{n_j\to\infty}\|x_{n_j}-q_1\| \\ &= \lim_{n\to\infty}\|x_n-q_1\|\end{aligned}$$

which is a contradiction so $q_1 = q_2$. Therefore, $\{x_n\}$ converges weakly to a common fixed point of T and S.

Theorem 3. Let E be a uniformly convex Banach space which has a Fréchet differentiable norm and let C, T, S and $\{x_n\}$ be taken as in Theorem 1. If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T and S.

Proof. By Lemma 4, $\langle p-q, J(p_1-p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$. Therefore $\|q_1-q_2\|^2 = \langle q_1-q_2, J(q_1-q_2)\rangle = 0$ implies $q_1 = q_2$. Consequently, $\{x_n\}$ converges weakly to a point of F and this completes the proof.

We have the following corollaries.

Corollary 1. Let E be a uniformly convex Banach space satisfying Opial condition and let C and T be taken as in Theorem 1. Suppose that $F(T) \neq \emptyset$. If the mapping $I-T$ is demiclosed at zero, then $\{x_n\}$ defined by (2) converges weakly to a fixed point of T.

Corollary 2. Let E be a uniformly convex Banach space which has a Fréchet differentiable norm and let C and T be taken as in Theorem 1. If $F(T) \neq \emptyset$, then $\{x_n\}$ defined by (2) converges weakly to a fixed point of T.

Corollary 3. Let E be a uniformly convex Banach space satisfying Opial condition and let C and T be taken as in Theorem 1. Suppose that $F(T) \neq \emptyset$. If the mapping $I-T$ is demiclosed at zero, then $\{x_n\}$ defined by (1) converges weakly to a fixed point of T.

Corollary 4. Let E be a uniformly convex Banach space which has a Fréchet differentiable norm and let C and T be taken as in Theorem 1. If $F(T) \neq \emptyset$, then $\{x_n\}$ defined by (1) converges weakly to a fixed point of T.

Theorem 4. Let E be a real Banach space and let $C, T, S, F, \{x_n\}$ be taken as in Theorem 1. Then $\{x_n\}$ converges to a point of F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$ where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. As proved in Lemma 3, $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F$, therefore $\lim_{n\to\infty} d(x_n, F)$ exists. But by hypothesis, $\liminf_{n\to\infty} d(x_n, F) = 0$, therefore we have $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists a positive integer n_0 such that

$$d(x_n, F) < \frac{\epsilon}{4}, \quad \forall n \geq n_0.$$

In particular, $\inf\{\|x_{n_0} - p\| : p \in F\} < \frac{\epsilon}{4}$. Thus there must exist $p^* \in F$ such that

$$\|x_{n_0} - p^*\| < \frac{\epsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq 2\,\|x_{n_0} - p^*\| \\ &< 2\left(\frac{\epsilon}{2}\right) = \epsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a Banach space E and so it must converge to a point q in C. Now, $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Since F is closed, so we have $q \in F$.

Applying Theorem 4, we obtain strong convergence of the process (4) under the Condition (A') as follows.

Theorem 5. Let E be a real uniformly convex Banach space and let $C, T, S, F, \{x_n\}$ be taken as in Theorem 1. Let T, S satisfy the Condition (A'), then $\{x_n\}$ converges strongly to a common fixed point of T and S .

Proof. We proved in Theorem 1 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0. \qquad (17)$$

From the Condition (A') and (17), either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Sx_n\| = 0,$$

Hence

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0,\ f(r) > 0$ for all $r \in (0, \infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Now all the conditions of Theorem 4 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of F.

Remarks. (i) Results using the iterative processes (1) and (2) can now be obtained as corollaries from Theorems 4 and 5.

(ii) The case of nonexpansive mappings using the iterative processes (1) , (2) and (4) can now be deduced from our above results.

(iii) Theorems of this paper can also be proved with error terms.

References

[1] R.P.Agarwal, Donal O'Regan and D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonliear Convex. Anal.**8(1)**(2007), 61–79.

[2] L.C. Deng, P. Cubiotti, J.C. Yao, *Approximation of common fixed points of families of nonexpansive mappings*, Taiwanese J. Math. 12 (2008) 487-500.

[3] L.C. Deng, P. Cubiotti, J.C. Yao, *An implicit iteration scheme for monotone variational inequalities and fixed point problems*, Nonlinear Anal. 69 (2008) 2445-2457.

[4] L.C. Deng, S. Schaible, J.C. Yao, *Implicit iteration scheme with perturbed mapping for equilibrium problems and fixed point problems of finitely many nonexpansive mappings*, J. Optim. Theory Appl. 139 (2008) 403-418.

[5] Y. J. , H. Y. Zhou and G. Guo, *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings,* Comput. Math. Appl. **47**(2004), 707-717.

[6] G. Das and J. P. Debata, *Fixed points of Quasi-nonexpansive mappings,* Indian J. Pure. Appl. Math., **17** (1986), 1263–1269.

[7] H. Fukhar-ud-din and S. H. Khan, *Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications,* J. Math. Anal. Appl. **328** (2007), 821–829.

[8] K.Goebel and W.A.Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc.35(1972), 171-174.

[9] S. H. Khan, Y.J. , M. Abbas, *Convergence to common fixed points by a modified iteration process*, Journal of Appl. Math. and Comput. doi: 10.1007/s12190-010-0381-z.

[10] S. H. Khan, J.K. Kim,*Common fixed points of two nonexpansive mappings by a modified faster iteration scheme* , Bull. Korean Math. Soc. 47 (2010), No. 5, pp. 973–985, DOI 10.4134/BKMS.2010.47.5.973

[11] S. H. Khan and W. Takahashi, *Approximating common fixed points of two asymptotically nonexpansive mappings*, Sci. Math. Jpn., **53(1)** (2001), 143–148.

[12] K. Nammanee, M.A. Noor, S. Suantai, *Convergence criteria of modifies Noor iterations with errors for asymptotically nonexpansive mappings*, J. Math. Anal. Appl. 314 (2006) 320-334.

[13] W. Nilsrakoo, S. Saejung, *A new three-step fixed point iteration scheme for asymptotically nonexpansive mappings*, Comput. Math. Appl. 181 (2006) 1026-1034.

[14] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73** (1967), 591–597.

[15] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. **43** (1991), 153-159.

[16] W. Takahashi and G.E.Kim, *Approximatig fixed points of nonexpansive mappings in Banach spaces* , Math.Japonoica 48 1(1998), 1–9.

[17] N. Shahzad, H. Zegeye, *Strong convergence of an implicit iteration process for a finite family of generalized asymptotically quasi-nonexpansive maps*, Appl. Math. Comput. 189 (2007) 1058-1065.

[18] W. Takahashi, *Iterative methods for approximation of fixed points and their applications*, J.Oper.Res.Soc. Jpn., **43(1)** (2000), 87 -108.

[19] W. Takahashi and T. Tamura, *Limit theorems of operators by convex combinations of nonexpansive retractions in Banach spaces,* J.Approx.Theory , **91(3)** (1997), 386 -397.

[20] K.K. Tan, H.K. Xu, *Approximating fixed points of nonexpansive mappings by Ishikawa iteration process*, J. Math. Anal. Appl. 178 (1993) 301-308.