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Three-player Domineering

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***Abstract*—**Domineering is a classic two-player combinatorial game usually played on a rectangular board. Three-player Domineering is the three-player version of Domineering played on a three dimensional board. Experimental results are presented for $x \times y \times z$ boards with $x + y + z < 10$ and $x, y, z \geq 2$. Also, some theoretical results are shown for $2 \times 2 \times n$ board with n even and $n \geq 4$.

***Keywords*—**Combinatorial games, Domineering, three-player games.

I. Introduction

THE game of Domineering, also known as Crosscram and Dominoes, is a typical two-player game with perfect information, proposed around 1973 by Göran Andersson [10], [9], [2]. The two players, usually denoted by Vertical and Horizontal, take turns in placing dominoes (2×1 tile) on a checkerboard. Vertical is only allowed to place its dominoes vertically and Horizontal is only allowed to place its dominoes horizontally on the board. Dominoes are not allowed to overlap and the first player that cannot find a place for one of its dominoes loses. After a time the remaining space may separate into several disconnected regions, and each player must choose into which region to place a domino.

Berlekamp [1] solved the general problem for $2 \times n$ board for odd n. The 8×8 board and many other small boards were recently solved by Breuker, Uiterwijk and van den Herik [3] using a computer search with a good system of transposition tables. Subsequently, Lachmann, Moore, and Rapaport solved the problem for boards of width 2, 3, 5, and 7 and other specific cases [11]. Finally, Bullock solved the 10×10 board [4].

Three-player Domineering is played on a three dimensional board with edges parallel to the x-, y-, and z-axes. We use $x \times y \times z$ to indicate an x by y by z board. Three players, denoted by X, Y, and Z, take turns in cyclic fashion (…, X, Y, Z, X, Y, Z, …) in placing three dimensional dominoes ($2 \times 1 \times 1$ tile) on a three dimensional board. X is only allowed to place its dominoes parallel to the x-axis, Y is only allowed to place its dominoes parallel to the y-axis, and Z is only allowed to place its dominoes parallel to the z-axis. Dominoes are not allowed to overlap and when one of the players cannot find a place for one of its dominoes, he/she leaves the game and the remaining player continue in alternation until one of them is unable to move. Then that player leaves the game and the remaining player is the winner. In other words, the player that is able to make the last move is the winner.

Three-player combinatorial games are difficult to analyze because of *queer* game [12], [5], [6], [7], [8], i.e., games where no player can force a win.

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TABLE I
OUTCOMES OF THREE-PLAYER DOMINEERING ON SMALL BOARDS

	X starts	Y starts	Z starts
$2 \times 2 \times 2$	Z wins	X wins	Y wins
$3 \times 2 \times 2$	Queer	X wins	Queer
$2 \times 3 \times 2^*$	Queer	Queer	Y wins
$2 \times 2 \times 3^*$	Z wins	Queer	Queer
$3 \times 3 \times 2$	Queer	Queer	Z wins
$2 \times 3 \times 3^*$	X wins	Queer	Queer
$3 \times 2 \times 3^*$	Queer	Y wins	Queer
$3 \times 3 \times 3$	Queer	Queer	Queer
$4 \times 2 \times 2$	Queer	Queer	Queer
$2 \times 4 \times 2^*$	Queer	Queer	Queer
$2 \times 2 \times 4^*$	Queer	Queer	Queer
$4 \times 3 \times 2$	Queer	Queer	Queer
$2 \times 4 \times 3^*$	Queer	Queer	Queer
$3 \times 2 \times 4^*$	Queer	Queer	Queer
$4 \times 2 \times 3$	Queer	Queer	Queer
$3 \times 4 \times 2^*$	Queer	Queer	Queer
$2 \times 3 \times 4^*$	Queer	Queer	Queer
$5 \times 2 \times 2$	Queer	Queer	Queer
$2 \times 5 \times 2^*$	Queer	Queer	Queer
$2 \times 2 \times 5^*$	Queer	Queer	Queer

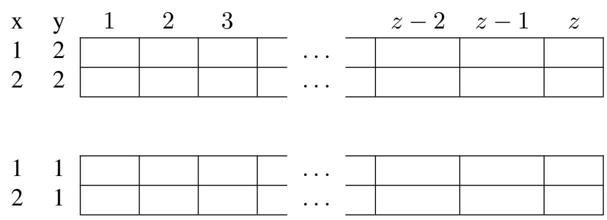

Fig. 1. The $2 \times 2 \times z$ board represented by the union of two rectangular boards.

II. Main Results

Table I shows the results obtained using an exhaustive search algorithm for $x \times y \times z$ boards with $x + y + z < 10$ and $x, y, z \geq 2$. The entries with * can be obtained by symmetry from the previous cases.

Theorem 1: Let $2 \times 2 \times z$ be a three-dimensional board with z even. If X starts the game, then X does not have a winning strategy.

Proof: Because of the difficulties of playing in three dimensions, we will represent the three dimensional board by the union of two rectangular boards as shown in Fig. 1. Each position will be indicated by a tern

$$(a, b, c)$$

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x	y	1	2	3	4	5	6	7	8
1	2		Y	Y		X	Y	Y	X
2	2	Z	Z	Z	Z	X			X
1	1	X	Y	Y	X		Y	Y	
2	1	X			X	Z	Z	Z	Z

Fig. 2. The $2 \times 2 \times 8$ board where X does not have a winning strategy when X starts the game.

x	y	1	2	3	4	5	6	7	8
1	2	Y	X	X	Y		X	X	
2	2		X	X		Y	X	X	Y
1	1	Y			Y	Z	Z	Z	Z
2	1	Z	Z	Z	Z	Y			Y

Fig. 3. The $2 \times 2 \times 8$ board where Y does not have a winning strategy when Y starts the game.

where $1 \leq a \leq 2$, $1 \leq b \leq 2$, $1 \leq c \leq z$. A move for X is

$$(1|2, b, c)$$

a move for Y is

$$(a, 1|2, c)$$

and a move for Z is

$$(a, b, d|d+1)$$

where $a, b \in \{1, 2\}$, $c \in \{1, 2, \ldots, z\}$, $d \in \{1, 2, \ldots, z-1\}$.

Whenever X moves into

$$(1|2, b, c)$$

Y and Z will reply as follows:

- If $b = 1$ and c is odd, then Y moves into
$$(1, 1|2, c+1)$$
and Z moves into
$$(2, 2, c|c+1)$$
- If $b = 1$ and c is even, then Y moves into
$$(1, 1|2, c-1)$$
and Z moves into
$$(2, 2, c-1|c)$$
- If $b = 2$ and c is odd, then Y moves into
$$(1, 1|2, c+1)$$
and Z moves into
$$(2, 1, c|c+1)$$
- If $b = 2$ and c is even, then Y moves into
$$(1, 1|2, c-1)$$
and Z moves into
$$(2, 1, c-1|c)$$

An example is shown in Fig. 2. ■

Theorem 2: Let $2 \times 2 \times z$ be a three-dimensional board with z even. If Y starts the game, then Y does not have a winning strategy.

Proof: Whenever Y moves into

$$(a, 1|2, c)$$

Z and X will reply as follows:

- If $a = 1$ and c is odd, then Z moves into
$$(2, 1, c|c+1)$$
and X moves into
$$(1|2, 2, c+1)$$
- If $a = 1$ and c is even, then Z moves into
$$(2, 1, c-1|c)$$
and X moves into
$$(1|2, 2, c-1)$$
- If $a = 2$ and c is odd, then Z moves into
$$(1, 1, c|c+1)$$
and X moves into
$$(1|2, 2, c+1)$$
- If $a = 2$ and c is even, then Z moves into
$$(1, 1, c-1|c)$$
and X moves into
$$(1|2, 2, c-1)$$

An example is shown in Fig. 3. ■

Theorem 3: Let $2 \times 2 \times z$ be a three-dimensional board with z even. If X starts the game, then Y does not have a winning strategy.

Proof: In the beginning, X move into

$$(1|2, 1, \overline{c})$$

with c odd. Whenever Y moves into

$$(a, 1|2, c)$$

with $c \neq \overline{c}$, Z and X will reply with the same strategy in Theorem 2. If Y moves into either

$$(1, 1|2, \overline{c}+1)$$

or

$$(2, 1|2, \overline{c}+1)$$

then Z moves respectively into either

$$(2, 2, \overline{c}|\overline{c}+1)$$

or

$$(1, 2, \overline{c}|\overline{c}+1)$$

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x	y	1	2	3	4		
1	2		X	Z	Z		...
2	2	Y	X		Y		...
1	1						...
2	1	Y	Z	Z	Y		...

Fig. 4. X does not have a winning strategy when Z starts the game (First case).

Successively, X and Z can continue with the same strategy used in the beginning of the game. ■

Theorem 4: Let $2 \times 2 \times z$ be a three-dimensional board with z even. If Z starts the game, then X does not have a winning strategy.

Proof: In the beginning, Z move into

$$(1, 1, \overline{c}|\overline{c} + 1)$$

with c odd. Whenever X moves into

$$(1|2, b, c)$$

with $c \neq \overline{c}, \overline{c} + 1$, Y and Z will reply with the same strategy in Theorem 1. If X moves into either

$$(1|2, 2, \overline{c})$$

or

$$(1|2, 2, \overline{c} + 1)$$

then Y moves respectively into either

$$(2, 1|2, \overline{c} + 1)$$

or

$$(2, 1|2, \overline{c})$$

Successively, Z and X can continue with the same strategy used in the beginning of the game. ■

Theorem 5: Let $2 \times 2 \times z$ be a three-dimensional board with z even and $z \geq 4$. If Y starts the game, then X does not have a winning strategy.

Proof: In the beginning, Y moves into

$$(2, 1|2, 1)$$

and Z moves into $(2, 1, 2|3)$. Whenever X moves into

$$(1|2, b, c)$$

with $c > 4$, Y and Z will reply with the same strategy in Theorem 1. If X moves into

$$(1|2, b, c)$$

with $c \leq 4$, then 4 different cases are possible:

1) If X moves into
$$(1|2, 2, 2)$$
then Y and Z will reply as shown in Fig. 4 and after X does not have a winning strategy by Theorem 1.
2) If X moves into
$$(1|2, 2, 3)$$

x	y	1	2	3	4		
1	2	Z	Z	X			...
2	2	Y		X	Y		...
1	1						...
2	1	Y	Z	Z	Y		...

Fig. 5. X does not have a winning strategy when Z starts the game (Second case).

x	y	1	2	3	4		
1	2		Y	Z	Z		...
2	2	Y					...
1	1		Y		X		...
2	1	Y	Z	Z	X		...

Fig. 6. X does not have a winning strategy when Z starts the game (Third case).

then Y and Z will reply as shown in Fig. 5 and after X does not have a winning strategy by Theorem 1.
3) If X moves into
$$(1|2, 1, 4)$$
then Y and Z will reply as shown in Fig. 6 and after X does not have a winning strategy by Theorem 1.
4) If X moves into
$$(1|2, 2, 4)$$
then Y and Z will reply as shown in Fig. 7. Successively, if X move into
$$(1|2, 1, 4)$$
then Y can reply into
$$(1, 1|2, 2)$$
and X does not have a winning strategy by Theorem 4. ■

Theorem 6: Let $2 \times 2 \times z$ be a three-dimensional board with z even and $z \geq 4$. If Z starts the game, then Y does not have a winning strategy.

Proof: In the beginning, Z moves into

$$(2, 1, 2|3)$$

and X moves into

$$(1|2, 1, 1)$$

x	y	1	2	3	4		
1	2	Y			X		...
2	2	Y	Z	Z	X		...
1	1	Y					...
2	1	Y	Z	Z			...

Fig. 7. X does not have a winning strategy when Z starts the game (Fourth case).

x	y	1	2	3	4		
1	2		Y	Z	Z		...
2	2						...
1	1	X	Y		X		...
2	1	X	Z	Z	X		...

Fig. 8. Y does not have a winning strategy when Z starts the game (First case).

x	y	1	2	3	4		
1	2	Z	Z	Y			...
2	2						...
1	1	X		Y	X		...
2	1	X	Z	Z	X		...

Fig. 9. Y does not have a winning strategy when Z starts the game (Second case).

Whenever Y moves into

$$(a, 1|2, c)$$

with $c > 4$, Z and X will reply with the same strategy in Theorem 2. If Y moves into

$$(a, 1|2, c)$$

with $c \leq 4$, then 4 different cases are possible:

1) If Y moves into
$$(1, 1|2, 2)$$
then Z and X will reply as shown in Fig. 8 and after Y does not have a winning strategy by Theorem 2.
2) If Y moves into
$$(1, 1|2, 3)$$
then Z and X will reply as shown in Fig. 9 and after Y does not have a winning strategy by Theorem 2.
3) If Y moves into
$$(2, 1|2, 4)$$
then Z and X will reply as shown in Fig. 10 and after Y does not have a winning strategy by Theorem 2.
4) If Y moves into
$$(1, 1|2, 4)$$

x	y	1	2	3	4		
1	2		X				...
2	2		X		Y		...
1	1	X		Z	Z		...
2	1	X	Z	Z	Y		...

Fig. 10. Y does not have a winning strategy when Z starts the game (Third case).

x	y	1	2	3	4		
1	2	X	Z	Z	Y		...
2	2	X					...
1	1	X			Y		...
2	1	X	Z	Z			...

Fig. 11. Y does not have a winning strategy when Z starts the game (Fourth case).

then Z and X will reply as shown in Fig. 11. Successively, if Y move into
$$(2, 1|2, 4)$$
then Z can reply into
$$(1, 1, 2|3)$$
and Y does not have a winning strategy by Theorem 3.

■

As a consequence of the previous 6 theorems, the following corollary holds.

Corollary 1: Let $2 \times 2 \times z$ be a three dimensional board with z even and $z \geq 4$. Then, either Z has a winning strategy or the game is queer.

Experimental results so far obtained seem indicate that the game is always queer but further efforts are necessary for a formal proof.

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