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Two Fourth-order Iterative Methods Based on Continued Fraction for Root-finding Problems

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Abstract—In this paper, we present two new one-step iterative methods based on Thiele's continued fraction for solving nonlinear equations. By applying the truncated Thiele's continued fraction twice, the iterative methods are obtained respectively. Analysis of convergence shows that the new methods are fourth-order convergent. Numerical tests verifying the theory are given and based on the methods, two new one-step iterations are developed.

Keywords—Iterative method, Fixed-point iteration, Thiele's continued fraction, Order of convergence.

I. Introduction

SOLVING non-linear equations with computers is one of the most important problems in numerical analysis. In this paper, we construct two iterative methods to find a simple root of a nonlinear equation $f(x) = 0$, where $f : X \to R$, $X \subset R$, is a scalar function.

Expanding $f(x)$ into a Taylor series about the point x_k gives

$$f(x) = f(x_k) + (x - x_k)f'(x_k) + \frac{1}{2!}(x - x_k)^2 f''(x_k) + \cdots .$$

If $f'(x_k) \neq 0$, we can obtain an approximate expression of the equation $f(x) = 0$ by substituting the linear part of above expansion for the function $f(x)$. Denote by x_{k+1} the root of the equation $f(x_k) + (x - x_k)f'(x_k) = 0$. Then we have

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{1}$$

This is the Newton's method (NM) [1], [2], [3] for root-finding of nonlinear equation, which converges quadratically.

Halley's method (HM) in [10], [11], [12] is well-known for its order of convergence three. The method can be written as below

$$x_{k+1} = x_k - \frac{2f(x_k)f'(x_k)}{2f'^2(x_k) - f(x_k)f''(x_k)}. \tag{2}$$

It is easy to see that Halley's method need to compute second derivative of the function $f(x)$.

The iterative methods with higher-order convergence are presented in some literature [13], [14], [15]. In [14], Abbasbandy gives an iterative method, called Abbasbandy's method (AM) provisionally, which is expressed as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{f^2(x_k)f''(x_k)}{2f'^3(x_k)} - \frac{f^3(x_k)f'''(x_k)}{6f'^4(x_k)}. \tag{3}$$

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It is pointed out that AM has nearly supercubic convergence. One can see easily that AM requires the evaluation of second and third derivatives of the function $f(x)$. Kou et al present some higher-order methods in [13], [15], [16], [17], but most of which are two-step methods.

In this paper, we use the Thiele's continued fraction to construct two new one-step iterative methods. It is shown that the new methods are at least fourth-order convergent. Numerical results demonstrate that the iteration schemes are efficient and superior to Newton's scheme, Halley's scheme and Abbasbandy's method.

II. The New Method

Given a set of real points $G = \{x_i | x_i \in R, i = 0, 1, 2, \cdots\}$ and $b_n \in R$, $n = 0, 1, 2, \cdots$.

Definition 1 A continued fraction of the following form

$$b_0 + \frac{x - x_0}{\lfloor b_1} + \frac{x - x_1}{\lfloor b_2} + \cdots + \frac{x - x_{n-1}}{\lfloor b_n} + \cdots$$

is called Thiele's continued fraction (see [4], [5]).

Definition 2 The following continued fraction

$$b_0 + \frac{x - x_0}{\lfloor b_1} + \frac{x - x_1}{\lfloor b_2} + \cdots + \frac{x - x_{n-1}}{\lfloor b_n}$$

is called the n-th truncated Thiele's continued fraction (see [4], [5]).

Viscovatov Algorithm Suppose that the function $f(x)$ has n-th derivative in the interval X. If $f(x)$ can be expanded into the following Thiele's continued fraction

$$f(x) = b_0 + \frac{x - x_k}{\lfloor b_1} + \frac{x - x_k}{\lfloor b_2} + \cdots + \frac{x - x_k}{\lfloor b_n} + \cdots,$$

then the coefficients $b_n, n = 0, 1, 2, \cdots$, can be computed by means of the Viscovatov algorithm as below

$$\begin{cases} b_0 = C_0^{(0)}, \\ b_1 = 1/C_1^{(0)}, \\ C_i^{(1)} = -C_{i+1}^{(0)}/C_1^{(0)}, i \geq 1, \\ b_l = C_1^{(l-2)}/C_1^{(l-1)}, l \geq 2, \\ C_i^{(l)} = C_{i+1}^{(l-2)} - b_l C_{i+1}^{(l-1)}, i \geq 1, l \geq 2, \end{cases}$$

where $C_i^{(0)} = \frac{f^{(i)}(x_k)}{i!}, i = 0, 1, 2, \cdots$ (see [4], [5]).

Now, we give two new iterative schemes as follows.

- **(A)** Replacing $f(x)$ with the second truncated Thiele's continued fraction in $f(x) = 0$ yields

$$b_0 + \frac{x - x_k}{\lfloor b_1} + \frac{x - x_k}{\lfloor b_2} = 0.$$

Solving the above equation gives

$$x - x_k = -\frac{b_0 b_1 b_2}{b_0 + b_2}. \tag{4}$$

Furthermore, substituting $f(x)$ with third truncated Thiele's continued fraction yields the following equation

$$b_0 + \frac{x - x_k|}{|\, b_1} + \frac{x - x_k|}{|\, b_2} + \frac{x - x_k|}{|\, b_3} = 0. \tag{5}$$

From (4) and (5), we have

$$b_0 + \frac{x - x_k|}{|\, b_1} + \frac{x - x_k|}{|\, b_2} + \frac{-\frac{b_0 b_1 b_2}{b_0 + b_2}|}{|\, b_3} = 0. \tag{6}$$

Let x_{k+1} denote the root of the equation (6). We have

$$x_{k+1} = x_k - \frac{b_0 b_1 (b_2 b_3 (b_0 + b_2) - b_0 b_1 b_2)}{(b_0 + b_2)^2 b_3 - b_0 b_1 b_2}. \tag{7}$$

Using Viscovatov algorithm, one gets

$$b_0 = f(x_k), \tag{8}$$

$$b_1 = \frac{1}{f'(x_k)}, \tag{9}$$

$$b_2 = -\frac{2(f'(x_k))^2}{f''(x_k)}, \tag{10}$$

$$b_3 = \frac{3(f''(x_k))^2}{2(f'(x_k))^2 f'''(x_k) - 3f'(x_k)(f''(x_k))^2}. \tag{11}$$

Substituting (8), (9), (10) and (11) into (7), we obtain

$$x_{k+1} = x_k - \tfrac{P_1(x_k)}{Q_1(x_k)}. \tag{12}$$

This is a new one-step iterative scheme based on Thiele's continued fraction, which is called Thiele's method (T-M1). Therefore the corresponding iteration function for TM1 can be defined by

$$\varphi_1(x) = x - \tfrac{P_1(x)}{Q_1(x)}, \tag{13}$$

where $P_1(x) = 4f(x)f'(x)(3f'^2(x)f''(x) - 3f(x)f''^2(x) + f(x)f'(x)f'''(x))$ and $Q_1(x) = 12f'^4(x)f''(x) - 18f(x)f'^2(x)f''^2(x) + 3f^2(x)f''^3(x) + 4f(x)f'^3(x)f'''(x)$.

- **(B)** Replacing $f(x)$ with the first truncated Thiele's continued fraction in $f(x) = 0$ gives

$$b_0 + \frac{x - x_k|}{|\, b_1} = 0.$$

Solving the above equation yields

$$x - x_k = -b_0 b_1. \tag{14}$$

Then, substituting (14) into (5) gets the following equation

$$b_0 + \frac{x - x_k|}{|\, b_1} + \frac{x - x_k|}{|\, b_2} + \frac{-b_0 b_1|}{|\, b_3} = 0. \tag{15}$$

Let x_{k+1} denote the root of the equation (15). We have

$$x_{k+1} = x_k - \frac{b_0 b_1 (b_2 b_3 - b_0 b_1)}{(b_0 + b_2) b_3 - b_0 b_1}. \tag{16}$$

Substituting (8), (9), (10) and (11) into (16), we obtain

$$x_{k+1} = x_k - \tfrac{P_2(x_k)}{Q_2(x_k)}. \tag{17}$$

This is another new one-step iterative scheme based on Thiele's continued fraction, which is also called Thiele's method (TM2). Therefore the corresponding iteration function for TM2 can be defined by

$$\varphi_2(x) = x - \tfrac{P_2(x)}{Q_2(x)}, \tag{18}$$

where $P_2(x) = f(x)(6f'^2(x)f''(x) - 3f(x)f''^2(x) + 2f(x)f'(x)f'''(x))$ and $Q_2(x) = 2f'(x)(3f'^2(x)f''(x) - 3f(x)f''^2(x) + f(x)f'(x)f'''(x))$.

III. Convergence Analysis

Theorem 1 Suppose that $Q_1(x) = 12f'^4(x)f''(x) - 18f(x)f'^2(x)f''^2(x) + 3f^2(x)f''^3(x) + 4f(x)f'^3(x)f'''(x)$. Let x^* be a root of the equation $f(x) = 0$. Then we have

Case (a): x^* is a single root of the equation $f(x) = 0$ if $Q_1(x^*) \neq 0$.

Case (b): x^* is a multiple root of the equation $f(x) = 0$ if $Q_1(x^*) = 0$.

Proof Suppose n is the multiplicity of the root x^*. Then $f(x)$ can be rewritten as the form

$$f(x) = (x - x^*)^n h(x), \tag{19}$$

where $h(x^*) \neq 0$. Computing first and second derivatives of (19), we have

$$f'(x) = (x - x^*)^n h'(x) + n(x - x^*)^{n-1} h(x), \tag{20}$$

$$\begin{aligned} f''(x) = {} & (x - x^*)^n h''(x) + 2n(x - x^*)^{n-1} h'(x) \\ & + n(n-1)(x - x^*)^{(n-2)} h(x). \end{aligned} \tag{21}$$

Substituting (19), (20) and (21) into $Q_1(x)$ and noticing that $f(x^*) = 0$, we get the following conclusions.
Case (a): $Q_1(x^*) = 24h^4(x^*)h'(x^*) \neq 0$ for $n = 1$, i.e., x^* is a single root of the equation $f(x) = 0$.
Case (b): $Q_1(x^*) = 0$ for $n \geq 2$, that is, x^* is a multiple root of the equation $f(x) = 0$.
The proof is completed.

Theorem 2 Suppose that $Q_1(x) = 12f'^4(x)f''(x) - 18f(x)f'^2(x)f''^2(x) + 3f^2(x)f''^3(x) + 4f(x)f'^3(x)f'''(x) \neq 0$ for an arbitrary $x \in [a, b] \subset X$. Then the equation $f(x) = 0$ has at most one single root in the interval $[a, b]$.

Proof According to Theorem 1, it is obvious that $f(x) = 0$ has only single roots if any. Let

$$\varphi(x) = f(x) e^{\int_a^x \frac{3f(t)f''^3(t) - 18f'^2(t)f''^2(t) + 4f'^3(t)f'''(t)}{12f'^3(t)f''(t)} dt}. \tag{22}$$

The root-finding problem $f(x) = 0$ can clearly be transformed into the equivalent problem $\varphi(x) = 0$. Computing first derivative of (22), we have

$$\varphi'(x) = K(x) e^{\int_a^x \frac{3f(t)f''^3(t) - 18f'^2(t)f''^2(t) + 4f'^3(t)f'''(t)}{12f'^3(t)f''(t)} dt},$$

where $K(x) = \frac{Q_1(x)}{12f'^3(x)f''(x)}$. It follows from $Q_1(x) \neq 0$ that

$$\varphi'(x) \neq 0.$$

Assume that the equation $f(x)=0$ has two different roots in the interval $[a,b]$. Then by Rolle mean-value theorem, at least a point $\xi \in [a,b]$ exists such that $f'(\xi)=0$ and $\varphi'(\xi)=0$, which contradicts with the fact that $\varphi'(x)\neq 0$ in the interval $[a,b]$. Therefore there is a unique single root in the interval $[a,b]$. Theorem 2 is proved.

Lamma 1 Suppose that x^* is a root of the equation $f(x)=x-\varphi(x)=0$, where $\varphi(x)=x-\Phi(f(x))$ and Φ is a continuous function such that $\Phi(0)=0$. Let $\varphi(x)$ have m-th derivative in the neighborhood of x^*, where $m\geq 2$. And assume that $\varphi(x)$ satisfies

$$\varphi^{(j)}(x^*)=0, j=1,2,\cdots,m-1, \varphi^{(m)}(x^*)\neq 0.$$

Then the convergence order of fixed-point iteration $x_k=\varphi(x_{k-1})$ is at least m.

Proof For the details of the proof, see the literature [1], [2].

Theorem 3 Suppose that x^* is a root of the equation $f(x)=0$ and let $f'(x^*)\neq 0$. If $f(x)$ is sufficiently smooth in the neighborhood of x^*, then the convergence order of Thiele's method (TM1) given in (12) is at least four.

Proof Using the iterative function defined by (13), one can easily verify that

$$\varphi_1'(x^*)=0,$$
$$\varphi_1''(x^*)=0,$$
$$\varphi_1'''(x^*)=0$$

and

$$\varphi_1^{(4)}(x^*)=\frac{3f''(x^*)f^{(4)}(x^*)-4(f'''(x^*))^2}{3f'(x^*)f''(x^*)}\neq 0.$$

Therefore, the convergence order of TM1 given in (12) is at least four according to Lemma 1.

Similarly, we give three theorems as below:

Theorem 4 Suppose that $Q_2(x)=2f'(x)(3f'^2(x)f''(x)-3f(x)f''^2(x)+f(x)f'(x)f'''(x))$. Let x^* be a root of the equation $f(x)=0$. Then we have

Case (a): x^* is a single root of the equation $f(x)=0$ if $Q_2(x^*)\neq 0$.

Case (b): x^* is a multiple root of the equation $f(x)=0$ if $Q_2(x^*)=0$.

Theorem 5 Suppose that $Q_2(x)=2f'(x)(3f'^2(x)f''(x)-3f(x)f''^2(x)+f(x)f'(x)f'''(x))\neq 0$ for an arbitrary $x\in[a,b]\subset X$. Then the equation $f(x)=0$ has at most one single root in the interval $[a,b]$.

Theorem 6 Suppose that x^* is a root of the equation $f(x)=0$ and let $f'(x^*)\neq 0$. If $f(x)$ is sufficiently smooth in the neighborhood of x^*, then the convergence order of Thiele's method (TM2) given in (17) is at least four.

IV. Numerical Examples

Now, we employ the new methods (TM1 and TM2) given in (12) and (17) to solve some non-linear equations and compare it with Newton's method (NM), Halley's method (HM) and Abbasbandy's method (AM). All computations are carried out with double arithmetic precision. All problems are solved with a given initial value x_0. Displayed in Table 1 are the number of iterations (k). We choose $|f(x_k)|<10^{-14}$ as stopping criteria so that the iterative process is terminated when the criteria is satisfied. x^* is the root of equation. We use the following functions, most of which are the same as in [6], [7], [8], [9], [16], [17].

$$f_1(x)=x^3+4x^2-25, x^*=2.035268481182.$$
$$f_2(x)=x^2-e^x-3x+2, x^*=0.257530285439.$$
$$f_3(x)=x^3-10, 2.154434690032.$$
$$f_4(x)=\cos(x)-x, x^*=0.739085133215.$$
$$f_5(x)=\sin^2(x)-x^2+1, x^*=1.404491648215.$$
$$f_6(x)=x^2+\sin(x/5)-1/4, x^*=0.409992017989.$$
$$f_7(x)=e^x-4x^2, x^*=0.714805912363.$$
$$f_8(x)=e^{-x}+\cos(x), x^*=1.746139530408.$$
$$f_9(x)=x\cos x+\sin x-1, x^*=0.555968430719.$$
$$f_{10}(x)=e^x\sin x-\ln(x^2+1), x^*=3.029169668013.$$

TABLE I
Comparison of NM, HM, AM and TM

$f(x)$	x_0	k				
		NM	HM	AM	TM1	TM2
$f_1(x)$	$x_0=-0.9$	29	10	9	9	6
	$x_0=3.5$	6	4	5	3	3
$f_2(x)$	$x_0=0.5$	4	3	3	3	3
	$x_0=-1$	5	3	3	3	3
$f_3(x)$	$x_0=1$	7	4	5	3	4
	$x_0=-0.5$	10	8	16	5	4
$f_4(x)$	$x_0=1$	4	3	3	2	2
	$x_0=0.5$	4	3	3	2	2
$f_5(x)$	$x_0=3.5$	6	4	4	4	4
	$x_0=1.2$	5	3	3	2	3
$f_6(x)$	$x_0=1$	6	4	4	3	3
	$x_0=0.2$	5	3	4	2	3
$f_7(x)$	$x_0=1$	5	3	3	2	2
	$x_0=0.4$	6	4	6	3	3
$f_8(x)$	$x_0=2.4$	5	4	4	3	3
	$x_0=0.5$	5	4	4	3	4
$f_9(x)$	$x_0=0$	5	4	4	3	3
	$x_0=-0.4$	5	4	6	3	3
$f_{10}(x)$	$x_0=3.5$	6	4	4	3	3
	$x_0=3$	4	3	3	2	2

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