

Two-dimensional Differential Transform Method for Solving Linear and Non-linear Goursat Problem

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Abstract—A method for solving linear and non-linear Goursat problem is given by using the two-dimensional differential transform method. The approximate solution of this problem is calculated in the form of a series with easily computable terms and also the exact solutions can be achieved by the known forms of the series solutions. The method can easily be applied to many linear and non-linear problems and is capable of reducing the size of computational work. Several examples are given to demonstrate the reliability and the performance of the presented method.

Keywords—Quadrature, Spline interpolation, Trapezoidal rule, Numerical integration, Error analysis.

I. INTRODUCTION

ONE-dimensional Differential Transform Method (DTM) was first introduced by Zhou [1] for solving linear and non-linear initial value problems in electrical circuit analysis. It has been also used in obtaining series solutions to a wide class of linear and non-linear ordinary differential equations [2-10]. Based on the same methodology, Chen and Ho [11] recently developed the two-dimensional DTM for solving the differential and integral equations. For example, in [12] this method is used for solving a system of differential equations and in [13] for differential-algebraic equations and in [14], [15] is applied to partial differential equations.

The subject of the presented paper is to apply the two-dimensional DTM for solving linear and non-linear Goursat problem which arises in physical phenomena and applied sciences. For this purpose, we consider the standard form of the Goursat problem [16-19]

$$\begin{aligned} u_{xt} &= f(x, t, u, u_x, u_t), \quad 0 \leq x \leq a, \quad 0 \leq t \leq b, \\ u(x, 0) &= g(x), \quad u(0, t) = h(t), \\ g(0) &= h(0) = u(0, 0). \end{aligned}$$

This equation has been examined by several numerical methods such as Runge-Kutta method, finite difference method, finite elements method, Adomian Decomposition Method (ADM), Variational Iteration Method (VIM) and geometric mean averaging of the functional values of $f(x, t, u, u_x, u_t)$. See for example [16-21] and references therein.

In this paper, we present the applicability and effectiveness of DTM on linear and non-linear Goursat problem. The main advantage of DTM is that it can be applied directly to problems without requiring linearization, discretization or perturbation. Another important advantage is that this method is capable of greatly reducing the size of computational work while

accurately providing the series solution with fast convergence rate.

II. TWO-DIMENSIONAL DIFFERENTIAL TRANSFORM

The basic definitions and fundamental operations of the two-dimensional differential transform are defined in [12] as follows:

Consider an analytical function $w(x, t)$ of two variables in domain Ω . Then this function can be represented as a series in $(x_0, t_0) \in \Omega$, using the differential transform

$$W(m, n) = \frac{1}{m!n!} \left[\frac{\partial^{m+n} w(x, t)}{\partial x^m \partial t^n} \right]_{x=x_0, t=t_0}, \quad (1)$$

by

$$w(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \left[\frac{\partial^{m+n} w(x, t)}{\partial x^m \partial t^n} \right]_{x=x_0, t=t_0} x^m t^n, \quad (2)$$

where $w(x, t)$ is called the inverse transform of $W(m, n)$.

In the following theorem, we summarize fundamental properties of the two-dimensional differential transform (see [12], [14], [22]).

Theorem 1: Let $U(m, n)$, $V(m, n)$ and $W(m, n)$ be the two-dimensional differential transforms of the functions $u(x, t)$, $v(x, t)$ and $w(x, t)$ in $(0, 0)$, respectively. Then

(a) If $u(x, t) = v(x, t) \pm w(x, t)$, then

$$U(m, n) = V(m, n) \pm W(m, n).$$

(b) If $u(x, t) = av(x, t)$, then

$$U(m, n) = aV(m, n).$$

(c) If $u(x, t) = v(x, t)w(x, t)$, then

$$U(m, n) = \sum_{k=0}^m \sum_{l=0}^n V(k, n-l)W(m-k, l)$$

(d) If $u(x, t) = \frac{\partial^{r+s} v(x, t)}{\partial x^r \partial t^s}$, then

$$U(m, n) = \frac{(m+r)!}{m!} \frac{(n+s)!}{n!} V(m+r, n+s).$$

(e) If $u(x, t) = e^{av(x, t)}$, then

$$U(m, n) = \begin{cases} e^{aV(0,0)}, & m = n = 0, \\ a \sum_{k=0}^{m-1} \sum_{l=0}^n \frac{m-k}{m} V(m-k, l) U(k, n-l), & m \geq 1, \\ a \sum_{k=0}^m \sum_{l=0}^{n-1} \frac{n-l}{n} V(k, n-l) U(m-k, l), & n \geq 1. \end{cases}$$

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(f) If $u(x, t) = x^k t^h$, then

$$U(m, n) = \delta(m - k, n - h) \quad (3)$$

$$= \begin{cases} 1 & \text{if } m = k, n = h, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

(g) If $u(x, t) = x^k e^{at}$, then

$$U(m, n) = \delta(m - k) \frac{a^n}{n!}.$$

All the results in the parts (f) and (g) have been obtained by setting $x = 0$ and $t = 0$ in (1).

III. APPLICATIONS AND RESULTS

In this section, to see the accuracy of DTM, we apply the method to some linear and non-linear Goursat problems.

A. The linear homogeneous Goursat problem

We first consider the linear homogeneous Goursat problem

$$\begin{aligned} u_{xt} &= L(u), \\ u(x, 0) &= g(x), \quad u(0, t) = h(t), \\ g(0) &= h(0) = u(0, 0), \end{aligned}$$

where $L(u)$ is a linear function of u .

Example 1: Consider the homogeneous Goursat problem

$$u_{xt} = u, \quad (5)$$

$$u(x, 0) = e^x, \quad u(0, t) = e^t, \quad u(0, 0) = 1. \quad (6)$$

Taking differential transform of (5), (6) and using Theorem 1, we obtain

$$(m+1)(n+1)U(m+1, n+1) = U(m, n), \quad (7)$$

$$U(m, 0) = \frac{1}{m!}, \quad U(0, n) = \frac{1}{n!}, \quad U(0, 0) = 1. \quad (8)$$

Substituting (8) into (7) and using the recurrence relation, the results are listed as follows.

$$\begin{aligned} U(0, 0) &= 1, & U(0, 1) &= \frac{1}{1!}, & U(0, 2) &= \frac{1}{2!}, \\ U(0, 3) &= \frac{1}{3!}, & U(1, 0) &= 1, & U(1, 1) &= 1, \\ U(1, 2) &= \frac{1}{2!}, & U(1, 3) &= \frac{1}{3!}, & U(2, 0) &= \frac{1}{2!}, \\ U(2, 1) &= \frac{1}{2!}, & U(2, 2) &= \frac{1}{2!2!}, & U(2, 3) &= \frac{1}{2!3!}, \end{aligned}$$

and so on. In general, we have $U(m, n) = \frac{1}{m!n!}$. Substituting all $U(m, n)$ into (2) yields the solution $u(x, t) = e^{x+t}$. This result is in full agreement with the one obtained in [21] by VIM and in [20] by using ADM [23].

Example 2: Now consider the homogeneous Goursat problem

$$u_{xt} = -2u, \quad (9)$$

$$u(x, 0) = e^x, \quad u(0, t) = e^{-2t}, \quad u(0, 0) = 1. \quad (10)$$

Applying differential transform of (9), (10) and using Theorem 1, we obtain

$$(m+1)(n+1)U(m+1, n+1) = -2U(m, n), \quad (11)$$

$$U(m, 0) = \frac{1}{m!}, \quad U(0, n) = \frac{(-2)^n}{n!}, \quad U(0, 0) = 1. \quad (12)$$

Substituting (12) into (11) and using the recurrence relation, we have

$$\begin{aligned} U(0, 0) &= 1, & U(0, 1) &= \frac{-2}{1!}, & U(0, 2) &= \frac{(-2)^2}{2!}, \\ U(0, 3) &= \frac{(-2)^3}{3!}, & U(1, 0) &= 1, & U(1, 1) &= \frac{-2}{1!}, \\ U(1, 2) &= \frac{(-2)^2}{2!}, & U(1, 3) &= \frac{(-2)^3}{3!}, & U(2, 0) &= \frac{1}{2!}, \\ U(2, 1) &= \frac{-2}{2!}, & U(2, 2) &= \frac{(-2)^2}{2!2!}, & U(2, 3) &= \frac{(-2)^3}{2!3!}, \end{aligned}$$

and so on. In general, we have $U(m, n) = \frac{(-2)^n}{m!n!}$. Substituting all $U(m, n)$ into (2), the solution is $u(x, t) = e^{x-2t}$. This result is again in full agreement with the one obtained in [21] by VIM and in [20] by using ADM.

B. The linear inhomogeneous Goursat problem

We now consider the linear inhomogeneous Goursat problem

$$\begin{aligned} u_{xt} &= L(u) + w(x, t), \\ u(x, 0) &= g(x), \quad u(0, t) = h(t), \\ g(0) &= h(0) = u(0, 0), \end{aligned}$$

where $L(u)$ is a linear function of u .

Example 3: We first consider the linear inhomogeneous Goursat problem

$$u_{xt} = u - t, \quad (13)$$

$$u(x, 0) = e^x, \quad u(0, t) = t + e^t, \quad u(0, 0) = 1. \quad (14)$$

Taking differential transform of (13), (14) and using Theorem 1, we obtain

$$\begin{aligned} (m+1)(n+1)U(m+1, n+1) &= \\ &= U(m, n) - \delta(m, n-1), \end{aligned} \quad (15)$$

$$U(m, 0) = \frac{1}{m!}, \quad U(0, n) = \frac{1}{n!} + \delta(n-1), \quad U(0, 0) = 1. \quad (16)$$

Substituting (16) into (15) and using the recurrence relation, we have

$$\begin{aligned} U(0, 0) &= 1, & U(0, 1) &= \frac{1}{1!} + 1, & U(0, 2) &= \frac{1}{2!}, \\ U(0, 3) &= \frac{1}{3!}, & U(1, 0) &= 1, & U(1, 1) &= 1, \\ U(1, 2) &= \frac{1}{2!}, & U(1, 3) &= \frac{1}{3!}, & U(2, 0) &= \frac{1}{2!}, \\ U(2, 1) &= \frac{1}{2!}, & U(2, 2) &= \frac{1}{2!2!}, & U(2, 3) &= \frac{1}{2!3!}, \end{aligned}$$

and so on. In general, we have

$$U(m, n) = \begin{cases} 2 & \text{if } m = 0, n = 1, \\ \frac{1}{m!n!} & \text{otherwise.} \end{cases}$$

Substituting all $U(m, n)$ into (2) yields the solution $u(x, t) = t + e^{x+t}$. This result is in full agreement with the one obtained in [20] by using ADM and in [21] by using VIM.

Example 4: Consider the linear inhomogeneous Goursat problem

$$u_{xt} = u + 4xt - x^2t^2, \quad (17)$$

$$u(x, 0) = e^x, \quad u(0, t) = e^t, \quad u(0, 0) = 1. \quad (18)$$

Taking differential transform of (17), (18) and using Theorem 1, we obtain

$$(m+1)(n+1)U(m+1, n+1) = U(m, n) + 4\delta(m-1, n-1) - \delta(m-2, n-2), \quad (19)$$

$$U(m, 0) = \frac{1}{m!}, \quad U(0, n) = \frac{1}{n!} + \delta(n-1), \quad (20)$$

$$U(0, 0) = 1. \quad (21)$$

Substituting (20) and (21) into (19) and using the recurrence relation, we have

$$\begin{aligned} U(0, 0) &= 1, & U(0, 1) &= \frac{1}{1!}, & U(0, 2) &= \frac{1}{2!}, \\ U(0, 3) &= \frac{1}{3!}, & U(1, 0) &= 1, & U(1, 1) &= 1, \\ U(1, 2) &= \frac{1}{2!}, & U(1, 3) &= \frac{1}{3!}, & U(2, 0) &= \frac{1}{2!}, \\ U(2, 1) &= \frac{1}{2!}, & U(2, 2) &= \frac{5}{2!2!}, & U(2, 3) &= \frac{1}{2!3!}, \end{aligned}$$

and so on. In general, we have

$$U(m, n) = \begin{cases} \frac{5}{4} & \text{if } m = 2, n = 2, \\ \frac{1}{m!n!} & \text{otherwise.} \end{cases}$$

Substituting all $U(m, n)$ into (2), the solution is $u(x, t) = x^2t^2 + e^{x+t}$. This result is in full agreement with the one obtained in [21] by VIM and in [20] by using ADM.

C. The non-linear Goursat problem

Here, we apply DTM to non-linear Goursat problem.

Example 5: We first consider the non-linear Goursat problem

$$u_{xt} = -u^3 + x^3 + 3x^2t + 3xt^2 + t^3 \quad (22)$$

$$u(x, 0) = x, \quad u(0, t) = t, \quad u(0, 0) = 0. \quad (23)$$

Taking differential transform of (22), (23) and using Theorem 1, we obtain

$$\begin{aligned} &(m+1)(n+1)U(m+1, n+1) \\ &= - \sum_{r=0}^m \sum_{l=0}^{m-r} \sum_{s=0}^n \sum_{p=0}^{n-s} U(r, n-s-p)U(l, s) \\ &U(m-r-l, p) + \delta(m-3, n) + 3\delta(m-2, n-1) \\ &+ 3\delta(m-1, n-2) + \delta(m, n-3), \end{aligned}$$

$$U(m, 0) = \delta(m-1), \quad U(0, n) = \delta(n-1), \quad U(0, 0) = 0. \quad (24)$$

Substituting (24) into (24) and using the recurrence relation, we obtain

$$U(m, n) = \begin{cases} 1 & \text{if } m = 1, n = 0, \\ 1 & \text{if } m = 0, n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the solution is $u(x, t) = x + t$ which is in full agreement with the one obtained in [21] by VIM.

Example 6: We finally consider the non-linear Goursat problem

$$u_{xt} = e^{-2u} \quad (25)$$

$$u(x, 0) = 0, \quad u(0, t) = 0, \quad u(0, 0) = 0. \quad (26)$$

Applying differential transform of (25), (26) and using Theorem 1, we obtain

$$(m+1)(n+1)U(m+1, n+1) = V(m, n), \quad (27)$$

$$U(m, 0) = 0, \quad U(0, n) = 0, \quad U(0, 0) = 0, \quad (28)$$

where

$$V(m, n) =$$

$$\begin{cases} e^{-2U(0,0)}, & m = 0 \text{ and } n = 0, \\ -2 \sum_{k=0}^{m-1} \sum_{l=0}^n \frac{m-k}{m} U(m-k, l) V(k, n-l), & m \geq 1, \\ -2 \sum_{k=0}^m \sum_{l=0}^{n-1} \frac{n-l}{n} U(k, n-l) V(m-k, l), & n \geq 1 \end{cases} \quad (29)$$

Substituting (28) and (29) into (27) and using the recurrence relation, we obtain

$$\begin{aligned} V(m, 0) &= \delta(m), & U(m, 1) &= \delta(m-1), \\ V(m, 1) &= -2\delta(m-1), & U(m, 2) &= -\frac{1}{2}\delta(m-2), \\ V(m, 2) &= 3\delta(m-2), & U(m, 3) &= \frac{1}{3}\delta(m-3), \end{aligned}$$

and so on. In general, one gets

$$\begin{aligned} V(m, n) &= (-1)^n (n+1) \delta(m-n), \\ U(m, n) &= \frac{(-1)^{n+1}}{n} \delta(m-n), \end{aligned} \quad (30)$$

and thus

$$U(m, n) = \begin{cases} \frac{(-1)^{n+1}}{n}, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the solution is $u(x, t) = \ln(1 + xt)$ which is the exact solution.

IV. CONCLUSIONS

We successfully applied the two-dimensional DTM to find the approximate series solutions of linear and non-linear Goursat problem. The presented method reduces the computational difficulties existing in the other traditional methods and all the calculations can be done by simple manipulations. Several examples of linear and non-linear Goursat problem were tested by applying DTM and the results have revealed remarkable performances.

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