International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:8, No:2, 2014

Treatment of Spin-1/2 Particle in Interaction with a Time-Dependent Magnetic Field by the Fermionic Coherent-State Path-Integral Formalism

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Abstract—We consider a spin-1/2 particle interacting with a time-dependent magnetic field using path integral formalism. The propagator is first of all written in the standard form replacing the spin by two fermionic oscillators via the Schwinger model. The propagator is then exactly determined, thanks to a simple transformation, and the transition probability is deduced.

Keywords—Path integral, formalism, Propagator.

I. Introduction

THE applications of the path-integral formalism have widely increased since a large class of potentials had been resolved [1]. However it is known that the most relativistic interactions are those where the spin, which is a very useful and very important notion in physics, is taken into account. In the framework of non-relativistic theory the phenomena of spin is automatically introduced by the Pauli equation which contains the Schrödinger Hamiltonian and a spin-field interaction. This motivates the research into the solvable Pauli equations which are inevitably useful in applied physics. For instance a well-known example of its direct application is the time-dependent field acting on an atom with two levels whose time-evolution is controlled by the Pauli-type equation. The solution for this equation has made clear the associated transition amplitudes [2]. This and similar [3], [4] types of interaction aside, there are little analytical and exact computations which treat the time-dependent spin-field interaction. Furthermore, if one replaces the time dependence of the exterior field by a space-time dependence or by only space dependence this becomes even more restrictive [5]–[10].

Moreover, the problem becomes nearly unsolvable if we try to build these solutions by the path integral formalism because the spin is a discrete quantity. The difficulty here is associated to the fact that the path integral lacks some classical ideas such as trajectories and up to now one does not know how to deal with this kind of technique in this important case. Thus some effort has been made to find a partial solution using the Schwinger model of spin and some explicit computations are then carried out [11]–[20].

In this paper we are devoted to this type of interaction by considering a problem which is treated by usual quantum mechanics [4], namely a spin-$1/2$ particle which interacts with a time-dependent magnetic field:

$$\mathbf{B}(t)=\begin{pmatrix} \frac{B}{\sqrt{1+\lambda^2t^2}}\sin\frac{\omega t}{1+\omega^2t^2} \\ -\frac{2B}{1+\lambda^2t^2}-B_0\frac{1-\omega^2t^2}{(1+\omega^2t^2)^2} \\ \frac{B}{\sqrt{1+\lambda^2t^2}}\cos\frac{\omega t}{1+\omega^2t^2}\end{pmatrix}. \tag{1}$$

Its dynamics is described by the Hamiltonian:

$$H=-\frac{g}{2}\sigma\mathbf{B}, \tag{2}$$

where g is the gyromagnetic ratio. Then the Hamiltonian becomes:

$$\begin{aligned} H \;=\; & -\frac{\lambda}{\sqrt{1+\lambda^2t^2}}\sin\frac{\omega t}{1+\omega^2t^2}\sigma_x+ \\ & +\left(\tfrac{\lambda}{1+\lambda^2t_n^2}+\frac{\omega}{2}\tfrac{1-\omega^2t_n^2}{(1+\omega^2t_n^2)^2}\right)\sigma_y \\ & -\frac{\lambda}{\sqrt{1+\lambda^2t^2}}\cos\frac{\omega t}{1+\omega^2t^2}\sigma_z, \end{aligned} \tag{3}$$

where $B=\lambda/g$ and $B_0=\omega/g$. The Pauli matrices are given by:

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}\ \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{4}$$

Considering this problem by the path integral formalism, our approach is the following. For interaction with the coupling of spin-field type, the propagator is first, by construction, written in the standard form $\sum_{\text{path}}\exp\left(i\mathcal{S}\left(\text{path}\right)/\hbar\right)$, where $\mathcal{S}$ is the action that describes the system. The discrete variable accounting for spin is inserted as a (continuous) path using fermionic coherent states. The knowledge of the propagator is essential in the determination of physical quantities such as the transition probability which is the aim of this paper.

This paper is organized as follows. In Section II, we introduce our notation and the necessary spin-coherent–state path-integral for a spin-$\frac{1}{2}$ system for our further computations. In Section III, after setting up a path integral formalism for the propagator, we perform the direct calculations. The integration over the spin variables is easily carried out thanks to simple transformations. The explicit result of the propagator is directly computed and the transition probability is then deduced. Finally, in Section IV, we present our conclusions.

II. Coherent-states formalism

First of all we introduce some definitions, properties and notations needed in this paper. As we are interested in the spin-field interaction we use the approach known as the

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Schwinger fermionic model of spin, whose recipie includes replacing the Pauli matrices σ_i by a pair of fermionic operators (u, d):

$$\sigma \longrightarrow \left(u^{\dagger}, d^{\dagger}\right) \sigma \left(\begin{array}{c} u \\ d \end{array} \right), \tag{5}$$

where the pair (u, d) describes a two-dimensional fermionic oscillators.

Incidentally, the spin eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ are generated from the fermionic vacuum state $|0,0\rangle$ by the action of the fermionic oscillators u^{+} and d^{+} through the following the relations:

$$u^{+} |0,0\rangle = |\uparrow\rangle, \qquad \text{and} \qquad d^{+} |0,0\rangle = |\downarrow\rangle, \tag{6}$$

and the action of u and d on this vacuum state is given by the vanishing results:

$$u |0,0\rangle = 0, \qquad \text{and} \qquad d |0,0\rangle = 0. \tag{7}$$

The pair of fermionic oscillators (u, d) and its adjoint (u^{+}, d^{+}) satisfy the usual fermionic algebra defined by the following anti-commutator relations:

$$\left[u, u^{+}\right]_{+} = 1, \qquad \left[d, d^{+}\right]_{+} = 1, \tag{8}$$

and all other anti-commutators vanish. The notation $[A, B]_{+}$ stands for:

$$[A, B]_{+} = AB + BA.$$

We now introduce the coherent states related to this fermionic-oscillator algebra. These states are generally defined as eigenvectors of the fermionic oscillators u and d:

$$u |\alpha, \beta\rangle = \alpha |\alpha, \beta\rangle, \qquad d |\alpha, \beta\rangle \ = \beta |\alpha, \beta\rangle, \tag{9}$$

where (α, β) is a pair of Grasmann variables which are anti-commuting with the fermionic oscillators and also with themselves, namely:

$$\left\{ \begin{array}{l} [\alpha, u]_{+} = [\alpha, u^{+}]_{+} = [\alpha, d]_{+} = [\alpha, d^{+}]_{+} = 0, \\ [\beta, u]_{+} = [\beta, u^{+}]_{+} = [\beta, d]_{+} = [\beta, d^{+}]_{+} = 0, \end{array} \right. \tag{10}$$

and which are commuting with vacuum states $|0,0\rangle, \langle 0,0|$:

$$\left\{ \begin{array}{ll} \alpha |0,0\rangle = |0,0\rangle \alpha, & \langle 0,0| \alpha = \alpha \langle 0,0| \\ \beta |0,0\rangle = |0,0\rangle \beta, & \langle 0,0| \beta = \beta \langle 0,0|. \end{array} \right. \tag{11}$$

The above definitions are in fact equivalent to the fact that these states are generated from the vacuum state according to the following relation:

$$|\alpha, \beta\rangle = \exp\left(-\alpha u^{+} - \beta d^{+}\right) |0,0\rangle. \tag{12}$$

The states exhibit the following properties:

- Completeness relation:

$$\int d\bar{\alpha} d\alpha d\bar{\beta} d\beta e^{-\bar{\alpha}\alpha - \bar{\beta}\beta} |\alpha, \beta\rangle \ \langle \alpha, \beta| = 1. \tag{13}$$

- Non-orthogonality:

$$\langle \alpha, \beta \mid \alpha', \beta' \rangle = e^{\bar{\alpha}\alpha' + \bar{\beta}\beta'}. \tag{14}$$

III. Path integral formulation

At this stage we shall provide a path-integral expression for the propagator for the Hamiltonian given in the expression (3). This can readily be achieved by exploiting the above model for spin which consequently converts the Hamiltonian into the following fermionic form:

$$\begin{aligned} H \ = \ & -\frac{\lambda}{\sqrt{1+\lambda^2 t^2}} \sin \frac{\omega t}{1+\omega^2 t^2} (u^{\dagger} d + d^{\dagger} u) \\ & + \left(\tfrac{\lambda}{1+\lambda^2 t_n^2} + \frac{\omega}{2} \tfrac{1-\omega^2 t_n^2}{(1+\omega^2 t_n^2)^2}\right) (-i u^{\dagger} d + i d^{\dagger} u) \\ & - \frac{\lambda}{\sqrt{1+\lambda^2 t^2}} \cos \frac{\omega t}{1+\omega^2 t^2} (u^{\dagger} u - d^{\dagger} d). \end{aligned} \tag{15}$$

Moreover, it is convenient to choose the quantum state as $|\alpha, \beta\rangle$, where (α, β) describes the spin variables.

According to the habitual construction procedure of the path integral, we define the propagator as the matrix element of the evolution operator between the initial state $|\alpha_i, \beta_i\rangle$ and the final state $|\alpha_f, \beta_f\rangle$:

$$\mathbf{K}(\alpha_{f,}\beta_f; \alpha_{i,}\beta_i; T) = \langle \alpha_f, \beta_f \mid U(T) \mid \alpha_i, \beta_i \rangle, \tag{16}$$

where

$$U(T) = \mathbf{T}_D \exp\left(-\frac{i}{\hbar} \int_0^T H(t) dt\right), \tag{17}$$

with $\mathbf{T}_D$ the Dyson chronological operator.

In order to move to the path-integral representation, we first subdivide the time interval $[0, T]$ into $N+1$ intermediate moments of length ε. Using the Trotter's formula we then introduce in eq. (16) the projectors according to these N intermediate instants, which are regularly distributed between 0 and T. We obtain the discrete path-integral form of the propagator:

$$\begin{aligned} & \mathbf{K}(\alpha_f, \beta_f, \alpha_i, \beta_i T) = \lim_{N \to \infty} \prod_{n=1}^{N} \int d\bar{\alpha}_n d\alpha_n d\bar{\beta}_n d\beta_n \\ & \times e^{-\bar{\alpha}_n \alpha_n - \bar{\beta}_n \beta_n} \exp \sum_{n=1}^{N+1} \left[\bar{\alpha}_n \alpha_{n-1} + \bar{\beta}_n \beta_{n-1}\right. \\ & i\varepsilon \tfrac{\lambda}{2\sqrt{1+\lambda^2 t_n^2}} \cos \tfrac{\omega t_n}{1+\omega^2 t_n^2} \left(\bar{\alpha}_n \alpha_{n-1} - \bar{\beta}_n \beta_{n-1}\right) + \\ & + i\varepsilon \tfrac{\lambda}{2\sqrt{1+\lambda^2 t_n^2}} \sin \tfrac{\omega t_n}{1+\omega^2 t_n^2} \left(\bar{\alpha}_n \beta_{n-1} + \bar{\beta}_n \alpha_{n-1}\right) \\ & - i\varepsilon \left(\tfrac{\lambda}{1+\lambda^2 t_n^2} + \frac{\omega}{2} \tfrac{1-\omega^2 t_n^2}{(1+\omega^2 t_n^2)^2}\right) \\ & \left. \times \left(-i\bar{\alpha}_n \beta_{n-1} + i\bar{\beta}_n \alpha_{n-1}\right)\right]. \end{aligned} \tag{18}$$

The formal continuous expression for the transition amplitude

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:8, No:2, 2014

(18) is found by taking the limit $N \to \infty$:

$$
\begin{aligned}
&\mathbf{K}(\alpha_f,\beta_f;\alpha_i,\beta_i;T) = \int \mathcal{D}\bar{\alpha}\mathcal{D}\alpha\mathcal{D}\bar{\beta}\mathcal{D}\beta \\
&e^{\frac{1}{2}\left(\bar{\alpha}_f\alpha_f+\bar{\beta}_f\beta_f+\bar{\alpha}_i\alpha_i+\bar{\beta}_i\beta_i\right)} \\
&\times \exp\int_0^\infty dt \left[-\frac{1}{2}\left(\bar{\alpha}\dot{\alpha}+\bar{\beta}\dot{\beta}-\dot{\bar{\alpha}}\alpha-\dot{\bar{\beta}}\beta\right)+\right. \\
&+i\omega_1\left(\bar{\alpha}\beta+\bar{\beta}\alpha\right)\tfrac{\lambda}{2\sqrt{1+\lambda^2t^2}}\sin\tfrac{\omega t}{1+\omega^2t^2} \\
&+i\left(\bar{\alpha}\alpha-\bar{\beta}\beta\right)\tfrac{\lambda}{2\sqrt{1+\lambda^2t^2}}\cos\tfrac{\omega t}{1+\omega^2t^2} \\
&i\left(\tfrac{\lambda}{1+\lambda^2t^2}+\frac{\omega}{2}\tfrac{1-\omega^2t^2}{(1+\omega^2t^2)^2}\right) \\
&\left.\times\left(-i\bar{\alpha}\beta+i\bar{\beta}\alpha\right)\right], \qquad (19)
\end{aligned}
$$

with $(\alpha_0,\beta_0)=(\alpha_i,\beta_i)$, and $\left(\bar{\alpha}_{N+1},\bar{\beta}_{N+1}\right)=\left(\bar{\alpha}_f,\bar{\beta}_f\right)$. This last formula represents the path integral of the propagator which has been the subject of many previous papers, and which has the advantage that it permits us to explicitly perform some concrete calculations.

IV. Calculation of the propagator

To begin, we first introduce new Grassmann variables via a unitary transformation in spin-coherent space which eliminates the angle $\omega t/(1+\omega^2t^2)$ present in the expression of the magnetic field:

$$
\left\{
\begin{array}{c}
(\alpha_n,\beta_n)\mapsto(\eta_n,\xi_n), \\
\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix} = e^{-\frac{i}{2}\frac{\omega t_n}{1+\omega^2t_n^2}\sigma_y}\begin{pmatrix}\eta_n\\ \xi_n\end{pmatrix}.
\end{array}
\right. \qquad (20)
$$

Then, it is easy to show that the measure and the infinitesimal action become respectively:

$$
\begin{aligned}
&\prod_{n=1}^{N}\left(d\bar{\alpha}_n d\alpha_n d\bar{\beta}_n d\beta_n e^{-\bar{\alpha}_n\alpha_n-\bar{\beta}_n\beta_n}\right) = \\
&\qquad = \prod_{n=1}^{N}\left(d\bar{\eta}_n d\eta_n d\bar{\xi}_n d\xi_n e^{-\bar{\eta}_n\eta_n d-\bar{\xi}_n\xi_n}\right), \qquad (21)
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar{\alpha}_n\alpha_{n-1}+\bar{\beta}_n\beta_{n-1} = \bar{\eta}_n\eta_{n-1}+\bar{\xi}_n\xi_{n-1} \\
&+i\frac{\omega}{2}\varepsilon\tfrac{1-\omega^2t_n^2}{(1+\omega^2t_n^2)^2}\left(-i\bar{\eta}_n\xi_{n-1}+i\bar{\xi}_n\eta_{n-1}\right)+O(\varepsilon^2) \\
&+i\varepsilon\left[\sin\tfrac{\omega t_n}{1+\omega^2t_n^2}\left(\bar{\alpha}_n\beta_{n-1}+\bar{\beta}_n\alpha_{n-1}\right)+\right. \\
&\left.\cos\tfrac{\omega t_n}{1+\omega^2t_n^2}\left(\bar{\alpha}_n\alpha_{n-1}-\bar{\beta}_n\beta_{n-1}\right)\right] \\
&= i\varepsilon\left(\bar{\eta}_n\eta_{n-1}-\bar{\xi}_n\xi_{n-1}\right). \qquad (22)
\end{aligned}
$$

The propagator as a function of the new Grassmann variables η and ξ becomes:

$$
\begin{aligned}
\mathbf{K}(f,i;T) \;=\; &\lim_{N\to\infty}\prod_{n=1}^{N} d\bar{\eta}_n d\eta_n d\bar{\xi}_n d\xi_n e^{-\bar{\eta}_n\eta_n-\bar{\xi}_n\xi_n} \\
&\prod_{n=1}^{N+1}\exp\left[\bar{\eta}_n\eta_{n-1}+\bar{\xi}_n\xi_{n-1}\right. \\
&+i\,\varepsilon\tfrac{\lambda}{2\sqrt{1+\lambda^2t_n^2}}\left(\bar{\eta}_n\eta_{n-1}-\bar{\xi}_n\xi_{n-1}\right) \\
&\left.-i\varepsilon\tfrac{\lambda}{1+\lambda^2t_n^2}\left(-i\bar{\eta}_n\xi_{n-1}+i\bar{\xi}_n\eta_{n-1}\right)\right]. \quad (23)
\end{aligned}
$$

Now using the following transformation:

$$
\varepsilon = -\frac{1+\lambda^2t_n^2}{2\lambda}\tau, \qquad \text{with} \qquad \tau = s_n - s_{n-1}, \qquad (24)
$$

where

$$
s_n = 2\arcsin\frac{1}{\sqrt{1+\lambda^2t_n^2}}, \qquad (25)
$$

and

$$
Z_n = \begin{pmatrix}\eta_n\\ \xi_n\end{pmatrix} \qquad \text{and} \qquad \bar{Z}_n = \left(\bar{\eta}_n,\bar{\xi}_n\right), \qquad (26)
$$

the propagator becomes:

$$
\begin{aligned}
\mathbf{K}(f,i;T) \;=\; &\lim_{N\to\infty}\prod_{n=1}^{N}\int d\bar{Z}_n dZ_n e^{-\bar{Z}_nZ_n} \\
&\prod_{n=1}^{N+1}\exp\left[\bar{Z}_nZ_{n-1}+i\tau\bar{Z}_nQ(n)Z_{n-1}\right], \qquad (27)
\end{aligned}
$$

where

$$
Q(n) = \begin{pmatrix} -\frac{1}{4\sin\frac{s_n}{2}} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{4\sin\frac{s_n}{2}} \end{pmatrix}. \qquad (28)
$$

Then we introduce new Grassmann variables Ψ via a unitary transformation in spin-coherent state-space defined by:

$$
Z_n = U(n)\Psi_n \qquad \bar{Z}_n = \Psi_nU^\dagger(n), \qquad (29)
$$

with

$$
U(n) = \exp\left(-\frac{i}{2}\ln\tan\frac{s_n}{4}\sigma_z\right), \qquad (30)
$$

which modify the expression (27) into the following form:

$$
\begin{aligned}
&\mathbf{K}(\alpha_f,\beta_f;\alpha_i,\beta_i;T) = \lim_{N\to\infty}\prod_{n=1}^{N}\int d\Psi_n d\Psi_n e^{-\Psi_n\Psi_n} \\
&\qquad\times\prod_{n=1}^{N+1}\exp\left[\overline{\Psi}_n\Psi_{n-1}+i\tau\Psi_nQ_1(n)\Psi_{n-1}\right], \quad (31)
\end{aligned}
$$

where

$$
Q_1(n) = \begin{pmatrix} 0 & -\frac{i}{2}e^{+\frac{i}{2}\ln\tan\frac{s_n}{4}\sigma_z} \\ \frac{i}{2}e^{-\frac{i}{2}\ln\tan\frac{s_n}{4}\sigma_z} & 0 \end{pmatrix}. \qquad (32)
$$

The next step consists of taking the diagonal form for the action in order to be able to perform the integration. Thus, we set a unit transformation over the Grassmann variables:

$$
\left\{
\begin{array}{c}
\Psi \longrightarrow \Phi, \\
\Psi = U_1(s)\Phi = \begin{pmatrix} A(s) & -B^*(s) \\ B(s) & A^*(s)\end{pmatrix}\Phi,
\end{array}
\right. \qquad (33)
$$

and

$$
\left\{
\begin{array}{c}
U_1(s)U_1^\dagger(s) = U_1^\dagger(s)U_1(s) = 1, \\
\det U_1(s) = 1,
\end{array}
\right. \qquad (34)
$$

with the initial conditions $A(t=0)=1$ and $B(t=0)=0$.

By means of a simple calculation including the following expansion:

$$
\begin{aligned}
U_1(s_{n-1}) \;&=\; U_1(s_n)-\tau\frac{dU_1(n)}{dt_n}+\mathcal{O}(\tau^2), \qquad (35) \\
U_1^\dagger(s_n)U_1(s_{n-1}) \;&=\; \mathbf{I}-\tau U_1^\dagger(s_n)\frac{dU_1}{dt}(s_n), \qquad (36)
\end{aligned}
$$

we obtain:

$$\mathbf{K}(\alpha_{f,}\beta_f;\alpha_{i,}\beta_i;T)=\lim_{N\to\infty}\prod_{n=1}^{N}\int d\Phi_n d\Phi_n e^{-\Phi_n\Phi_n}$$
$$\times\prod_{n=1}^{N+1}\exp\left[\overline{\Phi}_n\Phi_{n-1}+i\tau\Phi_n Q_2(n)\Phi_{n-1}\right],\quad(37)$$

where

$$Q_2(n)=iU_1^{\dagger}(s_n)\frac{dU_1}{ds}(s_n)+U_1^{\dagger}(s_n)Q_1(n)U_1(s_n).\quad(38)$$

Now we determine the unit transformation by fixing the diagonal form for the action, which leads us to the following condition:

$$Q_2(n)=0.\quad(39)$$

In order to perform the integration we have to write the expression for the propagator in the following appropriate form:

$$\mathbf{K}(\alpha_{f,}\beta_f;\alpha_{i,}\beta_i;T)=\int d\xi^{\dagger}d\xi\exp\left[-\xi^{\dagger}\xi+\mathbf{V}^{\dagger}\xi+\xi^{\dagger}\mathbf{W}\right],\quad(40)$$

where

$$\mathbf{V}^{\dagger}=\left(0,\cdots,\overline{\Phi}_{N+1}\right),\quad\xi=\begin{pmatrix}\Phi_1\\ \vdots\\ \Phi_1\end{pmatrix},\quad\mathbf{W}=\begin{pmatrix}\Phi_0\\ \vdots\\ 0\end{pmatrix}.\quad(41)$$

We absorb the linear terms in ξ and $\xi^{\dagger}$ thanks to the shift:

$$\begin{aligned}\xi&\to\xi+\mathbf{W},\\ \xi^{\dagger}&\to\xi^{\dagger}+\mathbf{V}^{\dagger},\end{aligned}\quad(42)$$

and we integrate over the Grassmann variables. Our result for the propagator for the spin-$1/2$ particle subject to a time-dependent magnetic field is finally written as follows:

$$\mathbf{K}(\alpha_{f,}\beta_f;\alpha_{i,}\beta_i;T)=e^{\overline{\Phi}_f\Phi_i}.\quad(43)$$

In terms of the variables (α,β) this is written as:

$$\mathbf{K}(\alpha_f,\beta_f,\alpha_{i,}\beta_i;T)=\exp\left(\bar{\alpha}_f,\bar{\beta}_f\right)\mathbf{R}(t)\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix},\quad(44)$$

where

$$\mathbf{R}(t)=e^{-\frac{i}{2}\frac{\omega t}{1+\omega^2t^2}\sigma_y}\times$$
$$\times e^{-\frac{i}{2}\ln\tan\frac{s(t)}{4}\sigma_z}\begin{pmatrix}A(s(t)) & B(s(t))\\ -B^*(s(t)) & A^*(s(t))\end{pmatrix}.\quad(45)$$

V. The transition probability

Let us calculate the matrix representation for the propagator (45) in the spin states. In what follow we only show how to evaluate the matrix element $K(\uparrow,\uparrow;T)$ while other matrix elements can easily be deduced following the same method. In fact, the propagator in the spin eigenstates is given by:

$$K(\uparrow,\uparrow;T)=\langle\uparrow|U(T)|\uparrow\rangle.\quad(46)$$

With the help of the completeness relations, this amplitude becomes:

$$\begin{aligned}K(\uparrow,\uparrow;T)&=\int d\overline{\alpha}_f d\alpha_f d\overline{\beta}_f d\beta_f d\overline{\alpha}_i d\alpha_i d\overline{\beta}_i d\beta_i\\ &\times e^{-\overline{\alpha}_f\alpha_f-\overline{\beta}_f\beta_f}e^{-\overline{\alpha}_i\alpha_i-\overline{\beta}_i\beta_i}\\ &\times\langle\uparrow|\,\alpha_f,\beta_f\rangle\langle\alpha_i,\beta_i\,|\uparrow\rangle\mathbf{K}(\alpha_f,\beta_f,\alpha_i,\beta_i;T).\end{aligned}\quad(47)$$

Then:

$$K(\uparrow,\uparrow;T)=\int d\bar{\alpha}_f d\alpha_f d\bar{\beta}_f d\beta_f d\bar{\alpha}_i d\alpha_i d\bar{\beta}_i d\beta_i$$
$$\times e^{-\bar{\alpha}_i\alpha_i-\bar{\beta}_i\beta_i}e^{-\bar{\alpha}_f\alpha_f-\bar{\beta}_f\beta_f}\langle\uparrow|\,\alpha_f,\beta_f\rangle\langle\alpha_i,\beta_i|\uparrow\rangle$$
$$\times\exp\left\{\left(\bar{\alpha}_f,\bar{\beta}_f\right)\begin{pmatrix}\mathbf{R}_{11}(t) & \mathbf{R}_{12}(t)\\ \mathbf{R}_{21}(t) & \mathbf{R}_{22}(t)\end{pmatrix}\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}\right\}.\quad(48)$$

Thanks to the features from Ref. [21]:

$$\langle\uparrow|\,\alpha_f,\beta_f\rangle=\alpha_f,\ \langle\alpha_i,\beta_i\,|\uparrow\rangle=\bar{\alpha}_i,\ \text{and}\ \alpha_f\bar{\alpha}_i=e^{-\bar{\alpha}_i\alpha_f}-1,\quad(49)$$

eq. (48) takes the following form:

$$K(\uparrow,\uparrow;T)=\int d\nu^{\dagger}d\nu\left[\exp\nu^{\dagger}M'\nu-\exp\nu^{\dagger}M\nu\right],\quad(50)$$

where the matrices M and M' are, respectively:

$$\mathbf{M}=\begin{pmatrix}-1 & 0 & 0 & 0\\ \mathbf{R}_{11} & -1 & \mathbf{R}_{11} & 0\\ 0 & 0 & -1 & 0\\ \mathbf{R}_{21} & 0 & \mathbf{R}_{22} & -1\end{pmatrix},\quad(51)$$

and

$$\mathbf{M}'=\begin{pmatrix}-1 & -1 & 0 & 0\\ \mathbf{R}_{11} & -1 & \mathbf{R}_{11} & 0\\ 0 & 0 & -1 & 0\\ \mathbf{R}_{21} & 0 & \mathbf{R}_{22} & -1\end{pmatrix},\quad(52)$$

with $\mathbf{R}_{nm}$ the matrix elements of $\mathbf{R}$, and

$$\nu=\begin{pmatrix}\alpha_i\\ \alpha_f\\ \beta_i\\ \beta_f\end{pmatrix},\quad\nu^{\dagger}=\left(\overline{\alpha}_i,\overline{\alpha}_f,\overline{\beta}_i,\overline{\beta}_f\right),\quad(53)$$

are vectors gathering the Grassmann variables.

The integration over the Grassmann variables is thus simple:

$$K(\uparrow,\uparrow;T)=\det\mathbf{M}'-\det\mathbf{M}.\quad(54)$$

Now since $\det\mathbf{M}'=1+\mathbf{R}_{11}$ and $\det\mathbf{M}=1$, the propagator matrix element for the up-up states is finally written:

$$K(\uparrow,\uparrow;T)=\mathbf{R}_{11}(t).\quad(55)$$

Repeating the calculations and considering all initial and final states of the spin, the propagator takes the following matrix form:

$$K(m_f,m_i;T)=\begin{pmatrix}\mathbf{R}_{11}(t) & \mathbf{R}_{12}(t)\\ \mathbf{R}_{21}(t) & \mathbf{R}_{22}(t)\end{pmatrix},\quad(56)$$

and hence the transition probability from a down spin state to an up spin state is given by:

$$\begin{aligned} P_{\downarrow\uparrow} &= |\mathbf{R}_{21}(t)|^2 \\ &= \left| \left[A(t) e^{-\frac{i}{2}\ln\tan\frac{s(t)}{4}} \sin\frac{1}{2}\left(\tfrac{\omega t}{1+\omega^2 t^2}\right) \right.\right. \\ &\quad \left.\left. - B^*(t) e^{\frac{i}{2}\ln\tan\frac{s(t)}{4}} \cos\frac{1}{2}\left(\tfrac{\omega t}{1+\omega^2 t^2}\right)\right]\right|^2 . \end{aligned} \quad (57)$$

Note that the matrix $U_1(s)$ introduced in (33) has been fixed by the condition (39), so it has to satisfy the following auxiliary equation:

$$i\frac{dU_1}{ds} + Q_1(s)U_1(s) = 0, \quad (58)$$

i.e. a system of two coupled equations:

$$\left\{ \begin{array}{l} \frac{dA}{ds} = -\frac{1}{2}B^* \exp(i\ln\tan\frac{s(t)}{4}) \\ \frac{dB^*}{ds} = \frac{1}{2}A\exp(-i\ln\tan\frac{s(t)}{4}) \end{array} \right. \text{ with } \left\{ \begin{array}{l} A(\pi) = 1, \\ B(\pi) = 0, \end{array} \right. \quad (59)$$

and whose solution determines the elements A and B of the matrix U_1. Let us decouple this system of equations. We have:

$$\frac{d^2B^*}{ds^2} + \frac{i}{2\sin\frac{s}{2}}\frac{dB^*}{ds} + \frac{1}{4}B^* = 0. \quad (60)$$

The solution to this equation is:

$$B^*(s) = \frac{i}{2}(i + \cos\frac{s}{2})h\left(s\right) \quad (61)$$

then

$$A(s) = 2e^{i\ln\tan\frac{s(t)}{4}}\left[-\frac{i}{4}h\left(s\right)\sin\frac{s}{2} + \frac{i}{2}(i+\cos\frac{s}{2})\frac{dh(s)}{ds}\right], \quad (62)$$

where

$$h\left(s\right) = \int_\pi^s \frac{e^{-i\ln\tan\frac{\tau}{4}}}{(i+\cos\frac{\tau}{2})^2}d\tau. \quad (63)$$

A straightforward calculation leads to the well-known Rabi-like formula:

$$\begin{aligned} P_{\downarrow\uparrow} = \frac{1+\cos^2\frac{s}{2}}{4}&\left|\left[-h(s)\cos\frac{1}{2}\left(\tfrac{\omega t}{1+\omega^2t^2}\right) + \right.\right. \\ 2\left[\frac{dh(s)}{ds} - \frac{\sin\frac{s}{2}}{2(i+\cos\frac{s}{2})}h\left(s\right)\right]&\left.\left.\sin\frac{1}{2}\left(\tfrac{\omega t}{1+\omega^2t^2}\right)\right]\right|^2 . \end{aligned} \quad (64)$$

This result coincides with that of Refs [12] and [4].

VI. Conclusions

By using the formalism of path integrals and the fermionic coherent-states approach, we have been able to calculate the explicit expression of the propagator for a spin-$1/2$ particle interacting with a time-dependent magnetic field. To treat the spin dynamics, we used the Schwinger recipe in which one replaces the Pauli matrices by a pair of fermionic oscillators. The introduction of particular rotations in coherent-state space eliminates the rotation angle of the magnetic field which then somewhat simplifies the Hamiltonian of the considered system. As a consequence, we have been able to integrate over the spin variables described by fermionic oscillators. The exactness of the result is manifested in the evaluation of the transition probability formula.

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