

# Theoretical Analysis of Capacities in Dynamic Spatial Multiplexing MIMO Systems

Imen Sfaihi, and Nouredine Hamdi

**Abstract**—In this paper, we investigate the study of techniques for scheduling users for resource allocation in the case of multiple input and multiple output (MIMO) packet transmission systems. In these systems, transmit antennas are assigned to one user or dynamically to different users using spatial multiplexing. The allocation of all transmit antennas to one user cannot take full advantages of multi-user diversity. Therefore, we developed the case when resources are allocated dynamically. At each time slot users have to feed back their channel information on an uplink feedback channel. Channel information considered available in the schedulers is the zero forcing (ZF) post detection signal to interference plus noise ratio. Our analysis study concerns the round robin and the opportunistic schemes.

In this paper, we present an overview and a complete capacity analysis of these schemes. The main results in our study are to give an analytical form of system capacity using the ZF receiver at the user terminal. Simulations have been carried out to validate all proposed analytical solutions and to compare the performance of these schemes.

**Keywords**—MIMO, scheduling, ZF receiver, spatial multiplexing, round robin scheduling, opportunistic.

## I. INTRODUCTION

It is known that multiple input and multiple output (MIMO) antennas enhance the capacity of wireless packet communication systems [1] [2] [3] [4]. In these systems, the allocation of transmit antennas to users can be in different ways. These antennas can be used in spatial diversity (SD) [1] or in spatial multiplexing (SM) [5] [6] [7] schemes. The receiver of each user estimates the channel state information (CSI) measured as the post detection signal to interference plus noise ratio. Detectors studied in this work are being zero-forcing (ZF) [8] [9] or minimum mean square error (MMSE) [10]. We develop in this study the system throughputs if, ZF is employed, and analytic solutions can be extended to system with other detectors.

Each user feeds back to the node station the CSI estimates on an uplink channel. With the SD schemes, the same data

stream is repeatedly sent through all the transmit antennas. This scheme would transform the transmitting medium to a rich fading environment. The SM schemes enhance the system throughput by multiplexing data streams on the transmit antennas. Therefore the downlink capacity would be enhanced compared to the SD.

The scheduler can assign (i) all  $M_t$  transmit antennas to one user or (ii) each transmit antenna to different users at each time slot. The last scheme allocates each antenna dynamically to users; hence the system resources are shared between users at each time slot. The Schedulers in these schemes would maximize system capacity by allocating each transmit antenna to user that experiences peak level SINR on that antenna. Hence in MIMO systems based on SM, each spatial channel or transmit antenna can be assigned to different users at each time slot. The use of schedulers as round robin (RR) and opportunistic depends on how the designer considers: system rate maximizing, fairness among users or a trade-off between the two goals [11] [12]. In this paper, we investigate the effective capacity using these different schedulers relative to the downlink channel of a single cell MIMO system.

This paper is organized as follows: section II describes the system and channel models. In Section III, we give the analytical expression of the system capacity. In section VI, we develop the post detection signal to interference plus noise ratio (SINR) using the ZF receiver and their distribution function. In section V, different schedulers are studied, the system capacity in analytical form is given and simulation results are discussed. In section IV, we compare the performance of the round robin scheduling and the opportunistic.

## II. ASSUMPTIONS AND SYSTEM MODEL

The considered scheme is a multi-user MIMO wireless packet transmission system. We consider the downlink transmission of a single cell system comprising a node station and  $N$  MIMO users. The node station is equipped with  $M_t$  transmit antennas, and each user terminal with  $M_r$  receive antennas. The node station serves the  $N$  active users in a time division multiplex (TDM) fashion. It is assumed that slow power control is employed to equally sharing the total transmitted power  $P_t$  on all transmit antennas at the transmitter. User data packets are loaded on transmit antennas

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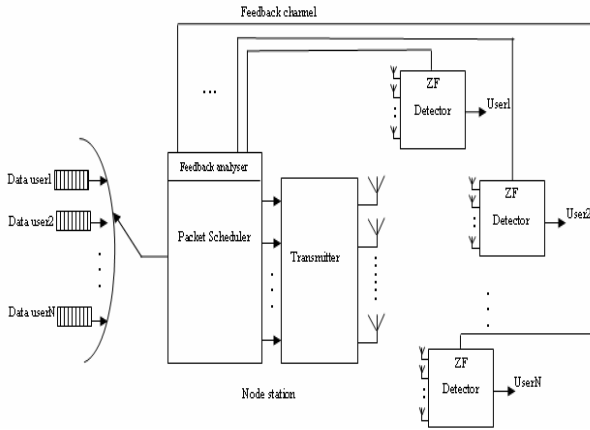


Fig. 1 Cellular system with multiple transmits and receives antennas

using spatial multiplexing (SM) technique. The ZF detector is considered in this study. The users' estimates of supportable rates are feed back to the node station through an uplink feedback channel as shown in Fig. 1.

It is assumed that transmit signals experience path loss, log-normal shadow fading, and multi-path fading. The CSI is measured by matrix  $H_k(t)$  which represents the short term fading CSI on all branches from the node station to the  $k^{th}$  user during the a time slot. According to the slow power control, (i)each entry of the matrix  $H_k$  is an independent and identically distributed complex Gaussian random variable  $CN(0; 1)$  representing short term fading; (ii) the CSI experiences flat fading during each time slot, and varies independently over slots.

We denote by  $H_k(t)$  the  $M_r \times M_t$  channel matrix,  $s(t)$  the  $M_t \times 1$  transmitted signal,  $n_k(t)$   $M_r \times 1$  the additive white Gaussian noise (AWGN) vector with distribution  $CN\left(0; \frac{N_0}{2} I_{M_r}\right)$  for each element, and  $y(t)$  the  $M_r \times 1$  received signal.

Where  $I_{M_r}$  is the identity matrix with dimension  $M_r$ .

Then, the received signal for the considered multi-user MIMO system in the slot  $t$  is represented as

$$y_k(t) = \sqrt{\frac{P_t}{M_t}} H_k(t) s(t) + n_k(t), k = 1 \dots N \quad (1)$$

### III. ANALYTIC EXPRESSION OF CAPACITY IN SM SYSTEMS

If we consider the assumptions and the system model explained before, the correspondent capacity is given by

$$C = \log_2 \left[ \det \left( I_r + \frac{\bar{\gamma}}{M_t} H H^H \right) \right] \quad (2)$$

Where  $\bar{\gamma}$  is the average SNR at each of the receive antennas and  $I_r$  is the  $M_r \times M_r$  identity matrix.

Equation (2) is equivalent to

$$C = \sum_{m=1}^{M_t} \log_2 \left( 1 + \frac{\bar{\gamma}}{M_t} \lambda_m \right) \quad (3)$$

Where  $\lambda_m (m=1, 2, \dots, M_t)$  are the positive eigen values of  $H H^H$ .

### IV. ANALYTIC EXPRESSION OF ZF POST DETECTION SINR AND THE RELATING DISTRIBUTION

According the assumptions specified below, we consider a high-rate data-stream. Each data stream is divided into sub-streams which would be transmitted through the  $M_t$  transmit antenna and different time slots. We develop here an analytical expression of the post detection of the signal to interference plus noise ratio (SINR) at the receiver that employ zero forcing (ZF) detectors.

#### A. The Post Detection SINR

The received signal for the considered multi-user MIMO system is given by (1). The output of the ZF detector can be expressed as [8]

$$\hat{s} = W_{ZF} y \quad (4)$$

To find the expression of a ZF receiver, we should first let the following equation

$$e = \hat{s} - s \quad (5)$$

Where  $e$  is the error vector between the transmitted signal  $s$  and the received signal  $\hat{s}$ .

The received detector is assumed to be ZF, and then if we use the orthogonal principle, we obtain the following equality

$$E[e s^H] = 0 \quad (6)$$

$(.)^H$  is the Hermitian transpose of  $(.)$ .

According to equations (1) to (4), we have

$$E[e s^H] = E \left[ \left( \left( \sqrt{\frac{P_t}{M_t}} W_{ZF} H - I \right) s + W_{ZF} n \right) s^H \right] = 0 \quad (7)$$

It is assumed that  $s$  and  $n$  are independent random variables. Where  $W_{ZF}$  is the invert of the channel obtained by a ZF equalizer.

This matrix may invert the channel yielding to the desired symbol stream, which is obtained by equation (7) and is expressed as follows

$$W_{ZF} = \sqrt{\frac{M_t}{P_t}} H^\dagger \quad (8)$$

$$W_{ZF} = \sqrt{\frac{M_t}{P_t}} (H^H H)^{-1} H^H$$

Where  $H^\dagger$  denotes the Moore-Penrose inverse of the channel matrix  $H$ .

Consequently, the corresponding SINR of the ZF receiver is equal to

$$\gamma_k = \frac{1}{\frac{M_t \sigma_n^2}{P_t} \left[ (H^H H)^{-1} \right]_{k,k}}$$

$$\gamma_k = \frac{1}{\frac{M_t}{\bar{\gamma}} \left[ (H^H H)^{-1} \right]_{k,k}} \quad (9)$$

Where  $\bar{\gamma}$  is the average SNR at each of the receive antennas.

### B. The Probability Distributed Function

According to the assumptions,  $H$  is a matrix of independent and identically distributed complex white Gaussian random variable. Gore et al have demonstrated in [8] and [9] that the post detection of the signal to interference plus noise ratio (SINR) using a zero forcing (ZF) detector on each of the  $M_t$  streams is distributed as a Chi-squared random variable with  $2(M_r - M_t + 1)$  degrees of freedom.

The probability distributed function (pdf) of  $\gamma_k$  can be given by

$$f_\gamma(x) = \frac{M_t}{\bar{\gamma}(M_r - M_t)!} e^{-\frac{M_t}{\bar{\gamma}}x} \left( \frac{M_t}{\bar{\gamma}}x \right)^{M_r - M_t} \quad (10)$$

The corresponding cumulative distribution function (cdf) of  $\gamma_k$  is given by

$$F_\gamma(x) = \int_0^x f_\gamma(t) dt \quad (11)$$

$$F_\gamma(x) = 1 - e^{-\frac{M_t}{\bar{\gamma}}x} \sum_{k=0}^{M_r - M_t} \frac{1}{k!} \left( \frac{M_t}{\bar{\gamma}}x \right)^k \quad (12)$$

Let  $y = \frac{M_t}{\bar{\gamma}}x$ ,  $M = M_r - M_t$  and  $\alpha = \frac{\bar{\gamma}}{M_t}$

$$\text{We have } \sum_{k=0}^M \frac{y^k}{k!} = \frac{e^y (1+M) \Gamma(1+M, y)}{\Gamma(M+2)}$$

Consequently, the compact form of the cumulative distribution function is

$$F_\gamma(x) = 1 - \Gamma_{inc}(M_r - M_t + 1, \alpha x) \quad (13)$$

Where  $\Gamma_{inc}(a, z) = \frac{1}{\Gamma(a)} \int_z^\infty t^{a-1} e^{-t} dt$  is the regularized incomplete gamma function.

## V. SCHEDULERS THROUGHPUTS

The essential goals of the scheduling are to maximize the system throughput and to provide fairness among users. The channel state knowledge dependent scheduling scheme maximizes the system throughput by the use of multi-user diversity and the dynamic allocation of the transmit antennas. The maximum throughput gain is achieved, if each spatial channel is allocated to a user that experiences the best channel condition for each time slot. This scheduler is known as the opportunistic scheduler. This scheme maximizes the throughput but shares the system resources in an unfair way. The round robin scheduling scheme provides fairness among users for a cost of non tolerable loss in throughput.

After that, we are developing an overview study on performances of these scheduler schemes, where analytical solutions for their throughputs in the case of zero forcing detectors are developed.

### A. The Round Robin Scheduling (RRS)

#### 1. Theoretical Analysis

Initially, users are listed in some fashion. Then, we apply the technique when one transmit antenna is assigned to one user. At each time slot,  $M_t$  users are selected from  $N$  users in a round robin fashion. For example, in the case of  $N=20$  and  $M_t=2$ , the selection of users for subsequent time slots are  $\{1, 2\}$ ,  $\{3, 4\}$ , ...,  $\{19, 20\}$ . To guarantee equal channel access chance to users as in the round robin scheduling (RRS) scheme, the selection of users is made in a round robin fashion, and each selected user is restricted to use one and only one spatial channel.

The system capacity for the time slot  $t$  may be calculated as

$$r_k(t) = \sum_{n=1}^{M_t} \log_2(1 + \gamma_{k,n}(t)) \quad (14)$$

Where  $\gamma_{k,n}(t)$  is the post detection SINR at the receiver of the channel from the  $n^{th}$  transmitting antenna to the  $k^{th}$  user.

The distribution function of the post detection SINR using a zero forcing detector is given by equation (9) and is distributed as (10)

If we apply the algorithm of the round robin scheduling scheme, we denote by  $Y = \text{rand}_k \gamma_{k,m}$  and the analytic average throughput per slot for  $k^{th}$  user is

$$C_{av}^{RRS} = \frac{1}{N} \int_0^\infty \log_2(1+u) f_Y(u) du \quad (15)$$

$$= \frac{1}{N \log(2)} \int_0^\infty \log(1+u) \frac{M_t}{\bar{\gamma}(M_r - M_t)!} e^{-\left(\frac{M_t}{\bar{\gamma}}u\right)} \left(\frac{M_t}{\bar{\gamma}}u\right)^{M_r - M_t} du$$

If we use the method of integrating by parts; we have in the

end the expression that given by

$$C_{av}^{RRS} = \frac{e^{\frac{M_t}{\bar{\gamma}}}}{N \log(2)} \left[ E_1\left(\frac{M_t}{\bar{\gamma}}\right) + \sum_{k=1}^{M_t-M_t} \frac{1}{k!} \sum_{i=1}^k \binom{k}{i} \left(-\frac{M_t}{\bar{\gamma}}\right)^i \Gamma\left(k-i, \frac{M_t}{\bar{\gamma}}\right) \right]$$

Then the analytical solution for the system capacity would be

$$C_{av}^{RRS} = \frac{e^{\frac{M_t}{\bar{\gamma}}}}{\ln(2)} \left[ E_1\left(\frac{M_t}{\bar{\gamma}}\right) + \sum_{k=1}^{M_t-M_t} \frac{1}{k!} \sum_{i=1}^k \binom{k}{i} \left(-\frac{M_t}{\bar{\gamma}}\right)^i \Gamma\left(k-i, \frac{M_t}{\bar{\gamma}}\right) \right] \quad (16)$$

Where  $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$  is the incomplete gamma function.

And where  $E_1(x) = \int_x^\infty t^{-1} e^{-t} dt$  is the exponential integral function.

## 2. Validation and Numerical Results

In this subsection, the performance of the RRS is evaluated and compared with the theoretical results in terms of system throughput. Fig. 2 shows the effect of the variation of the numbers of transmit antenna and receive antenna in system throughput. This figure plots the system average capacity expected on a time moving window  $t_c$  of 500 time slots. The average  $\bar{\gamma}$  used for this simulation is 5dB and the number of active users  $N$  varies from 5 to 40.

We note that the system throughput is nearly independent of the active users' number. Then, we can conclude that the round robin scheduling cannot take advantages of multi-user diversity. These two simulation results which are carried out for  $M_t=M_r=4$  and  $M_t=M_r=2$ , illustrate that system capacity is enhanced in antenna number growth. The theoretical solution is near the empirical solution obtained by simulations which

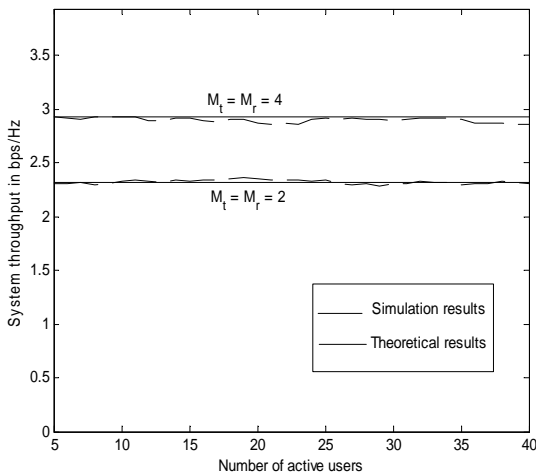


Fig. 2 Round robin scheduling throughput when  $M_t = 2, M_t = 4$ ;  $t_c = 500$

use  $t_c$  realizations. In the following section, we do similar

analysis for opportunistic scheduler.

## B. Opportunistic Scheduler

### 1. Theoretical Analysis

The basic idea is to maximize the capacity of each transmit antenna. In the opportunistic scheme all users can compete independently for each transmit antenna. At each time slot, each antenna is assigned to user which experience peak level channel.

Hence, the metric used in this case is

$$k_s = \arg \max_k (\gamma_{k,m}(t)) \quad (17)$$

When  $r_k(t)$  is given by (12), the supportable system capacity  $r_{k_s}(t)$  fed back from the select user may be expressed as

$$r_{k_s}(t) = \sum_{m=1}^{M_t} \log_2 \left( 1 + \max_k (\gamma_{k,m}(t)) \right) \quad (18)$$

Where  $\gamma_{k,m}(t)$  is the SINR at the receiver of the channel from the  $m^{th}$  transmitting antenna to the  $k^{th}$  user. The opportunistic scheduling scheme in dynamic SM tends to maximize the capacity of each transmit antenna.

The expression of the post detection SINR using a ZF detector is given by (9) and is distributed as (10)

We denote by  $Y_{max}$  the maximum of the sum of  $\gamma_{k,m}$

$$Y_{max} = \max_k \gamma_{k,m} \quad (19)$$

The probability distributed function and the cumulative distribution function of this stochastic process are respectively

$$f_{Y_{max}}(y) = N(F_\gamma(y))^{N-1} f_\gamma(y) \quad (20)$$

$$F_{Y_{max}}(y) = (F_\gamma(y))^N \quad (21)$$

Where,  $F_\gamma(y)$  is given by (11) and  $N$  is the number of active users.

The analytical average capacity per slot and per antenna is

$$C_{av}^{OPP} = \int_0^\infty \log_2(1+u) f_{Y_{max}}(u) du \quad (22)$$

$$C_{av}^{OPP} = \frac{1}{\log(2)} \int_0^\infty \log(1+y) N(F_\gamma(y))^{N-1} f_\gamma(y) dy$$

If we use the method of integrating by parts; we have in the end the analytical solution for the system capacity that is given by

$$C_{av}^{OPP} = \sum_{r=1}^N \binom{N}{r} (-1)^r \int_1^{\infty} \frac{(\Gamma_{inc}(1+M, \alpha(u-1)))^r}{u} du \quad (23)$$

$$\text{Let } I_r(x) = \int_x^{\infty} \frac{(\Gamma_{inc}(M+1, \alpha(u-1)))^r}{u} du$$

Then

$$C_{av}^{OPP} = \sum_{r=1}^N \binom{N}{r} (-1)^r I_r(1)$$

#### Particular case If, $M_r = M_t$ ,

The probability distributed function and the cumulative distribution function of the SINR will be

$$f_\gamma(x) = \frac{M_t}{\gamma} e^{-\frac{M_t}{\gamma} x} \quad (24)$$

$$F_\gamma(x) = 1 - e^{-\frac{M_t}{\gamma} x} \quad (25)$$

The analytic average capacity expression per slot for the  $k^{th}$  user is given by

$$C_{av}^{OPP} = \int_0^{\infty} \log_2(1+y) f_\gamma(y) dy$$

The system capacity would be

$$C_{av}^{OPP} = \frac{1}{\log(2)} \sum_{r=0}^{N-1} \binom{N-1}{r} (-1)^r \frac{e^{-\frac{(r+1)M_t}{\gamma}} E_1\left(\frac{(r+1)M_t}{\gamma}\right)}{r+1} \quad (26)$$

#### 2. Validation and Numerical Results

In this numerical validation, the performance of the opportunistic scheme is evaluated and compared with the theoretical results in terms of system throughput. Fig. 3 shows the effect of the variation of transmit antenna  $M_t$  and receive antenna  $M_r$  in system throughput with  $t_c$  is of 500,  $\bar{\gamma}$  used in simulation and in analytical solution is 5dB and the number of users is of 40.

Fig. 3 shows the system throughput of simulation and theoretical results versus the number of active users. We note that the system capacity is nearly to saturate for  $N > 10$  for all graphs. The system capacity in analytical form and that obtained by simulation are in good concordance. To confirm this, two simulations are carried out for  $M_t=M_r=4$  and

$M_t=M_r=2$ . According to these comparisons, we can conclude that the proposed analytical solution is the real average system capacity.

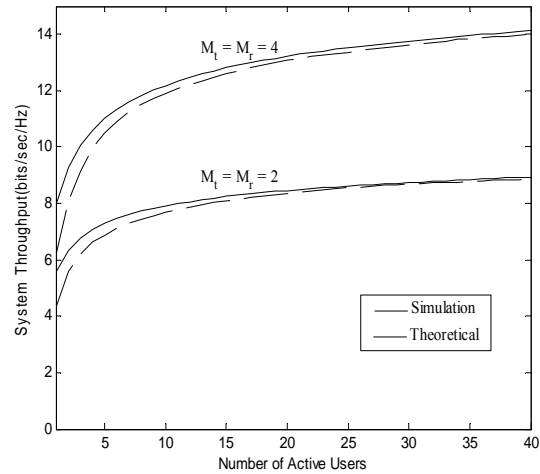


Fig. 3 Opportunistic throughput when  $M_t = 2$ ,  $M_t = 4$ ;  $t_c = 500$

#### VI. COMPARISON OF SCHEDULERS PERFORMANCES

Finally and after the study and the representation of simulation results of the round robin scheduling and the opportunistic, we compare the performance given by these schemes. Fig. 4 shows the effect of the variation of the average SNR in system capacity with  $t_c$  of 500, the number of users is of 40 and the number of transmit and receive antennas is of 2.

Fig. 4 shows the system throughput of simulation results versus the number of active users when the average SNR is of -5dB and of 5dB for the round robin and the opportunistic schedulers.

We note that the system capacity of the round robin scheduling is nearly constant for all number of active users. The system capacity of the opportunistic increases with the number of active users and saturates before  $N > 10$ . It is clear that the system capacity increase with number of average SNR. The coefficient of increasing is similar for the two schedulers. But it is clear that the opportunistic scheduling gives a number of capacities in the two simulations better than the round robin scheduling. According to these comparisons, we can conclude that the opportunistic is the best and the optimal scheduler.

These results are very clear because the round robin scheduling cannot take advantages of multi-user diversity. However, the opportunistic scheduler provides maximum multi-user diversity gain by allocating resources to different users having the best channel then the optimum capacity is achieved.

#### VII. CONCLUSION

The throughputs of round robin scheduling and the opportunistic scheme have been analyzed in this paper. In these schemes the transmit antenna are assigned to different users at each time slot. Simulations have been used to validate

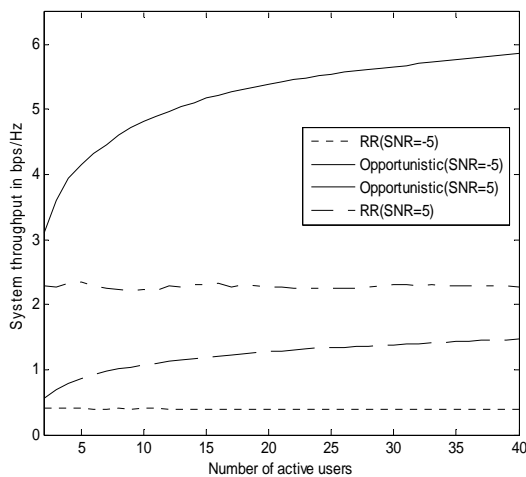


Fig. 4 RR scheduling and opportunistic throughput when  $\bar{\gamma} = -5$  dB and  $\bar{\gamma} = 5$  dB

the analytical solution of the capacity of MIMO systems that employ opportunistic and round robin scheduling.

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