

# Two Different Computing Methods of the Smith Arithmetic Determinant

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**Abstract**—The Smith arithmetic determinant is investigated in this paper. By using two different methods, we derive the explicit formula for the Smith arithmetic determinant.

**Keywords**—Elementary row transformation, Euler function, Matrix decomposition, Smith arithmetic determinant.

## I. INTRODUCTION

**S**UPPOSE  $A$  is an  $n \times n$  matrix,  $A = [a_{i,j}] = [(i,j)]$ , ( $i, j = 1, 2, \dots, n$ ), and  $(i, j)$  denotes the greatest common divisor of integer  $i$  and  $j$ . We call the determinant of this kind of matrix the *Smith arithmetic determinant* ([3],[4],[5],[6]), and we use the symbol  $S_n$  to denote the value of the Smith arithmetic determinant of order  $n$ .

A proof has been given in reference[3]. In this paper, we present another two different methods to derive the explicit formula of the Smith arithmetic determinant.

Firstly, we introduce some definitions and several basic results.

**Definition 1** ([1]). Suppose  $a, b$  are two integers that not all of them equal to zero. If a nonzero integer  $d$  divides both  $a$  and  $b$ , then we call  $d$  a common divisor of  $a$  and  $b$ ; Moreover, if any other common divisors of  $a$  and  $b$  is the divisor of  $d$ , we call  $d$  a greatest common divisor of  $a$  and  $b$ . Using symbol  $(a, b)$  to denote the positive greatest common divisor of  $a$  and  $b$ .

Obviously we have:

1)  $(1, i) = 1$ ;

2) Suppose  $p$  is a prime. If  $n$  is not a multiple of  $p$ , then  $(p, n) = 1$ ; else  $(p, n) = p$ .

**Theorem 1** ([1, Euclid Algorithm]). Suppose  $a$  and  $b \neq 0$  are two integers, and

$$a = q_1 b + r_1, 0 < r_1 < b,$$

$$b = q_2 r_1 + r_2, 0 < r_2 < r_1,$$

$$r_1 = q_2 r_2 + r_3, 0 < r_3 < r_2,$$

⋮

$$r_{n-1} = q_n r_n + r_{n+1}, 0 < r_{n+1} < r_n,$$

$$r_n = q_{n+1} r_{n+1}.$$

Then  $(a, b) = r_{n+1}$ .

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**Theorem 2** ([1]). Suppose integer  $n \geq 2$ , then there exist such primes that

$$0 < p_1 \leq p_2 \leq \dots \leq p_r, n = p_1 p_2 \dots p_r.$$

Furthermore, if there are other primes

$$0 < q_1 \leq q_2 \leq \dots \leq q_s, n = q_1 q_2 \dots q_s,$$

then we have  $r = s$  and  $p_i = q_i$  for  $1 \leq i \leq s$ .

**Definition 2** ([2]). Suppose  $n$  is a positive integer, and let  $\phi(n)$  denote the number of positive integers which are prime to  $n$  and less than  $n$ . Then we call  $\phi(n)$  the Euler function of  $n$ .

Apparently if  $p$  is a prime, then  $\phi(p) = p - 1$ .

In the following, we will give a classical result about Euler function, which will be used in the proof.

**Theorem 3** ([2]). Suppose  $m$  is a positive integer, then

$$\sum_{d|m} \phi(d) = m.$$

## II. ELEMENTARY ROW TRANSFORMATION ON THE SMITH ARITHMETIC DETERMINANT

Firstly we give two examples.

*Example 1.* Consider the computation of the Smith arithmetic determinant of order 4. By the definition,

$$S_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 4 \end{vmatrix}.$$

This determinant can be computed as follows: Multiply the first row  $R_1$  by  $-1$  and add it to every other row. Then multiply the second row  $R_2$  by  $-1$  and add it to the fourth row  $R_{2 \times 2} = R_4$ . As a result, we have a determinant of an upper triangular matrix, of which the dominating diagonal elements are  $\phi(1), \phi(2), \phi(3), \phi(4)$  in order. Therefore, the value of this Smith arithmetic determinant is  $\prod_{i=1}^4 \phi(i)$ .

The process above can be written as follows:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} \phi(1) & \phi(1) & \phi(1) & \phi(1) \\ 0 & \phi(2) & 0 & \phi(2) \\ 0 & 0 & \phi(3) & 0 \\ 0 & 0 & 0 & \phi(4) \end{vmatrix} = \prod_{i=1}^4 \phi(i).$$

*Example 2.* Consider the computation of the Smith arithmetic determinant of order 6. According to the definition,

$$S_6 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 2 & 3 & 2 & 1 & 6 \end{vmatrix}.$$

Inspired by the method above, we can compute it with the following method: multiply the first row  $R_1$  by  $-1$  and add it to every other row. Then multiply the second row  $R_2$  by  $-1$  and add it to the fourth row  $R_{2 \times 2}$ , the sixth row  $R_{2 \times 3}$ . Finally, we multiply the third row  $R_3$  by  $-1$  and add it to the sixth row  $R_{3 \times 2}$ . As a result, we have a determinant of an upper triangular matrix, the value of which is  $\prod_{i=1}^6 \phi(i)$ .

The process above can be written as follows:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 2 & 3 & 2 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 2 & 1 & 0 & 5 \end{vmatrix} \\ = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{vmatrix},$$

of which the diagonal elements are  $\phi(1), \phi(2), \phi(3), \phi(4), \phi(5), \phi(6)$  in order. Thus, we have  $S_6 = \prod_{i=1}^6 \phi(i)$ .

A general method of the computation of the Smith arithmetic determinant can be concluded from the two examples above: Multiply  $R_1$  by  $-1$  and add it to every other row, then multiply  $R_2$  by  $-1$  and add it to  $R_{2 \times i} (i \geq 2)$ , then multiply  $R_3$  by  $-1$  and add it to  $R_{3 \times i} (i \geq 2)$ , ..., until a determinant of an upper triangular matrix is yielded.

As the examples illustrated, we give out a determinant of an upper triangular matrix in which the elements on the diagonal are  $\phi(1), \phi(2), \dots, \phi(n)$  in order. Therefore, we have

$$S_n = \prod_{i=1}^n \phi(i).$$

Before giving a complete proof, we firstly introduce several lemmas.

**Lemma 1.** 1) *The sequence composed by the elements of each row of the Smith arithmetic determinant is periodic, and the minimal positive period is the row index number;*

2) *After the elementary row transformation used above, each row is still periodic, and the minimal positive period remains the same.*

*Proof.* 1) It is equivalent to prove  $(t, i) = (t, kt + i)$ . According to the Euclid algorithm, it is obviously true. Moreover, there is a one to one correspondence between the

sequence  $(t, i + 1), (t, i + 2), \dots, (t, i + t)$  and the sequence  $(t, 1), (t, 2), \dots, (t, t)$ , so the minimal positive period is  $t$ .

2) For each row  $R_a$ , according to the algorithm above, only when  $d|a$  that the row  $R_d$  will be timed by  $-1$  and added to  $R_a$ . The period of  $R_d$  and  $R_a$  are respectively  $d$  and  $a$ , and  $d|a$ , so  $a$  is also a period of  $R_d$ . It means that  $R_a$  and  $R_d$  have the same period  $a$ .

Therefore, when  $R_d$  is timed by  $-1$  and added to  $R_a$ , the period of  $R_a$  does not change.

If the index number of a row is a prime, then this row will be called *prime row*. Otherwise this row will be called *composite row*.

It is obvious that each element of the first row as well as the first column equals to 1 and each element of the row  $R_p$  (suppose  $p$  is a prime) equals to 1 or  $p$  (which occurs only when  $p$  divides its corresponding column index number).

Now we investigate the expression of the Smith arithmetic determinant of order  $n$ :

$$S_n = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & \dots & 1 & \dots & * \\ 1 & 1 & 3 & 1 & 1 & 3 & \dots & 1 & \dots & * \\ 1 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & \dots & 1 & \dots & * \\ 1 & 1 & 1 & 1 & 5 & 1 & \dots & 1 & \dots & * \\ 1 & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & \dots & 1 & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots & p & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & a_{n2} & a_{n3} & a_{n4} & a_{n5} & a_{n6} & \dots & a_{np} & \dots & n \end{vmatrix},$$

where  $a_{ij} = (i, j)$ .

Because of the periodicity in each row  $R_i$ , we only need to consider the elements from  $a_{i,1}$  to  $a_{i,i}$ .

*Elementary row transformation on the Smith arithmetic determinant of order  $n$ :*

*Step 1:* Multiply  $R_1$  by  $-1$  and add it to all the prime rows, and the result is:

$$S_n = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & \dots & 0 & \dots & * \\ 0 & 0 & 2 & 0 & 0 & 2 & \dots & 0 & \dots & * \\ 1 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & \dots & 1 & \dots & * \\ 0 & 0 & 0 & 0 & 4 & 0 & \dots & 0 & \dots & * \\ 1 & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & \dots & 1 & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & p-1 & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & a_{n2} & a_{n3} & a_{n4} & a_{n5} & a_{n6} & \dots & a_{np} & \dots & n \end{vmatrix}.$$

Because we have  $\phi(1) = 1, \phi(p) = p - 1$  (suppose  $p$  is a prime), we replace  $p - 1$  by  $\phi(p)$  in  $R_p$ , and the determinant

above can be written into:

$$\begin{pmatrix} \phi(1) & \phi(1) & \phi(1) & \phi(1) & \phi(1) & \phi(1) & \dots & 1 & \dots & * \\ 0 & \phi(2) & 0 & \phi(2) & 0 & \phi(2) & \dots & 0 & \dots & * \\ 0 & 0 & \phi(3) & 0 & 0 & \phi(3) & \dots & 0 & \dots & * \\ 1 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & \dots & 1 & \dots & * \\ 0 & 0 & 0 & 0 & \phi(5) & 0 & \dots & 0 & \dots & * \\ 1 & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & \dots & 1 & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \phi(p) & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & a_{n2} & a_{n3} & a_{n4} & a_{n5} & a_{n6} & \dots & a_{np} & \dots & n \end{pmatrix},$$

where every prime row has the form  $(0, 0, \dots, 0, \phi(p), 0, \dots)$ , or

$$a_{pt} = \begin{cases} \phi(p), & \text{if } p \text{ divides } t; \\ 0, & \text{otherwise.} \end{cases}$$

Step 2: We compute this determinant with the following method: if a row  $R_b$  is determined and the rows above  $R_b$  have the form

$$a_{it} = \begin{cases} \phi(i), & \text{if } i \text{ divides } t; \\ 0, & \text{otherwise.} \end{cases} \quad (i = 1, 2, \dots, p-1),$$

we multiply  $R_d$  by  $-1$  and add it to  $R_b$  where  $d|b$  and  $d < b$ , and we have a new row  $R'_b$ . It is obvious to know that  $R_4$  is the first row to be transformed, and all the rows to be transformed are composite rows. We assert that:

**Property 1.** After the transformation,  $R'_b$  has the form

$$a'_{bt} = \begin{cases} \phi(b), & \text{if } b \text{ divides } t; \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Because of the periodicity in  $R_b$ , and  $a_{bb} = b$ , we only need to consider the result after the transformation on  $a_{bi}$  ( $i = 1, 2, \dots, b$ ). For each  $a_{bi} = (b, i)$ , according to the method,

$$a'_{bi} = a_{bi} - \sum_{d|b, d < b} a_{di},$$

where  $a'_{bi}$  is the result after the transformation on  $a_{bi}$ . We prove it in four different cases:

1) If  $b$  and  $i$  are prime to each other, i.e.  $(b, i) = 1$ . Because  $d|b$  and  $(b, i) = 1$ , we have  $(d, i) = 1$ . If  $d \neq 1$ ,  $d$  will not divide  $i$ , so  $a_{di} = 0$ ; else if  $d = 1$  then  $a_{di} = 1$ . Therefore,

$$a'_{bi} = a_{bi} - 1 = (b, i) - 1 = 0.$$

2) If  $b$  and  $i$  are not prime to each other and  $b > i$ . Suppose  $(b, i) = r$ , then  $r|b$  and  $r|i$ , and for each common divisor  $e$  of  $b$  and  $i$ , we have  $e|r$ . Because we have

$$a'_{bi} = a_{bi} - \sum_{d|b, d < b} a_{di}, \text{ and } a_{di} = \phi(d) \text{ or } 0,$$

which depends on whether or not  $d$  divides  $i$ , so only when  $d$  is a common divisor of  $b$  and  $i$ , i.e.  $d|r$ ,  $a_{di} = \phi(d)$ , otherwise  $a_{di} = 0$ . Then we have

$$\sum_{d|b, d < b} a_{di} = \sum_{i|r} \phi(i) = r.$$

Therefore,

$$a'_{bi} = a_{bi} - \sum_{d|b, d < b} a_{di} = (b, i) - r = 0.$$

3) If  $b$  and  $i$  are not prime to each other and  $b = i$ , then

$$a'_{bb} = a_{bb} - \sum_{d|b, d < b} a_{db} = b - \sum_{d|b, d < b} \phi(d).$$

Because

$$b = \sum_{d|b} \phi(d) = \sum_{d|b, d < b} \phi(d) + \phi(b),$$

then

$$a'_{bb} = b - \sum_{d|b, d < b} \phi(d) = \phi(b).$$

4) If  $b$  and  $i$  are not prime to each other and  $b < i$ .

According to the periodicity in each row, the elements behind the diagonal element are the same as the elements before it.

In summary, after the transformation on  $R_b$ , the result  $R'_b$  has the form

$$a'_{bi} = \begin{cases} \phi(b), & \text{if } b \text{ divides } i, \\ 0, & \text{otherwise,} \end{cases}$$

which means that  $R'_b$  has the same form as the rows above, so the conclusion is true.

Step 3: Based on the method, the original determinant can be transformed into

$$\begin{pmatrix} \phi(1) & \phi(1) & \phi(1) & \phi(1) & \phi(1) & \phi(1) & \dots & 1 & \dots & * \\ 0 & \phi(2) & 0 & \phi(2) & 0 & \phi(2) & \dots & 0 & \dots & * \\ 0 & 0 & \phi(3) & 0 & 0 & \phi(3) & \dots & 0 & \dots & * \\ 0 & 0 & 0 & \phi(4) & 0 & 0 & \dots & 0 & \dots & * \\ 0 & 0 & 0 & 0 & \phi(5) & 0 & \dots & 0 & \dots & * \\ 0 & 0 & 0 & 0 & 0 & \phi(6) & \dots & 0 & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \phi(p) & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \dots & \phi(n) \end{pmatrix}.$$

The value of this determinant is  $\prod_{i=1}^n \phi(i)$ . Therefore, the value

of order  $n$  Smith arithmetic determinant is  $\prod_{i=1}^n \phi(i)$ , i.e.  $S_n =$

$$\prod_{i=1}^n \phi(i).$$

### III. COMPUTING THE SMITH ARITHMETIC DETERMINANT BY MATRIX DECOMPOSITION

Firstly we consider two examples of smaller order Smith arithmetic determinant.

*Example 3.* Consider the computation of the Smith arithmetic determinant of order 4. According to the definition, we have

$$S_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 4 \end{vmatrix}.$$

The matrix corresponding to this determinant is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 4 \end{pmatrix}.$$

We find that this matrix can be regarded as the multiplicative product of two matrices

$$\begin{pmatrix} \phi(1) & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & 0 \\ \phi(1) & 0 & \phi(3) & 0 \\ \phi(1) & \phi(2) & 0 & \phi(4) \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$S_4 = \begin{vmatrix} \phi(1) & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & 0 \\ \phi(1) & 0 & \phi(3) & 0 \\ \phi(1) & \phi(2) & 0 & \phi(4) \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ = \prod_{i=1}^4 \phi(i)$$

*Example 4.* Consider the computation of the Smith arithmetic determinant of order 6. The matrix corresponding to the 6 order Smith arithmetic determinant is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 2 & 3 & 2 & 1 & 6 \end{pmatrix}.$$

This matrix can be decomposed into the multiplicative product of

$$\begin{pmatrix} \phi(1) & 0 & 0 & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & 0 & 0 & 0 \\ \phi(1) & 0 & \phi(3) & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & \phi(4) & 0 & 0 \\ \phi(1) & 0 & 0 & 0 & \phi(5) & 0 \\ \phi(1) & \phi(2) & \phi(3) & 0 & 0 & \phi(6) \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$S_6 = \begin{vmatrix} \phi(1) & 0 & 0 & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & 0 & 0 & 0 \\ \phi(1) & 0 & \phi(3) & 0 & 0 & 0 \\ \phi(1) & \phi(2) & 0 & \phi(4) & 0 & 0 \\ \phi(1) & 0 & 0 & 0 & \phi(5) & 0 \\ \phi(1) & \phi(2) & \phi(3) & 0 & 0 & \phi(6) \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = \prod_{i=1}^6 \phi(i).$$

The examples illustrated above reveal that a Smith arithmetic determinant can be decomposed into two determinants that are simple enough to be computed.

Now we present the decomposition of a general Smith arithmetic determinant.

We define two new matrices  $B$  and  $C$  of order  $n$  in the following:

$$B = \begin{pmatrix} \phi(1) & 0 & 0 & 0 & 0 & 0 & \dots \\ \phi(1) & \phi(2) & 0 & 0 & 0 & 0 & \dots \\ \phi(1) & 0 & \phi(3) & 0 & 0 & 0 & \dots \\ \phi(1) & \phi(2) & 0 & \phi(4) & 0 & 0 & \dots \\ \phi(1) & 0 & 0 & 0 & \phi(5) & 0 & \dots \\ \phi(1) & \phi(2) & \phi(3) & 0 & 0 & \phi(6) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each element  $b_{ik}$  of row  $R_i$  satisfies

$$b_{ik} = \begin{cases} \phi(k), & \text{if } k \text{ divides } i; \\ 0, & \text{otherwise.} \end{cases}$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each element  $c_{kj}$  of column  $C_j$  satisfies

$$c_{kj} = \begin{cases} 1, & \text{if } k \text{ divides } j; \\ 0, & \text{otherwise.} \end{cases}$$

**Property 2.** Suppose the matrix corresponding to the Smith arithmetic determinant of  $n$  order is  $A$ , then  $A = B \cdot C$ .

*Proof.* Suppose each element of  $A$  is  $a_{ij}$ , and each element of  $B \cdot C$  is  $a'_{ij}$ . What is needed to be proved is  $a_{ij} = a'_{ij}$ .

Because  $a_{ij} = \sum_{k=1}^n b_{ik}c_{kj}$  and according to the definition of  $b_{ik}$  and  $c_{kj}$ , we can deduce that

$$\sum_{k=1}^n b_{ik}c_{kj} = \sum_{k|(i,j)} \phi(k) = (i, j) = a_{ij},$$

which means that  $a_{ij} = a'_{ij}$ , i.e.  $A = B \cdot C$ .

*Note:* For each element of the expression  $\sum_{k=1}^n b_{ik}c_{kj}$ , only when  $b_{ik} \neq 0$  and  $c_{kj} \neq 0$ , this element does not equal to 0, and when  $b_{ik} \neq 0$  and  $c_{kj} \neq 0$ , we have  $k|i$  and  $k|j$ , so  $k$  is a common divisor of  $i$  and  $j$ . Moreover, the value of  $k$  ranges from 1 to  $n$ , so the value of expression  $b_{ik}c_{kj}$  also ranges over every positive divisors of  $(i, j)$ . Therefore,

$$\sum_{k=1}^n b_{ik}c_{kj} = \sum_{k|(i,j)} \phi(k) = (i, j) = a_{ij}.$$

As a consequence of Property 2, we have

$$S_n = \det(B) \cdot \det(C) = \prod_{i=1}^n \phi(i).$$

So

$$S_n = \begin{vmatrix} \phi(1) & 0 & 0 & 0 & 0 & 0 & \dots \\ \phi(1) & \phi(2) & 0 & 0 & 0 & 0 & \dots \\ \phi(1) & 0 & \phi(3) & 0 & 0 & 0 & \dots \\ \phi(1) & \phi(2) & 0 & \phi(4) & 0 & 0 & \dots \\ \phi(1) & 0 & 0 & 0 & \phi(5) & 0 & \dots \\ \phi(1) & \phi(2) & \phi(3) & 0 & 0 & \phi(6) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = \prod_{i=1}^n \phi(i).$$

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