

# Three-Dimensional Generalized Thermoelasticity with Variable Thermal Conductivity

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**Abstract**—In this paper, a three-dimensional model of the generalized thermoelasticity with one relaxation time and variable thermal conductivity has been constructed. The resulting non-dimensional governing equations together with the Laplace and double Fourier transforms techniques have been applied to a three-dimensional half-space subjected to thermal loading with rectangular pulse and traction free in the directions of the principle co-ordinates. The inverses of double Fourier transforms, and Laplace transforms have been obtained numerically. Numerical results for the temperature increment, the invariant stress, the invariant strain, and the displacement are represented graphically. The variability of the thermal conductivity has significant effects on the thermal and the mechanical waves.

**Keywords**—Thermoelasticity, three-dimensional, Laplace transforms, Fourier transforms, thermal conductivity.

## I. INTRODUCTION

THE uncoupled thermoelasticity which is called the classical thermoelasticity theory, predicts that the phenomena are incompatible with the behavior of the thermoelastic materials. The heat conduction equation is separated and does not contain any elastic effect while the fact that the elastic changes lead to heat changes and the heat conduction equation is of the parabolic type which generates infinite speeds of propagation for heat waves.

Biot introduced the theory of coupled thermoelasticity (CTE) to fix the shortcoming in the uncoupled theory of thermoelasticity [1]. In the context of the CTE, the equations of motion are hyperbolic type, and heat conduction equation is of diffusion type which generates infinite speeds of thermal changes which are not compatible with the physical behavior. To fix the second paradox, many modifications of dynamic thermoelasticity theories were introduced by [2]-[5] based on second sound phenomena. Many problems based on the above theories have been solved [6]-[12].

State-space methods are one of the essentials of the modern control theory and it is the foundation for many studies in stochastic systems, nonlinear systems, and optimal control. There is no limit to the order as it works for any number of independent first-order differential equations [13]-[15].

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Moreover, this approach is useful because the linear systems based on time parameter could be analyzed as time-invariant linear systems and problems formulated could be programmed on a computer easily. High-order linear systems can be analyzed by state-space methods where multiple input and multiple output systems can be treated quickly. Solving thermoelastic problems by using the state-space approach in which the governing equations of the problem are rewritten in context of state-space variables, namely, the temperature increment, the displacement, or the strain and their gradients, has been developed by [13].

Godfrey has reported decreases of up to 45% in the thermal conductivity of various samples of silicon nitride between 1 and 400 °C. So, we have to know the effects of these variations on the stress and displacement distributions in metal components [16]. So, the temperature dependence of material properties must be taken into consideration in the thermal stress analysis of these elements. Many applications have been introduced assuming variable thermal conductivity [10], [17], [18].

## II. THE GOVERNING EQUATIONS

The governing equation of an isotropic and homogeneous elastic medium in the context of the generalized thermoelasticity with one relaxation time without any heat sources or any external body forces in general co-ordinates take the following form:

The equations of motion [7], [19]:

$$\rho \ddot{u}_i = (\lambda + \mu) u_{j,j} + \mu u_{i,jj} - \gamma \theta_{,i} \quad (1)$$

where  $\theta = (T - T_0)$  is the temperature increment and  $T_0$  is the reference temperature such that.

The heat equation [10], [18]:

$$(K(\theta) \theta_{,i})_{,i} = \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \left[ \frac{K(\theta) \dot{\theta}}{\kappa} + \gamma T_0 \dot{\epsilon} \right] \quad (2)$$

The constitutive relations in the form

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk} \delta_{ij} - \gamma \theta \delta_{ij} \quad (3)$$

The strain-displacement relation in the form

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (4)$$

where  $i, j = 1, 2, 3$  refer to generalized coordinates.

We will use the mapping [10], [19]:

$$\vartheta = \frac{1}{K_o} \int_0^\vartheta K(\xi) d\xi \quad (5)$$

where  $K_o$  is the thermal conductivity in the normal case.

Differentiating (5) with respect to the coordinates, we get

$$K_o \vartheta_{,i} = K(\theta) \theta_{,i} \quad (6)$$

Differentiating (6) again with respect to the coordinates, we obtain

$$K_o \vartheta_{,ii} = [K(\theta) \theta_{,i}]_{,i} \quad (7)$$

Differentiating (5) with respect to time, we get

$$K(\theta) \dot{\theta} = K_o \dot{\vartheta}, \quad \frac{\partial}{\partial t} [K(\theta) \dot{\theta}] = K_o \ddot{\vartheta} \quad (8)$$

Hence, we obtain

$$K_o \vartheta_{,ii} = \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \left[ \frac{K_o}{\kappa} \vartheta + \gamma T_o e \right] \quad (9)$$

We approximate the variation of the thermal diffusivity with temperature by the linear law [10], [18]:

$$K(\theta) = K_o (1 + K_1 \theta) \quad (10)$$

where  $K_1$  is a small constant is called parameter of the thermal conductivity change and it takes positive or negative values according to the material properties where the thermal conductivity of some materials increases and other decreases while heating.

Substituting from (10) into the mapping in (5), we get

$$\vartheta = \theta + \frac{K_1}{2} \theta^2 \quad (11)$$

which gives

$$\vartheta_{,i} = \theta_{,i} + K_1 \theta \theta_{,i} \quad (12)$$

Substitute in (1) and (3), we get

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} - \gamma \vartheta_{,i} + \gamma K_1 \theta \theta_{,i} = \rho \ddot{u}_i \quad (13)$$

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk} \delta_{ij} - \gamma \vartheta \delta_{ij} + \gamma \frac{K_1}{2} \theta^2 \delta_{ij} \quad (14)$$

For linearity, we will neglect the small nonlinear terms in

(13) and (14). Hence, we obtain

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} - \gamma \vartheta_{,i} = \rho \ddot{u}_i \quad (15)$$

and

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk} \delta_{ij} - \gamma \vartheta \delta_{ij} \quad (16)$$

### III. FORMULATION OF THE PROBLEM

Consider an isotropic, homogeneous and elastic body in three dimensional occupies the region  $\Omega$  which is defined by  $\Omega = \{x, y, z: 0 \leq x < \infty, -\infty < y < \infty, -\infty < z < \infty\}$  where the body is quiescent initially and has been shocked thermally moreover tractions free on the bounding of the surface  $x=0$  in the directions of the principal axis. The governing equations will be taken in the context of the generalized thermoelasticity when the body has no heat sources or any external forces. By using the Cartesian coordinates  $(x, y, z)$  and the components of the displacement  $u_i = (u, v, w)$ , we can write them as follows:

The equations of motion:

$$\rho \ddot{u} = (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + \mu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + (\lambda + \mu) \left( \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right) - \gamma \frac{\partial \vartheta}{\partial x} \quad (17)$$

$$\rho \ddot{v} = (\lambda + 2\mu) \frac{\partial^2 v}{\partial y^2} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} \right) + (\lambda + \mu) \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial z \partial y} \right) - \gamma \frac{\partial \vartheta}{\partial y} \quad (18)$$

$$\rho \ddot{w} = (\lambda + 2\mu) \frac{\partial^2 w}{\partial z^2} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + (\lambda + \mu) \left( \frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 v}{\partial z \partial y} \right) - \gamma \frac{\partial \vartheta}{\partial z} \quad (19)$$

The heat equation:

$$K_o \left( \frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2} + \frac{\partial^2 \vartheta}{\partial z^2} \right) = \frac{K_o}{\kappa} \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \vartheta + \gamma T_o \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \quad (20)$$

The constitutive relations are in the forms:

$$\sigma_{xx} = 2\mu e_{xx} + \lambda e - \gamma \vartheta \quad (21)$$

$$\sigma_{yy} = 2\mu e_{yy} + \lambda e - \gamma \vartheta \quad (22)$$

$$\sigma_{zz} = 2\mu e_{zz} + \lambda e - \gamma \vartheta \quad (23)$$

$$\sigma_{xy} = 2\mu e_{xy}, \quad \sigma_{xz} = 2\mu e_{xz}, \quad \sigma_{yz} = 2\mu e_{yz} \quad (24)$$

The strain-displacement relations are in the forms:

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{zz} = \frac{\partial w}{\partial z}, \quad (25)$$

$$e_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad e_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \quad e_{yz} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \quad (26)$$

and

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = e_{xx} + e_{yy} + e_{zz} \quad (27)$$

We can rewrite (17)-(19) by using (27) to be in the forms:

$$\rho \frac{\partial \ddot{u}}{\partial x} = \mu \nabla^2 \frac{\partial u}{\partial x} + (\lambda + \mu) \frac{\partial^2 e}{\partial x^2} - \gamma \frac{\partial^2 \mathcal{G}}{\partial x^2} \quad (28)$$

$$\rho \frac{\partial \ddot{v}}{\partial y} = \mu \nabla^2 \frac{\partial v}{\partial y} + (\lambda + \mu) \frac{\partial^2 e}{\partial y^2} - \gamma \frac{\partial^2 \mathcal{G}}{\partial y^2} \quad (29)$$

$$\rho \frac{\partial \ddot{w}}{\partial z} = \mu \nabla^2 \frac{\partial w}{\partial z} + (\lambda + \mu) \frac{\partial^2 e}{\partial z^2} - \gamma \frac{\partial^2 \mathcal{G}}{\partial z^2} \quad (30)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The heat equation takes the form:

$$K_0 \nabla^2 \mathcal{G} = \frac{K_0}{\kappa} \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \mathcal{G} + \gamma T_0 \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) e \quad (31)$$

Now, we will use the following dimensionless variables [7]:

$$(x', y', z') = c\eta(x, y, z), \quad (t', \tau_0') = c^2\eta(t, \tau_0), \quad \mathcal{G}' = \frac{\gamma \mathcal{G}}{(\lambda + 2\mu)},$$

$$\sigma'_{ij} = \frac{\sigma_{ij}}{\lambda + 2\mu}, \quad c = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \eta = \frac{1}{\kappa}$$

Applying the above dimensionless variables, we obtain

$$\frac{\partial \ddot{u}}{\partial x} = \beta \nabla^2 \frac{\partial u}{\partial x} + (1 - \beta) \frac{\partial^2 e}{\partial x^2} - \frac{\partial^2 \mathcal{G}}{\partial x^2} \quad (32)$$

$$\frac{\partial \ddot{v}}{\partial y} = \beta \nabla^2 \frac{\partial v}{\partial y} + (1 - \beta) \frac{\partial^2 e}{\partial y^2} - \frac{\partial^2 \mathcal{G}}{\partial y^2} \quad (33)$$

$$\frac{\partial \ddot{w}}{\partial z} = \beta \nabla^2 \frac{\partial w}{\partial z} + (1 - \beta) \frac{\partial^2 e}{\partial z^2} - \frac{\partial^2 \mathcal{G}}{\partial z^2} \quad (34)$$

$$\nabla^2 \mathcal{G} = \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \mathcal{G} + \varepsilon \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) e \quad (35)$$

$$\sigma_{xx} = 2\beta e_{xx} + (1 - 2\beta)e - \mathcal{G} \quad (36)$$

$$\sigma_{yy} = 2\beta e_{yy} + (1 - 2\beta)e - \mathcal{G} \quad (37)$$

$$\sigma_{zz} = 2\beta e_{zz} + (1 - 2\beta)e - \mathcal{G} \quad (38)$$

$$\sigma_{xy} = 2\beta e_{xy}, \quad \sigma_{xz} = \beta e_{xz}, \quad \sigma_{yz} = 2\beta e_{yz} \quad (39)$$

where

$$\beta = \frac{\mu}{\lambda + 2\mu}, \quad \varepsilon = \frac{\gamma^2 T_0 K_0}{\kappa(\lambda + 2\mu)}.$$

For simplicity, we dropped the primes.

By summing (32)-(34) and using (27), we get

$$\ddot{e} = \nabla^2 e - \nabla^2 \mathcal{G} \quad (40)$$

We will consider the invariant stress  $\sigma$  to be the mean value of the principal stresses as follows:

$$\sigma = \frac{1}{3} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \quad (41)$$

By using (36)-(38), we obtain

$$\sigma = \alpha e - \mathcal{G} \quad (42)$$

where

$$\alpha = 1 - \frac{4}{3}\beta.$$

Using Laplace transform defined for any function  $g(t)$  as:

$$\bar{g}(s) = \int_0^\infty g(t) e^{-st} dt \quad (43)$$

Applying the above transform for both sides of the above equations, we obtain

$$s^2 \frac{\partial \bar{u}}{\partial x} = s^2 \bar{e}_{xx} = \beta \nabla^2 \frac{\partial \bar{u}}{\partial x} + (1 - \beta) \frac{\partial^2 \bar{e}}{\partial x^2} - \frac{\partial^2 \bar{\mathcal{G}}}{\partial x^2} \quad (44)$$

$$s^2 \frac{\partial \bar{v}}{\partial y} = s^2 \bar{e}_{yy} = \beta \nabla^2 \frac{\partial \bar{v}}{\partial y} + (1 - \beta) \frac{\partial^2 \bar{e}}{\partial y^2} - \frac{\partial^2 \bar{\mathcal{G}}}{\partial y^2} \quad (45)$$

$$s^2 \frac{\partial \bar{w}}{\partial z} = s^2 \bar{e}_{zz} = \beta \nabla^2 \frac{\partial \bar{w}}{\partial z} + (1-\beta) \frac{\partial^2 \bar{e}}{\partial z^2} - \frac{\partial^2 \bar{g}}{\partial z^2} \quad (46)$$

$$s^2 \bar{e} = \nabla^2 \bar{e} - \nabla^2 \bar{g} \quad (47)$$

$$\nabla^2 \bar{g} = (s + \tau_0 s^2) \bar{g} + \varepsilon_1 (s + \tau_0 s^2) \bar{e} \quad (48)$$

$$\bar{\sigma}_{xx} = 2\beta \bar{e}_{xx} + (1-2\beta) \bar{e} - \bar{g} \quad (49)$$

$$\bar{\sigma}_{yy} = 2\beta \bar{e}_{yy} + (1-2\beta) \bar{e} - \bar{g} \quad (50)$$

$$\bar{\sigma}_{zz} = 2\beta \bar{e}_{zz} + (1-2\beta) \bar{e} - \bar{g} \quad (51)$$

$$\bar{\sigma}_{xy} = 2\beta \bar{e}_{xy}, \quad \bar{\sigma}_{xz} = \beta \bar{e}_{xz}, \quad \bar{\sigma}_{yz} = \beta \bar{e}_{yz} \quad (52)$$

$$\bar{\sigma} = \alpha \bar{e} - \bar{g} \quad (53)$$

and the initial conditions of the state functions are

$$\begin{aligned} u(x, y, z, t) \Big|_{t=0} &= v(x, y, z, t) \Big|_{t=0} = w(x, y, z, t) \Big|_{t=0} = 0 \\ \frac{\partial u(x, y, z, t)}{\partial t} \Big|_{t=0} &= \frac{\partial v(x, y, z, t)}{\partial t} \Big|_{t=0} = \frac{\partial w(x, y, z, t)}{\partial t} \Big|_{t=0} = 0 \\ \mathcal{G}(x, y, z, t) \Big|_{t=0} &= \frac{\partial \mathcal{G}(x, y, z, t)}{\partial t} \Big|_{t=0} = 0 \end{aligned} \quad (54)$$

By eliminating  $\bar{e}$  between (47), (48) and (53), we get

$$\nabla^2 \bar{\sigma} = \alpha_1 \bar{g} + \alpha_2 \bar{\sigma} \quad (55)$$

and

$$\nabla^2 \bar{\theta} = \alpha_3 \bar{g} + \alpha_4 \bar{\sigma} \quad (56)$$

where

$$\begin{aligned} \alpha_1 &= \frac{s^2 \alpha - (s + \tau_0 s^2)(1-\alpha)(\alpha + \varepsilon)}{\alpha}, \\ \alpha_2 &= \frac{s^2 \alpha - \varepsilon(1-\alpha)(s + \tau_0 s^2)}{\alpha}, \quad \alpha_3 = \frac{(s + \tau_0 s^2)(\alpha + \varepsilon)}{\alpha}, \\ \alpha_4 &= \frac{\varepsilon(s + \tau_0 s^2)}{\alpha}. \end{aligned}$$

Using double Fourier transform for any function  $f(x, y, z)$  which is defined as follows [7]:

$$\begin{aligned} F[\bar{f}(x, y, z, s)] &= \tilde{\bar{f}}(x, p, q, s) = \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f}(x, y, z, s) e^{-i(py+qz)} dy dz \end{aligned} \quad (57)$$

where the inversion transform of the double Fourier transform takes the form

$$\begin{aligned} F^{-1}[\tilde{\bar{f}}(x, p, q, s)] &= \bar{f}(x, y, z, s) = \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\bar{f}}(x, p, q, s) e^{i(py+qz)} dp dq \end{aligned} \quad (58)$$

Thus, we have

$$F[\nabla^2 \bar{f}(x, y, z, s)] = \left( \frac{d^2}{dx^2} - q^2 - p^2 \right) \tilde{\bar{f}}(x, p, q, s). \quad (59)$$

Applying the above transform, we have the following system of ordinary differential equations

$$\frac{d^2 \tilde{\bar{\sigma}}}{dx^2} = \alpha_1 \tilde{\bar{g}} + \beta_1 \tilde{\bar{\sigma}} \quad (60)$$

and

$$\frac{d^2 \tilde{\bar{\theta}}}{dx^2} = \beta_2 \tilde{\bar{g}} + \alpha_4 \tilde{\bar{\sigma}} \quad (61)$$

where  $\beta_1 = q^2 + p^2 + \alpha_2$  and  $\beta_2 = q^2 + p^2 + \alpha_3$

#### IV. STATE SPACE FORMULATION

We shall take as state variables in the physical domain the quantities  $\tilde{\bar{g}}, \tilde{\bar{\sigma}}, \frac{d\tilde{\bar{g}}}{dx}$ , and  $\frac{d\tilde{\bar{\sigma}}}{dx}$ .

Regarding the Laplace transform of these four variables, the equations can be written as [13], [14]:

$$\frac{d}{dx} \begin{bmatrix} \tilde{\bar{\sigma}} \\ \tilde{\bar{g}} \\ \tilde{\bar{\sigma}}' \\ \tilde{\bar{g}}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta_1 & \alpha_1 & 0 & 0 \\ \alpha_4 & \beta_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\bar{\sigma}} \\ \tilde{\bar{g}} \\ \tilde{\bar{\sigma}}' \\ \tilde{\bar{g}}' \end{bmatrix} \quad (62)$$

which takes the form

$$\frac{d \tilde{\bar{V}}(x, p, q, s)}{dx} = A(p, q, s) \tilde{\bar{V}}(x, p, q, s) \quad (63)$$

where  $\tilde{\bar{V}}(x, p, q, s) = \begin{bmatrix} \tilde{\bar{\sigma}} & \tilde{\bar{g}} & \tilde{\bar{\sigma}}' & \tilde{\bar{g}}' \end{bmatrix}^{Transpos}$ , is the state vector in the Laplace and Fourier transform domain and  $A = A(p, q, s)$  is the  $4 \times 4$  matrix.

The formal solution of (63) is given by:

$$\tilde{\bar{V}}(x, p, q, s) = \exp[A(p, q, s)x] \tilde{\bar{V}}(0, p, q, s) \quad (64)$$

The characteristic equation of matrix A takes the form

$$k^4 - L k^2 + M = 0 \quad (65)$$

where  $L = \beta_1 + \beta_2$  and  $M = \beta_1\beta_2 - \alpha_1\alpha_4$ , and satisfy the relations

$$k_1^2 + k_2^2 = \beta_1 + \beta_2, \quad k_1^2 k_2^2 = \beta_1\beta_2 - \alpha_1\alpha_4 \quad (66)$$

$\pm k_1^2$  and  $\pm k_2^2$  are the roots of the characteristic equation (65).

The Taylor series expansion for the matrix exponential in (64) is given by:

$$\exp[A(s)x] = \sum_{n=0}^{\infty} \frac{[A(s)x]^n}{n!} \quad (67)$$

By using the Cayley-Hamilton theorem, this infinite series can be reduced to

$$\exp[A(p,q,s)x] = a_0(p,q,s)I + a_1(p,q,s)A(s) + a_2(p,q,s)A^2(s) + a_3(p,q,s)A^3(s) \quad (68)$$

where I is the identity matrix of order 4.

To determine the coefficient,  $a_0, a_1, a_2, a_3$  we use the Cayley-Hamilton theorem again as follows:

$$\exp(k_1 x) = a_0 + a_1 k_1 + a_2 k_1^2 + a_3 k_1^3 \quad (69)$$

$$\exp(-k_1 x) = a_0 - a_1 k_1 + a_2 k_1^2 - a_3 k_1^3 \quad (70)$$

$$\exp(k_2 x) = a_0 + a_1 k_2 + a_2 k_2^2 + a_3 k_2^3 \quad (71)$$

$$\exp(-k_2 x) = a_0 - a_1 k_2 + a_2 k_2^2 - a_3 k_2^3 \quad (72)$$

The solution of the above system is given by

$$a_0 = \frac{k_1^2 \cosh k_2 x - k_2^2 \cosh k_1 x}{k_1^2 - k_2^2} \quad (73)$$

$$a_1 = \frac{(k_1^2 / k_2) \sinh k_2 x - (k_2^2 / k_1) \sinh k_1 x}{k_1^2 - k_2^2} \quad (74)$$

$$a_2 = \frac{\cosh k_1 x - \cosh k_2 x}{k_1^2 - k_2^2} \quad (75)$$

$$a_3 = \frac{(\sinh k_2 x) / k_1 - (\sinh k_1 x) / k_2}{k_1^2 - k_2^2} \quad (76)$$

Substituting (73)-(76) into (68) and computing  $A^2$  and  $A^3$  we obtain after some lengthy algebraic manipulations the following:

$$\exp[A(p,q,s)x] = L_{ij}(x,p,q,s), \quad i, j = 1, 2, 3, 4 \quad (77)$$

where the elements  $L_{ij}(x,p,q,s)$  are given by:

$$\begin{aligned} L_{11} &= a_0 + \beta_1 a_2, \quad L_{12} = \alpha_1 a_2, \quad L_{13} = a_1 + \beta_1 a_3, \quad L_{14} = \alpha_1 a_3 \\ L_{21} &= \alpha_4 a_2, \quad L_{22} = a_0 + \beta_2 a_2, \quad L_{23} = \alpha_4 a_3, \quad L_{24} = a_1 + \beta_2 a_3 \\ L_{31} &= \beta_1 a_1 + (\alpha_1 \alpha_4 + \beta_1^2) a_3, \quad L_{32} = \alpha_1 a_1 + \alpha_1 (\beta_1 + \beta_2) a_3, \\ L_{33} &= a_0 + \beta_1 a_2, \quad L_{34} = \alpha_1 a_2, \quad L_{41} = \alpha_4 a_1 + \alpha_4 (\beta_1 + \beta_2) a_3, \\ L_{42} &= \beta_2 a_1 + (\alpha_1 \alpha_4 + \beta_2^2) a_3, \quad L_{43} = \alpha_4 a_2, \quad L_{44} = a_0 + \beta_2 a_2 \end{aligned} \quad (78)$$

Hence, we obtain the solution in the form

$$\tilde{V}(x,p,q,s) = L_{ij}(x,p,q,s) \tilde{V}(0,p,q,s) \quad (79)$$

Since the solution is bounded at infinity, the expressions for  $L_{ij}(i, j = 1, 2, 3, 4)$  can be obtained by suppressing the positive exponential in (73)-(76) which is equivalent to replacing each  $\cosh k_i x$  by  $\left(\frac{1}{2} \exp(-k_i x)\right)$  and  $\sinh k_i x$  by  $\left(-\frac{1}{2} \exp(-k_i x)\right)$ .

To complete the solution we have to know the vector matrix  $\tilde{V}(0,p,q,s)$ , so we have to apply certain boundary conditions, so we consider that the bounding plane to the surface  $x=0$  has no traction on the principal axis ( $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$ ) and thermally shocked, which gives

$$\tilde{\sigma}(0,q,p,s) = 0 \quad (80)$$

and

$$\theta(0,y,z,t) = \theta_0 H(t) G(y,z) \quad (81)$$

where  $\theta_0$  (constant) is the intensity of the thermal shock and  $H(t)$  is the Heaviside unit step function. Thus, from (11) and (81), we have

$$\mathcal{G}(0,y,z,t) = \theta_0 H(t) G(y,z) + \frac{K_1}{2} \theta_0^2 H(t) H(t) G^2(y,z) \quad (82)$$

After using the Laplace and the double Fourier transforms, we obtain

$$\tilde{\mathcal{G}}(0,p,q,s) = \mathcal{G}_0 = \frac{\theta_0}{s} \tilde{G}(p,q) + \frac{K_1 \theta_0^2}{2s} \tilde{G}_1(p,q) \quad (83)$$

where  $\tilde{G}_1(p,q)$  is the double Fourier transforms  $G^2(y,z)$ . Hence, we have

$$\tilde{V}(0,p,q,s) = \begin{bmatrix} 0 & \mathcal{G}_0 & \tilde{\sigma}'(0,p,q,s) & \tilde{\mathcal{G}}'(0,p,q,s) \end{bmatrix}^{Transpos} \quad (84)$$

To get  $\tilde{\sigma}'(0, p, q, s)$  and  $\tilde{g}'(0, p, q, s)$ , we use (79) when  $x = 0$  as:

$$\begin{bmatrix} 0 \\ \mathcal{G}_0 \\ \tilde{\sigma}'(0, p, q, s) \\ \tilde{g}'(0, p, q, s) \end{bmatrix} = L_y(0, p, q, s) \begin{bmatrix} 0 \\ \mathcal{G}_0 \\ \tilde{\sigma}'(0, p, q, s) \\ \tilde{g}'(0, p, q, s) \end{bmatrix} \quad (85)$$

After simplifications, we get

$$\tilde{\sigma}'(0, p, q, s) = \frac{\alpha_1 \mathcal{G}_0}{(k_1 + k_2)}, \quad \tilde{g}'(0, p, q, s) = \frac{(\beta_2 + k_1 k_2) \mathcal{G}_0}{(k_1 + k_2)} \quad (86)$$

Then, we have

$$\tilde{V}(0, p, q, s) = \mathcal{G}_0 \begin{bmatrix} 0 & 1 & \frac{\alpha_1}{(k_1 + k_2)} & \frac{(\beta_2 + k_1 k_2)}{(k_1 + k_2)} \end{bmatrix}^{T_{Transpos}} \quad (87)$$

Thus, we can write the final solutions in the Laplace and Fourier transforms domain as:

$$\tilde{g}(x, p, q, s) = \frac{\mathcal{G}_0}{(k_1^2 - k_2^2)} \left[ (k_1^2 - \beta_1) e^{-k_1 x} - (k_2^2 - \beta_1) e^{-k_2 x} \right] \quad (88)$$

$$\tilde{\sigma}(x, p, q, s) = \frac{\alpha_1 \mathcal{G}_0}{(k_1^2 - k_2^2)} \left[ e^{-k_1 x} - e^{-k_2 x} \right] \quad (89)$$

$$\tilde{e}(x, p, q, s) = \frac{\mathcal{G}_0}{(k_1^2 - k_2^2) \alpha} \left[ \gamma_1 e^{-k_1 x} - \gamma_2 e^{-k_2 x} \right] \quad (90)$$

where  $\gamma_1 = k_1^2 - \beta_1 + \alpha_1$  and  $\gamma_2 = k_2^2 - \beta_1 + \alpha_1$ . To get one of the components of displacement  $\tilde{u}(x, p, q, s)$ , we write (44) after using the double Fourier transform in the form:

$$\left( \frac{d^2}{dx^2} - \lambda_u^2 \right) \tilde{u} = \frac{1}{\beta} \frac{\partial \tilde{g}}{\partial x} - \frac{(1-\beta)}{\beta} \frac{\partial \tilde{e}}{\partial x} \quad (91)$$

where  $\lambda_u^2 = p^2 + q^2 + \frac{s^2}{\beta}$ . Substituting from (88) and (90),

we get

$$\left( \frac{d^2}{dx^2} - \lambda_u^2 \right) \tilde{u} = \ell_1 e^{-k_1 x} - \ell_2 e^{-k_2 x} \quad (92)$$

where

$$\ell_1 = \frac{k_1 \mathcal{G}_0}{(k_1^2 - k_2^2) \beta} \left[ \frac{(1-\beta) \gamma_1}{\alpha} - (k_1^2 - \beta_1) \right] \text{ and}$$

$$\ell_2 = \frac{k_2 \mathcal{G}_0}{(k_1^2 - k_2^2) \beta} \left[ \frac{(1-\beta) \gamma_2}{\alpha} - (k_2^2 - \beta_1) \right]$$

The solution of the differential equation (92) takes the form

$$\tilde{u}(x, p, q, s) = C e^{-\lambda_u x} + \frac{\ell_1}{k_1^2 - \lambda_u^2} e^{-k_1 x} - \frac{\ell_2}{k_2^2 - \lambda_u^2} e^{-k_2 x} \quad (93)$$

where  $k_1^2 \neq k_2^2 \neq \lambda_u^2$  and C is constant and has to be determined.

From (49) and (53) and after applying the double Fourier transform, we have

$$2\beta \frac{\partial \tilde{u}}{\partial x} = \tilde{\sigma}_{xx} - \frac{(1-2\beta)}{\alpha} \tilde{\sigma} - \left[ \frac{(1-2\beta)}{\alpha} - 1 \right] \tilde{g} \quad (94)$$

When  $x = 0$  and by using the boundary conditions (80) and (83), we get

$$\left. \frac{\partial \tilde{u}(x, q, p, s)}{\partial x} \right|_{x=0} = \frac{(\alpha + 2\beta - 1) \mathcal{G}_0}{2\alpha \beta} \quad (95)$$

Applying the condition in (95) into (93); we get the constant C in the following form

$$C = \frac{1}{\lambda_u} \left[ \frac{(1-\alpha-2\beta) \mathcal{G}_0}{2\alpha \beta} - \frac{k_1 \ell_1}{(k_1^2 - \lambda_u^2)} + \frac{k_2 \ell_2}{(k_2^2 - \lambda_u^2)} \right]$$

which gives the displacement component  $\tilde{u}$  in the transformed domain as follows:

$$\tilde{u}(x, p, q, s) = \frac{1}{\lambda_u} \left[ \frac{(1-\alpha-2\beta) \mathcal{G}_0}{2\alpha \beta} - \frac{k_1 \ell_1}{(k_1^2 - \lambda_u^2)} + \frac{k_2 \ell_2}{(k_2^2 - \lambda_u^2)} \right] e^{-\lambda_u x} + \frac{\ell_1}{k_1^2 - \lambda_u^2} e^{-k_1 x} - \frac{\ell_2}{k_2^2 - \lambda_u^2} e^{-k_2 x} \quad (96)$$

## V. INVERSION OF THE LAPLACE AND THE DOUBLE FOURIER TRANSFORMS

To obtain the solution of the problem in the physical domain, we have to invert the double Fourier and Laplace transforms in (88)-(90) and (96). These expressions may be formally expressed as functions of x, and the parameters of the Fourier and Laplace transforms  $q$ ,  $p$ , and  $s$ , of the form  $\tilde{f}(x, q, p, s)$ .

First, we invert the double Fourier transform using the inversion formula which is given in (58). This gives the expression  $\bar{f}(x, y, z, s)$  in the Laplace transform domain.

The function  $G(y, z)$  was taken in the form of rectangular pulse as:

$$G(y, z) = f(y)g(z) \quad (97)$$

where

$$f(y) = \begin{cases} 1 & -a \leq y \leq a \\ 0 & |y| > a \end{cases}, g(z) = \begin{cases} 1 & -b \leq z \leq b \\ 0 & |z| > a \end{cases} \quad (98)$$

where  $a$  and  $b$  are constants. It means, the rectangular thermal pulse acts on a band of width  $2a$  centered around the  $y$ -axis and  $2b$  centered around the  $z$ -axis on the surface of the half-space  $x=0$  and is zero elsewhere as in Fig. 1. Applying double Fourier transform to the functions in (98), we get

$$\tilde{G}(q, p) = \tilde{G}_1(p, q) = \frac{2}{\pi} \frac{\sin(qa)\sin(pb)}{qp} \quad (99)$$

which gives

$$g_0 = \frac{\theta_0(2 + K_1\theta_0)\sin(qa)\sin(pb)}{\pi sp} \quad (100)$$

To get the inversion of the Laplace transform, the Riemann-sum approximation method is used. In this method, any function in Laplace domain may be inverted to the time domain as [20]:

$$g(t) = \frac{e^{\kappa t}}{t} \left[ \frac{1}{2} \bar{g}(\kappa) + \text{Re} \sum_{n=1}^N (-1)^n \bar{g} \left( \kappa + \frac{in\pi}{t} \right) \right] \quad (101)$$

where  $\text{Re}$  is the real part and  $i$  is imaginary number unit. For faster convergence, numerous numerical experiments have shown that the value of  $\kappa$  satisfies the relation  $\kappa t \approx 4.7$  [20]. After obtaining  $\mathcal{G}(x, y, z, t)$ , we can get the original temperature increment  $\theta(x, y, z, t)$  by solving (11) as [10], [18]:

$$\theta(x, y, z, t) = \frac{-1 + \sqrt{1 + 2K_1 \mathcal{G}(x, y, z, t)}}{K_1} \quad (102)$$

## VI. NUMERICAL RESULTS AND DISCUSSION

The copper material was chosen for purposes of numerical evaluations, and the constants of the problem were taken as [7], [10], [18]:

$$K = 386 \text{ N / K sec}, \alpha_t = 1.78 \times 10^{-5} \text{ K}^{-1}, C_E = 383.1 \text{ m}^2 / \text{K}, \\ \eta = 8886.73 \text{ m / sec}^2, T_0 = 293 \text{ K}, \mu = 3.86 \times 10^{10} \text{ N / m}^2, \\ \lambda = 7.76 \times 10^{10} \text{ N / m}^2, \rho = 8954 \text{ kg/m}^3, \tau_0 = 0.33 \times 10^{-15} \text{ sec}.$$

Thus, we get the following non-dimensional parameters

$$\tau_0 = 0.002, \beta = 0.25, \alpha = 0.67, \varepsilon = 0.0168.$$

The computations were carried out for non-dimensional time  $t = 0.1$ , length  $a = b = 1.0$ , and thermal shock intensity  $\theta_0 = 5.0$ . The temperature, the stress, the strain, and the displacement distributions are represented graphically.

Figs. 1-4 represent the temperature increment distribution, the invariant stress distribution, the invariant strain distribution, and the displacement component  $u$  distribution. Various values of parameter of the thermal conductivity  $K_1 = (-0.1, 0.0, 1.0)$  based on a wide range of distance  $x (0 \leq x \leq 1)$  when  $y = z = 0.0$  and  $y = z = 0.5$ , respectively, to stand on the effects of the parameter  $K_1$ . The parameter  $K_1$  has a significant effect on the thermal and the mechanical waves. The absolute values of all the state functions increase when the value of  $K_1$  increases for both positions  $y = z = 0.0$  and  $y = z = 0.5$ . In three dimensions, Figs. 5-8 represent the temperature increment distribution, the invariant stress distribution, the invariant strain distribution, and the displacement component  $u$  distribution based on a wide range of distance  $x (0 \leq x \leq 1)$  and  $K_1 (-1 \leq K_1 \leq 1)$  when  $y = z = 0.0$ . The parameter  $K_1$  has a significant effect on the thermal and the mechanical waves. The absolute values of all the state functions increase when the value of  $K_1$  increases.

## VII. CONCLUSION

A three-dimensional model based on generalized thermoelasticity theorem with one relaxation time and variable thermal conductivity has been constructed and solved. The variability of the thermal conductivity has significant effects on the mechanical and the thermal waves.

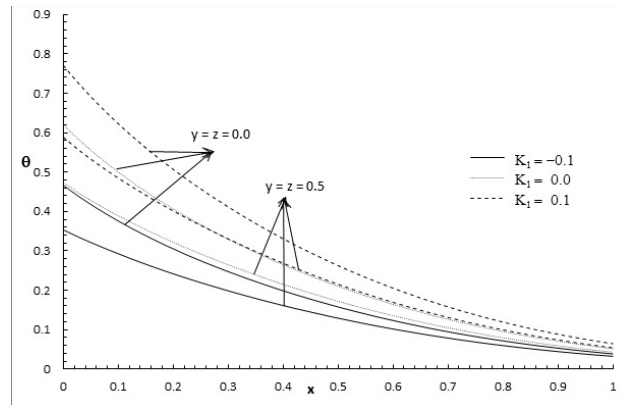


Fig. 1 The temperature increment distribution with various cases of thermal conductivity

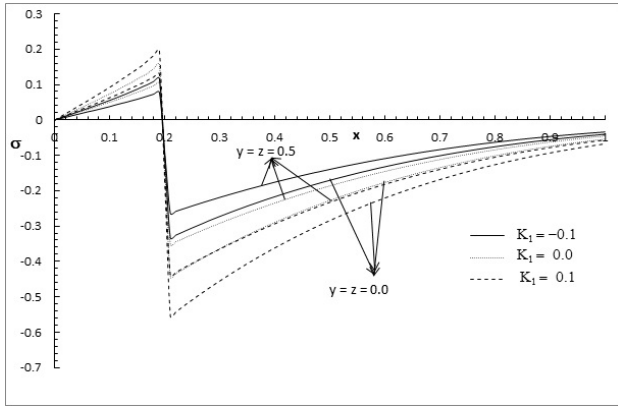


Fig. 2 The invariant stress distribution with various cases of thermal conductivity

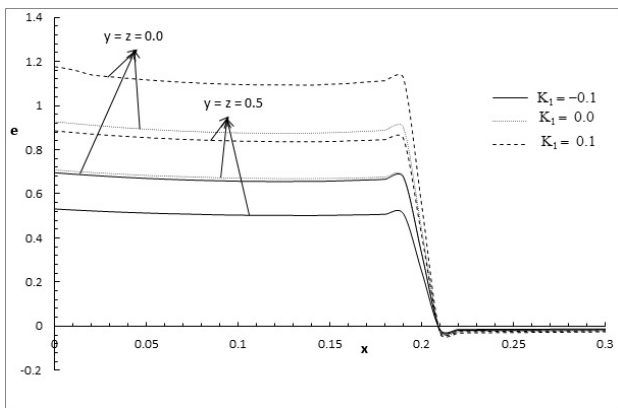


Fig. 3 The invariant strain distribution with various cases of thermal conductivity

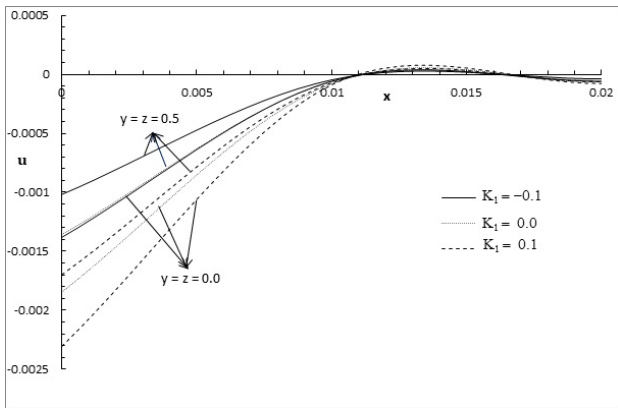


Fig. 4 The displacement component u distribution with various cases of thermal conductivity

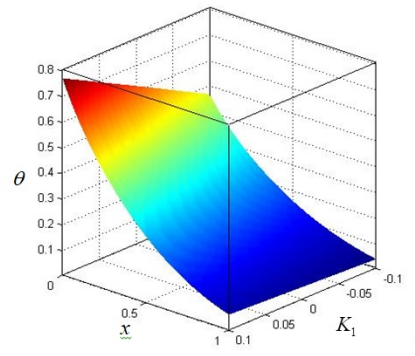


Fig. 5 The temperature increment distribution based on different value of  $K_1$ ,  $y = z = 0.0$

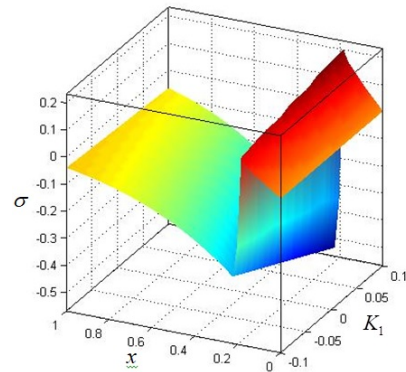


Fig. 6 The invariant stress distribution based on different value of  $K_1$  and  $y = z = 0.0$

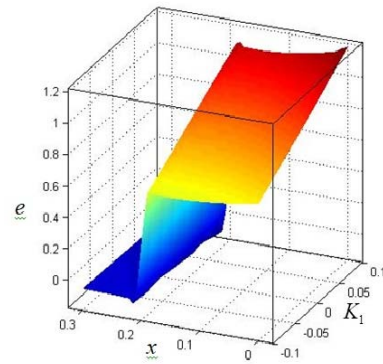


Fig. 7 The invariant strain distribution based on different value of  $K_1$  and  $y = z = 0.0$



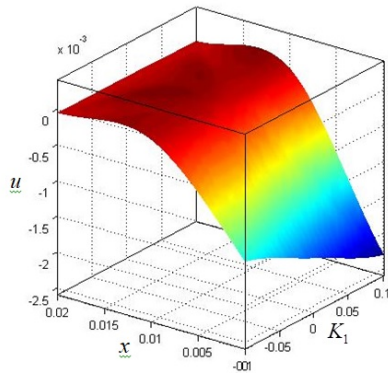


Fig. 8 The displacement component  $u$  distribution based on different value of  $K_1$  and  $y = z = 0.0$

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