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Traffic Flow on Road Junctions

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***Abstract*—The paper deals with a mathematical model for fluid dynamic flows on road networks which is based on conservation laws. This nonlinear framework is based on the conservation of cars. We focus on traffic circle, which is a finite number of roads that meet at some junctions. The traffic circle with junctions having either one incoming and two outgoing or two incoming and one outgoing roads. We describe the numerical schemes with the particular boundary conditions used to produce approximated solutions of the problem.**

***Keywords*—boundary conditions, conservation laws, finite difference Schemes, traffic flow.**

I. INTRODUCTION

THE main purpose of this paper is to investigate the numerical approximation of solutions to some nonlinear conservation laws. A mathematical model for fluid-dynamic flows on networks presented and based on Lighthill & Whitham and Richards (LWR) traffic flows model, dealing with a concave flux. Traffic models are represented by minimization of congestions, accidents, pollution, and the maximization of safety. Classical simulation models are based on stationary assumption, which results inadequate to deal with heavily congested traffic circle. Simulation of vehicular traffic can be treated in different ways referring to microscopic, mesoscopic or macroscopic models.

The main assumption of the car following models is that an individual car's motion only depends on the car ahead. The idea is to apply analogous concepts to traffic, since the density of traffic along a segment of road can vary in time only due to more traffic flowing in than out. As traffic jams display sharp discontinuities, there is a correspondence between traffic jams and shock waves. The drivers arriving at the junction distribute on the outgoing roads according to some known coefficients.We use the LWR model on the roads with time varying traffic distribution coefficients. For heavy incoming traffic, we analyze the probability of the junction to get stuck in dependence of the various parameters of the problem.

For heavy traffic the right of way parameters of traffic circle can be set so to avoid a complete jam. To describe a road junction as a finite collection of roads that meet at some junctions.

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In order to obtain a unique solution of the Riemann problem at junctions (problem with constant initial data on each road), we need to assume some rules, so we can construct solutions via wave-front tracking technique. Fluid dynamic models for traffic flow seem the most appropriate to detect some phenomena as shocks formation and propagation on roads, since they can develop discontinuities in a finite time even starting from smooth data. The modeling of a traffic circle, that is characterized by junctions either with two incoming and one outgoing roads or with one incoming and two outgoing roads, suggests a suitable use of the rules (A)-(B) and (C) at the junctions. The main interest is covered by the tuning of the right of way parameters in order to improve the performances.

The paper is organized as follows. In Section II, the model for traffic flow on a road network is described. Section III deals with Traffic circle and Numerical Method use for the traffic flow problem. Finally, we discuss the simulation results.

II. FLUID- DYNAMIC MODEL FOR TRAFFIC SIMULATION

Consider a road network as a finite number of roads, modeled by intervals $I_i = 1,......., N, a_i < b_i$, with one of the endpoints that can be infinite. The roads are connected by some junctions, and each junction J has a finite number of incoming and outgoing roads. On each road the problem agrees with equation (1).

$$\partial_t \rho + \partial_x f(\rho) = 0 \tag{1}$$

where $\rho = \rho(x,t) \in [0, \rho_{\max}], (x,t) \in \Re^2$ is the density of cars, $\rho_{\max}$ is the maximal density of cars, $f(\rho) = \rho\upsilon$ is the flux and υ the average velocity. We further assume that υ is a smooth decreasing function of the density ρ and f is concave. Such a conservation law describes a fluid-dynamic approach useful to perform macroscopic phenomena as shock waves formation and propagation. Fluid- dynamical approaches were extended to flows on urban networks based on the LWR model. For the junction setting, the simple LWR model, [4], [5] is sufficient to describe most of the important traffic behaviors features and it is the only one for which a fairly complete theory and numerics are available.

We make the following assumptions on the flux function $f : [0, \rho_{\max}] \to \Re$ is a smooth, strictly concave function, $f(0) = f(\rho_{\max}) = 0, |f'(x)| \le C < +\infty$. Hence, there exists a unique $\sigma \in [0, \rho_{\max}]$ such that $f'(\sigma) = 0$ The velocity function is:

$$\upsilon(\rho) = \upsilon_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)$$

where $\upsilon_{\max}$ is the maximal velocity of cars, which travel along the road. Then the flux is given by

$$f(\rho) = \upsilon_{max}\ \rho\left(1-\frac{\rho}{\rho_{max}}\right)$$

For a single conservation law (1), a Riemann Problem (RP) is a Cauchy problem for an initial data piecewise constant with only one discontinuity. The solutions are either formed by continuous waves (rarefactions) or by traveling discontinuities (shocks). The condition at the junctions (*Rankine-Hugoniot* relation) holds:

$$\sum_{i=1}^{n} f(\rho_i(t,b_i^+)) = \sum_{j=n+1}^{n+m} f(\rho_j(t,a_j^-)),$$

where $\rho_i, i=1,\ldots\ldots\ldots,n$, are the incoming densities and $\rho_j, j=n+1,\ldots\ldots\ldots,n+m$, the outgoing ones. It represents a different way of writing the conservation of cars: it expresses the equality of incoming and outgoing fluxes. Riemann Problems at junctions are under- determined even after prescribing the conservation of cars. Existence and Uniqueness of solution are guarantee by three following rules:

(A) There are some fixed coefficients representing the drivers' preferences. These coefficients denote the traffic's distribution from incoming to outgoing roads. For this reason, it's useful to define a traffic distribution matrix:

$$A=\{\alpha_{ji}\}, i=1,\ldots, n\ ,\ j=n+1,\ldots,n+m \in \Re^{m+n}\ ,$$

such that $0 \le \alpha_{j,i} \le 1\ ,\ \sum_{j=n+1}^{n+m} \alpha_{j,i} = 1$

(B) Respecting (A), the drivers choose roads such that the flux can be maximized, that is we suppose that no car can stop without cross the junction.

(C) Assuming that $m<n$ (m =1 and n = 2), let C be the amount of cars that can enter the outgoing road. We fix a right of way parameter $q \in [0, \rho_{\max}]$. Then qC cars come from the first incoming road and (1–qC) cars come from the second one.

A. RIEMANN SOLVER

In this section, we construct the Riemann solver at Junctions, which satisfy rules (A), (B) and (C). Particularly, we treat two case studies: junctions of type (two incoming roads and one outgoing road) and (one incoming road and two outgoing roads).

Proposition. Let $(\rho_{1,0}, \rho_{2,0}, \ldots\ldots\ldots, \rho_{n+m,0})$ be the initial densities of a RP at junction and $\gamma_i^{\max}, i=1,\ldots\ldots\ldots,n$ and be the maximum fluxes that can be obtained on incoming roads and outgoing ones.

Then:

$$\gamma_i^{\max} = \begin{cases} f(\rho_{i,0})\ , & if\ \rho_{i,0} \in [0,\sigma]\ , i=1,\ldots\ldots, n. \\ f(\sigma)\ , & if\ \rho_{i,0} \in [\sigma,1]\ , \end{cases} \qquad (2)$$

$$\gamma_j^{\max} = \begin{cases} f(\sigma)\ , & if\ \rho_{j,0} \in [0,\sigma]\ , j=n+1,\ldots.,n.+m. \\ f(\rho_{j,0})\ , & if\ \rho_{j,0} \in [\sigma,1], \end{cases} \qquad (3)$$

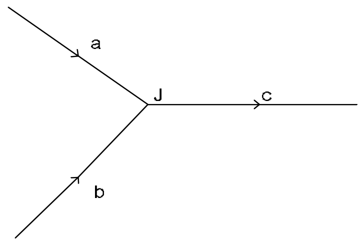

Fig. 1. A junction with two incoming and one outgoing roads

Consider a junction, *a* and *b* are the incoming roads and *c* is the only outgoing road. Considering rule (C), the solution to the Riemann problem with initial data $(\rho_{a,0}, \rho_{b,0}, \rho_{c,0})$ is constructed in the following way. Since we want to maximize the through traffic (rule (B)), we set:

$$\hat{\gamma}_c = \min\ \{\gamma_a^{\max} + \gamma_b^{\max}, \gamma_c^{\max}\},$$

where $\gamma_i^{\max}$, i=a, b, is defined as in (2) and $\hat{\gamma}_c$ as in (3).In fact, $\hat{\gamma}_c$ the maximal through flux, which can respect the Rankine-Hugoniot condition at the junction, i.e. then conservation of cars through the junction.

In this case the matrix *A* (or rule (A)) is simply given by the column vector (1, 1), thus it gives no additional restriction. This is due to the fact that there is a single outgoing road, so cars must flow to that outgoing road necessarily.

Consider now the space (γ_a, γ_b) and the line:

$$\gamma_b = \frac{1-q}{q}\gamma_a\ , \qquad (4)$$

defined according to the rule (C). Let P be the point of intersection of the line (4) with the line $\gamma_a + \gamma_b = \hat{\gamma}_c$ .The final fluxes must belong to the region:

$$\Omega=\{(\gamma_a,\gamma_b)\ : 0 \le \gamma_i \le \gamma_i^{\max},\ 0 \le \gamma_a+\gamma_b \le \hat{\gamma}_c\ ,\ i=a,b\}$$

There are two different cases:

- P belongs to Ω,
- P does not belong to Ω.

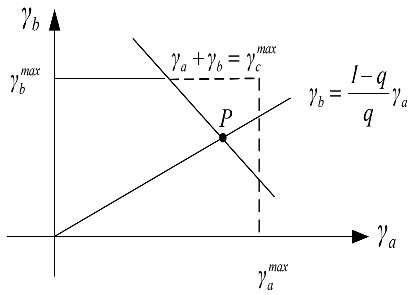

Fig. 2. First case

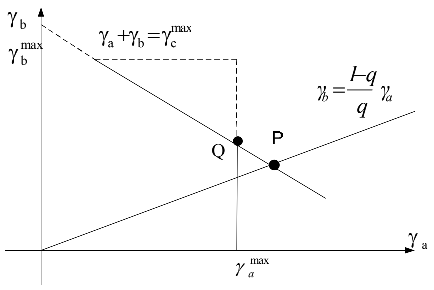

Fig. 3. Second case

The two cases are represented in Figures 2 and 3. In the first case, we set $(\hat{\gamma}_a,\hat{\gamma}_b) = P$, while in the second case we set $(\hat{\gamma}_a,\hat{\gamma}_b) = Q$, where Q is the point of the closest to the line (4). Once we have determined $\hat{\gamma}_a$ and $\hat{\gamma}_b$ (and $\hat{\gamma}_c$), we can find a unique way $\hat{\rho}_i, i \in \{a,b,c\}$ .This is gain due to restrictions on waves velocities.

Consider the junction with one incoming *a* and two outgoing (*b*, *c*) roads.

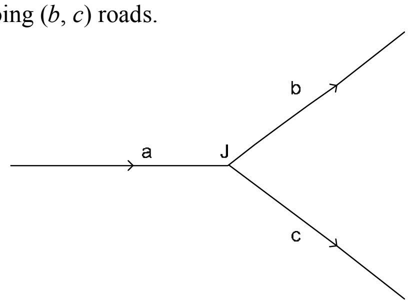

Fig. 4. A junction with one incoming and two outgoing roads

Road *a* is the only incoming road while *b* and *c* are the outgoing roads. No additional rule is needed thus only rules (A) and (B) are used. The distribution matrix, of rule (*A*), takes the form

$$A = \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} ,$$

where $\alpha\in[0,1]$ and $(1-\alpha)$ denotes the percentage of cars which, from road a, goes to road b and c, respectively.
Rule (B), the solution to a RP is:

$$\hat{\gamma}=(\hat{\gamma}_a,\hat{\gamma}_b,\hat{\gamma}_c)=(\hat{\gamma}_a,\alpha\hat{\gamma}_a,(1-\alpha)\hat{\gamma}_a),$$

where $$\hat{\gamma}_a=\min\left\{\gamma_a^{max},\frac{\gamma_b^{max}}{\alpha},\frac{\gamma_c^{max}}{1-\alpha}\right\}$$

Once we have obtained $\hat{\gamma}_a$, $\hat{\gamma}_b$ and $\hat{\gamma}_c$, it is possible to find in a unique way $\hat{\rho}_i, i\in\{a,b,c\}$.

B. GODUNOV'S METHOD

Scalar conservation laws for variable ρ with flux function f , consider piecewise constant data $\rho^n(x,t_n)$ at time t_n in each cell lies $x_{j-\frac{1}{2}}<x<x_{j+\frac{1}{2}}$.

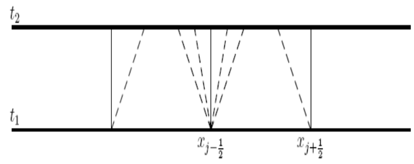

Fig. 5.Characteristics for Computing Flux

$$\rho_t+F(\rho)_x=0 \quad , \quad x\in\Re\ ,\ t\in(0,T)$$

with initial data

$$\rho(x,0)=\rho_0(x)=\begin{cases}\rho_l & if\ x<0\\ \rho_r & if\ x>0\end{cases}$$

is self similar, that is to say

$$\rho(x,t)=W_R\left(\frac{x}{t};\rho_l,\rho_r\right)$$

where W_R depends only on the flux function F and consists of the two constant states ρ_l and ρ_r separated by various waves starting from the origin whose speeds are bounded by

$$\max\ \{|F'(\xi)|, \xi\ between\ \ \rho_l\ and\ \ \rho_r\}$$

Given an initial data $\rho_0(x)$ we approximate it by ρ^Δ that represents a piecewise constant function defined in $\Re\times(0,+\infty)$.
The initial data is approximated by the sequence $\rho^0=(\rho_m^0)$ in the following way

$$\rho_j^{n+1}=\frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}\rho^n(x,t_{n+1})\,dx$$

We use the integral form of the conservation law in the cell to obtain

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}\rho^n(x,t_{n+1})\,dx=\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}\rho^n(x,t_n)\,dt+\int_{t_n}^{t_{n+1}}f(\rho^n(x_{j-\frac{1}{2}},t)\,dt$$

$$-\int_{t_n}^{t_{n+1}}f(\rho^n(x_{j+\frac{1}{2}},t))\,dx$$

Defining $$F(\rho_j^n,\rho_{j+1}^n)=\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}f(\rho^n(x_{j+\frac{1}{2}},t))\,dt$$

$$\rho_j^{n+1}=\rho_j^n-\frac{\Delta t}{\Delta x}\left[F(\rho_j^n,\rho_{j+1}^n)-F(\rho_{j-1}^n,\rho_j^n)\right].$$

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III. TRAFFIC CIRCLE

We introduce the traffic regulation problem: given a junction with some incoming roads and some outgoing ones, is it preferable to regulate the flux via a traffic light or via a traffic circle on which the incoming traffic enters continuously? Assuming that drivers arriving at the junction distribute on the outgoing roads according to some known coefficients our purpose is to understand which solution performs better from the point of view of total amount of cars going through the junction.

In order to treat this problem we need a model that describes the above situation and provides an accurate analysis. To this aim we consider the fluid dynamic model based on Equation (1) and proposed in [4] adapted in a suitable way in order to treat the case of traffic circles.

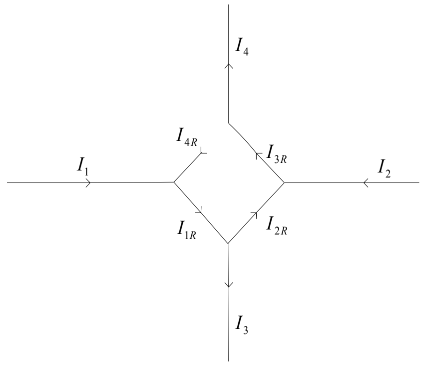

Fig 6.Traffic Circle

Consider a general network, junctions with having either one incoming and two outgoing or two incoming and one outgoing roads. Once the solution to Riemann's problems is fixed then we can introduce the definition of admissible solutions. More precisely, given a set of parameters q_k for all junctions J_k with two incoming and one outgoing roads, a solution ρ on the road network is admissible if for a.e. t with $\rho(t)$ of bounded variation the Riemann's problem at each junction J_k is solved in the correct way corresponding to the parameter q_k.

A. SINGLE LANE TRAFFIC CIRCLE WITH LOW TRAFFIC

. In this section a traffic circle assume that there is low traffic, in the sense that the number of cars reaching the circle is less than the capacity of the circle.

There are four roads, named, I_1, I_2, I_3, I_4 the first two incoming in the circle and the other two outgoing. Moreover there are four roads $I_{1R}, I_{2R}, I_{3R}, I_{4R}$ that form the circle.

The roads are parameterized by $[a_i, b_i]$, $i = 1,2,3,4$, and $[a_{iR}, b_{iR}]$, $i = 1,2,3,4.$ It is to assign a traffic distribution matrix A to describe how traffic coming from roads I_1, I_2 choose to exit to roads I_3, I_4. The roads of the circle are intermediate towards the final destination. Thus we assume to have two fixed parameters: $\alpha, \beta \in [0,1]$ so that:

(C1) If M cars reach the circle from road I_1, then αM drive to road I_3 and $(1-\alpha)M$ drive to road I_4.

(C2)If M cars reach the circle from road I_2, then βM drive to road I_4 and $(1 - \beta)M$ drive to road I_3.

Let $\bar{\rho}_1$ and $\bar{\rho}_2$ be constant densities from the roads I_1 and I_2.

$$\rho_1(t, a_1) = \bar{\rho}_1, \quad \rho_2(t, a_2) = \bar{\rho}_2 \tag{5}$$

If the roads I_3 and I_4 can absorb all incoming traffic.

$$f(\bar{\rho}_1) + f(\bar{\rho}_2) \le f(\sigma), \tag{6}$$

If the network is initially empty, the boundary data are given by Equation (5), then firstly the cars from road I_1 and I_2 reach road I_3 and I_4 and the coefficients should be simply set as:

$$\alpha_{1R,3} = \alpha, \alpha_{1R,2R} = (1-\alpha), \alpha_{3R,4} = \beta, \alpha_{3R,4R} = (1-\beta).$$

B. SINGLE LANE TRAFFIC CIRCLE WITH HEAVY TRAFFIC

In this case the condition (6) is violated. Traffic jams is possible under one of the following conditions:

$$f(\bar{\rho}_1) + (1-\beta) f(\bar{\rho}_2) > f(\sigma), \tag{7}$$

$$(1-\alpha) f(\bar{\rho}_1) + f(\bar{\rho}_2) > f(\sigma), \tag{8}$$

We set:

$$f_l = f(\bar{\rho}_1), \quad f_2 = f(\bar{\rho}_2)$$

$$q_1 = q(1, 4R, 1R), \quad q_2 = q(2, 2R, 3R)$$

For the junction $(1, 4R, 1R)$ we have

$$\gamma_1^{\max} = f_1 \, \gamma_{4R}^{\max} = (1-\beta) f_2, \ \gamma_{4R}^{\max} = f(\sigma) = 1.$$

Then we have $\hat{\gamma}_{1R} = f(\sigma) = 1$, $\hat{\rho}_{1R} = \sigma$. Depending on the value of q_1 there are three cases:

(a) $q_1 \le 1 - \dfrac{(1-\beta) f_2}{f(\sigma)}$, then $\hat{\gamma}_1 = f(\sigma) - (1-\beta) f_2$ and $\hat{\gamma}_{4R} = (1-\beta) f_2$;

(b) $1 - \dfrac{(1-\beta)}{f(\sigma)} < q_1 < \dfrac{f_1}{f(\sigma)}$, then $\hat{\gamma}_1 = q f(\sigma)$ and $\hat{\gamma}_{4R} = (1-q) f(\sigma)$;

(c) $q_1 \ge \dfrac{F_1}{f(\sigma)}$, then $\hat{\gamma}_1 = f_1$ and $\hat{\gamma}_{4R} = f(\sigma) - f_1$.

In case (a) a shock is produced on road 1 and no wave on road 4R, in case (c) a shock is produced on road 4R and no wave on road 1, finally in case (b) a shock is produced on both roads. An analogous analysis can be done for junction

(I_2, I_{2R}, I_{3R}). First we put in case (a) for both junctions (I_1, I_{4R}, I_{1R}) and (I_2, I_{2R}, I_{3R})., so that rarefactions are generated on roads I_{1R} and I_{3R}. We need to assign the corresponding distribution coefficients $\alpha = 0.5$, $(\alpha_{1R,3}, \alpha_{1R,2R})$ and $(\alpha_{3R,4}, \alpha_{3R,4R})$ we need to fix a right of way parameter $q \in [0,1]$.

IV. NETWORKS CASE STUDIES

We present some simulations reproducing a simple traffic circle composed by 8 roads and 4 junctions. The numerical solutions have been generated by finite difference method for h = 0.025.

We consider the initial data

$\rho_{1,0} = 0.25$, $\rho_{2,0} = 0.4$, $\rho_{3,0} = \rho_{4,0} = 0.5$,

$\rho_{1R} = 0.5$, $\rho_{2R} = 0.5$, $\rho_{3R} = 0.5$, $\rho_{4R} = 0.5$

and, for roads entering the circle, we impose the following boundary conditions:

$\rho_{1,b}(t) = 0.25$, $\rho_{2,b}(t) = 0.4$

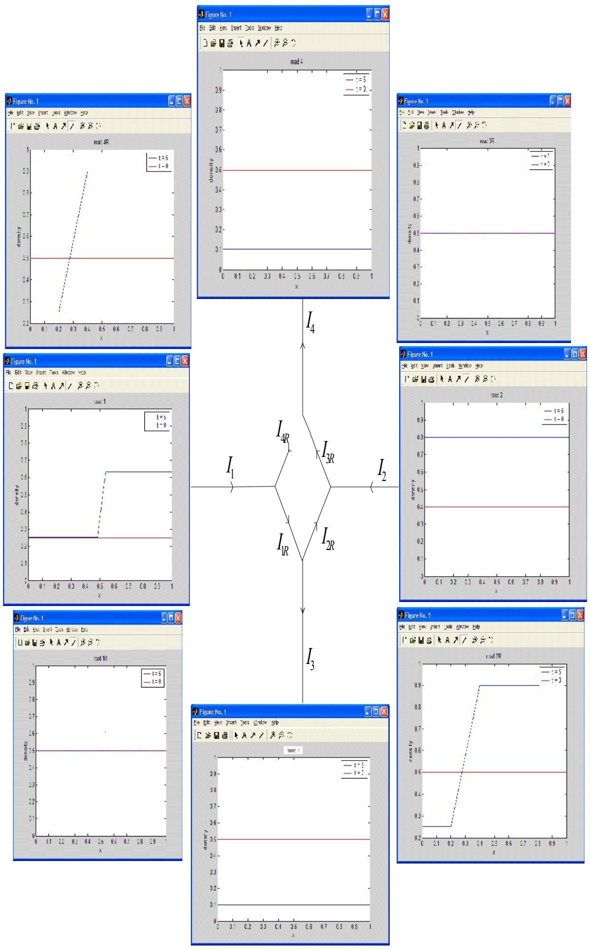

Fig. 7. Traffic circle with $q_1 = q_2 = 0.65$

Consider a Traffic circle with two incoming and one outgoing roads or one incoming and two outgoing roads. In this case we use case (b) and the right way of parameter $q_1 = q(I_1, I_{4R}, I_{1R}) = 0.65$, $q_2 = q(I_2, I_{2R}, I_{3R}) = 0.65$, then road I_1 is the through respect to road I_{4R} and road I_2 is the through street respect to I_{2R}. The distribution coefficients are assumed to be constant and all equal to $\alpha = 0.5$. Then rarefactions are produced on roads I_{1R} , I_{3R} and shocks on the other roads.

The evolution in time of traffic is reported in above figure. At time $t = 5$ shocks are generated on the entering roads I_1. The density on roads I_{2R}, I_{4R} increases and shocks are propagating backwards on roads. Roads I_3 *and* I_4 show a very low density of cars at time $t = 5$. Hence, in that case, the choice of the right of way parameter determines a situation of completely blocked traffic. Roads I_{1R} and I_{3R} maintain the same level of density. While rarefaction waves traveling in the sense of traffic created on roads I_{4R}, I_{3R}, I_3, I_4.

V. CONCLUSION

In this paper we deal with traffic problems on road network according with fluid-dynamic approach that analyzes traffic by means of conservation law on each road of networks. Then we show some simulations carried out by a simulation prototype.

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