

# The Spanning Laceability of $k$ -ary $n$ -cubes when $k$ is Even

Yuan-Kang Shih, Shu-Li Chang, and Shin-Shin Kao

**Abstract**— $Q_n^k$  has been shown as an alternative to the hypercube family. For any even integer  $k \geq 4$  and any integer  $n \geq 2$ ,  $Q_n^k$  is a bipartite graph. In this paper, we will prove that given any pair of vertices,  $w$  and  $b$ , from different partite sets of  $Q_n^k$ , there exist  $2n$  internally disjoint paths between  $w$  and  $b$ , denoted by  $\{P_i \mid 0 \leq i \leq 2n-1\}$ , such that  $\bigcup_{i=0}^{2n-1} P_i$  covers all vertices of  $Q_n^k$ . The result is optimal since each vertex of  $Q_n^k$  has exactly  $2n$  neighbors.

**Keywords**—container, Hamiltonian,  $k$ -ary  $n$ -cube,  $m^*$ -connected.

## I. INTRODUCTION

The  $k$ -ary  $n$ -cube, denoted by  $Q_n^k$ , has been proposed as an alternative to the hypercube since it shares many nice properties of  $Q_n$  such as regular degrees, vertex symmetry, edge symmetry, recursive structure, etc.. The underlying topology of many machines is based on  $k$ -ary  $n$ -cubes, such as the Cray T3E, the iWARP, the Cray T3D and so on. Please see [1], [4], [11], [17]. Many researchers have been working on  $k$ -ary  $n$ -cubes. For example, Stewart and Xiang [20] proved that the  $k$ -ary  $n$ -cube is edge-bipancyclic and bipanconnected for  $k \geq 3$  and  $n \geq 2$  and  $k$  being even. Namely, any edge of a  $k$ -ary  $n$ -cube  $Q_n^k$  lies on a cycle of any even length  $r$  for  $4 \leq r \leq |Q_n^k|$ , where  $|Q_n^k|$  is the total number of vertices of  $Q_n^k$ . Besides, given two vertices  $u$  and  $v$  of  $Q_n^k$ , there exists a path of any even length  $r$  between  $u$  and  $v$  for  $d(u, v) \leq r \leq |Q_n^k|$ , where  $d(u, v)$  is the distance between  $u$  and  $v$ . Other studies about fault tolerance on  $k$ -ary  $n$ -cubes can be found in [8], [23]. Recently, there are many studies about the spanning connectivity for interconnection networks and graphs [9]. A graph  $H = (B \cup W, E)$  is *bipartite* if  $V(H)$  is the union of two disjoint sets  $B$  and  $W$  such that every edge joins  $B$  with  $W$ . It is easy to see that any bipartite graph with at least three vertices is not hamiltonian connected except  $K_2$ . Note that any (nontrivial) bipartite graph except  $K_2$  cannot be hamiltonian connected, whereas a bipartite graph is *hamiltonian laceable* if there exists a hamiltonian path between any two vertices  $u, v$  with  $u \in B$  and  $v \in W$  [22]. A graph  $H = (B \cup W, E)$  is a *balanced* bipartite graph if  $|V(B)| = |V(W)|$ . Throughout this thesis, we only work on  $Q_n^k$  with  $k \geq 4$  an even integer and  $n \geq 2$ , which are balanced bipartite graphs. A bipartite graph  $H = (B \cup W, E)$  is  *$m^*$ -laceable* if given a white vertex  $w \in B$  and a black vertex  $b \in W$ , there exist(s)  $m$  internal disjoint paths between  $w$  and  $b$ , denoted by  $P_i$  for

$0 \leq i \leq m-1$ , such that  $\bigcup_{i=0}^{m-1} P_i$  covers  $V$ . The *spanning laceability* of a graph  $H$ ,  $\kappa^*(H)$ , is the largest integer  $k$  such that  $H$  is  $m^*$ -laceable for every  $m$  with  $1 \leq m \leq k$ . A higher spanning connectivity/laceability of the interconnection network implies a more efficient communication between processors. About the spanning connectivity and the spanning laceability, readers can refer to [6], [7], [12]–[15].

In this paper, we want to show the spanning laceability of  $k$ -ary  $n$ -cubes for any even integer  $k \geq 4$ . More precisely, we show that given a white vertex  $w$  and a black vertex  $b$  of a  $k$ -ary  $n$ -cube  $Q_n^k$ , there exist(s)  $m$  internally disjoint path(s) between  $w$  and  $b$  whose union covers all vertices of  $Q_n^k$  for  $1 \leq m \leq 2n$ . The result is optimal since any vertex in  $Q_n^k$  has exactly  $2n$  neighbors. This paper is organized as follows. In Section 2, we introduce the graph terminologies and symbols that will be used in the paper and the definition of  $Q_n^k$ . In Section 3, we show our main results.

## II. PRELIMINARIES

Throughout this paper, we follow [3] for the graph definitions and notations. The sets of vertices and edges of a graph  $G$  are denoted by  $V(G)$  and  $E(G)$ , respectively. If  $u, v$  are vertices of a graph  $G$  such that there is an edge  $e = (u, v) \in E(G)$  between  $u$  and  $v$ , then we say that the vertices  $u$  and  $v$  are *adjacent* in  $G$ . The *degree* of any vertex  $x$  is the number of distinct vertices adjacent to  $x$ . A *path*  $P$  between two vertices  $v_0$  and  $v_k$  is represented by  $P = \langle v_0, v_1, \dots, v_k \rangle$ , where each pair of consecutive vertices are connected by an edge. We use  $P^{-1}$  to denote the path  $\langle v_k, v_{k-1}, v_{k-2}, \dots, v_0 \rangle$ . We also write the path  $P = \langle v_0, v_1, \dots, v_k \rangle$  as  $\langle v_0, v_1, \dots, v_i, Q, v_j, v_{j+1}, \dots, v_k \rangle$ , where  $Q$  denotes the path  $\langle v_i, v_{i+1}, \dots, v_j \rangle$ . A *hamiltonian path* between  $u$  and  $v$ , where  $u$  and  $v$  are two distinct vertices of  $G$ , is a path joining  $u$  to  $v$  that visits every vertex of  $G$  exactly once. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of  $G$  is a cycle that traverses every vertex of  $G$  exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle. A graph  $G$  is *connected* if there is a path between any two distinct vertices in  $G$  and is *hamiltonian connected* if there is a hamiltonian path between any two distinct vertices in  $G$  [18]. A graph  $H = (W \cup B, E)$  is *bipartite* if  $V(H) = W \cup B$  and  $E(H)$  is a subset of  $\{(w, b) \mid w \in W, b \in B\}$ . A bipartite graph  $H$  is *hamiltonian laceable* if there is a hamiltonian path between any two distinct vertices from different partite sets in  $H$ .

A graph  $G$  is  *$k$ -connected* if there exists  $V' \subseteq V(G)$  with  $|V'| = k$  such that  $G - V'$  is disconnected and  $G - V''$  is

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connected for any  $V'' \subseteq V(G)$  with  $|V''| < k$ . It follows from Menger's Theorem [16] that for every  $k$ -connected graph  $G$ , there exist  $k$  internally vertex-disjoint paths between any pair of distinct vertices of  $G$ . A  $k$ -container  $C(u, v)$  in a graph  $G$  is a set of  $k$  internally vertex-disjoint paths between two distinct vertices  $u$  and  $v$ . We say that a graph  $G$  has a spanning  $k$ -container between  $u$  and  $v$ , denoted by  $C(u, v)$ , if  $C(u, v)$  is a  $k$ -container that covers all vertices of  $G$ . A spanning  $k$ -container is also abbreviated as a  $k^*$ -container for simplicity. A graph  $G$  is  $k^*$ -connected if there is a  $k^*$ -container between any pair of vertices of  $G$ . Obviously, a graph  $G$  is hamiltonian connected if and only if  $G$  is  $1^*$ -connected, and  $G$  is hamiltonian if and only if  $G$  is  $2^*$ -connected. Lin et al. [13] defined the concept of *spanning connectivity*. The *spanning connectivity* of a graph  $G$ ,  $\kappa^*(G)$ , is the largest integer  $k$  such that  $G$  is  $w^*$ -connected for all  $1 \leq w \leq k$ . Similarly, a bipartite graph  $H$  is  $k^*$ -laceable if there is a  $k^*$ -container between any pair of two vertices from different partite sets of  $H$ . Also, a bipartite graph  $H$  is hamiltonian laceable if and only if  $H$  is  $1^*$ -laceable, and  $H$  is hamiltonian if and only if  $H$  is  $2^*$ -laceable. So, the *spanning laceability* of a bipartite graph  $H$ ,  $\kappa^*(H)$ , is the largest integer  $k$  such that  $H$  is  $m^*$ -laceable for all  $1 \leq m \leq k$ .

The  $k$ -ary  $n$ -cube,  $Q_n^k$ , is defined for all integers  $k \geq 2$  and  $n \geq 1$ . The subclass  $Q_n^2$  is the well-studied hypercube family. The subclass  $Q_n^k$  with  $k \geq 3$  is defined as the cycle of length  $k$ . The  $k$ -ary  $n$ -cube,  $Q_n^k$ , for  $k \geq 3$  and  $n \geq 2$  is defined as follows. Let  $u \in V(Q_n^k)$  be represented by  $(u(0), u(1), \dots, u(n-1))$ , where  $0 \leq u(i) \leq k-1$ . Two vertices  $u$  and  $v$  are adjacent if and only if  $|u(i) - v(i)| = 1$  or  $k-1$  for some  $i$  and  $u(j) = v(j)$  for any  $0 \leq j \leq n-1$  with  $j \neq i$ . It is shown that  $Q_n^k$  is bipartite if  $k$  is even [10]. Here we mention some properties of  $Q_n^k$  that will be used in this paper.

$Q_n^k$  is *vertex symmetric* (and *edge symmetric*) [10]. It means that given any two distinct vertices  $v$  and  $v'$  of  $Q_n^k$ , there is an automorphism of  $Q_n^k$  mapping  $v$  to  $v'$ . Note that each vertex of  $Q_n^k$  is represented by a  $n$ -bit tuple. We will call the  $d$ th-bit the  $d$ th dimension. We can partition  $Q_n^k$  over dimension  $d$  by fixing the  $d$ th element of any vertex tuple at some value  $a$  for every  $a \in \{0, 1, \dots, k-1\}$ . This results in  $k$  copies of  $Q_{n-1}^k$ , denoted by  $Q_{n-1}^{k,0}, Q_{n-1}^{k,1}, \dots, Q_{n-1}^{k,k-1}$ , with corresponding vertices in  $Q_{n-1}^{k,0}, Q_{n-1}^{k,1}, \dots, Q_{n-1}^{k,k-1}$  joined in a cycle of length  $k$  (in dimension  $d$ ) [19].

In this article, we always partition  $Q_n^k$  over the 0-th dimension by letting  $V(Q_{n-1}^{k,i}) = \{(i, v(1), v(2), \dots, v(n-1)) \mid 0 \leq v(j) \leq k-1, \forall 1 \leq j \leq n-1\}$  for  $0 \leq i \leq k-1$ . Given a vertex  $x = (x(0), x(1), \dots, x(n-1)) \in V(Q_n^k)$ , the symbol  $x^j = ((j, x(1), x(2), \dots, x(n-1)))$ , where  $0 \leq j \leq k-1$ , is defined to be the vertex corresponding to  $x$  in  $Q_{n-1}^{k,j}$  for simplicity. So, if  $P = \langle x_0, x_1, \dots, x_{n-1} \rangle$ ,  $P^j$  is represented by  $\langle x_0^j, x_1^j, \dots, x_{n-1}^j \rangle$ . Throughout this paper, let  $n \geq 2$  be an integer and  $k \geq 4$  an even integer.

**Theorem 1.** [10] For any even integer  $k \geq 4$ ,  $Q_n^k$  is hamiltonian laceable for  $n \geq 2$ . In other words,  $Q_n^k$  is  $1^*$ -laceable.

**Theorem 2.** [5] The graph  $Q_n^k$  is hamiltonian. In other words,

$Q_n^k$  is  $2^*$ -laceable.

### III. MAIN RESULTS

**Lemma 1.** Given  $Q_n^k$  and its  $k$  subcubes,  $Q_{n-1}^{k,i}$ , where  $0 \leq i \leq k-1$ . Let  $j$  and  $j'$  be two integers satisfying  $0 \leq j \leq j' \leq k-1$ ,  $w \in V(Q_{n-1}^{k,j})$  an arbitrary white vertex, and  $b \in V(Q_{n-1}^{k,j'})$  an arbitrary black vertex. Then there exists a path between  $w$  and  $b$  that visits each vertex in  $Q_{n-1}^{k,j}, Q_{n-1}^{k,j+1}, Q_{n-1}^{k,j+2}, \dots, Q_{n-1}^{k,j'}$  exactly once.

*Proof:* There are three cases.

**Case 1.**  $j = j'$ . W.L.O.G., let  $j = j' = 0$ . By Theorem 1,  $Q_{n-1}^{k,0}$  is hamiltonian laceable. Thus, there is a hamiltonian path between  $w$  and  $b$  that visits each vertex of  $Q_{n-1}^{k,0}$  exactly once.

**Case 2.**  $j - j' = 1$ . W.L.O.G., we can let  $j = 0$  and  $j' = 1$ . Let  $w$  be a white vertex in  $Q_{n-1}^{k,0}$  and  $b$  a black vertex in  $Q_{n-1}^{k,1}$ . We can find a pair of adjacent vertices  $x^0$  and  $x^1$  where  $x^0$  is a black vertex of  $Q_{n-1}^{k,0}$  and  $x^1$  is a white vertex of  $Q_{n-1}^{k,1}$ . By Theorem 1, there exists a hamiltonian path  $P_0$  of  $Q_{n-1}^{k,0}$  between  $w$  and  $x^0$ , and a hamiltonian path  $P_1$  of  $Q_{n-1}^{k,1}$  between  $x^1$  and  $b$ . Let  $P = \langle w, P_0, x^0, x^1, P_1, b \rangle$ . Hence  $P$  is the path between  $w$  and  $b$  that visits every vertex of  $Q_{n-1}^{k,0}$  and  $Q_{n-1}^{k,1}$  exactly once.

**Case 3.**  $j - j' \geq 2$ . Let  $w$  be a white vertex in  $Q_{n-1}^{k,j}$  and  $b$  be a black vertex in  $Q_{n-1}^{k,j'}$ . There are  $j - j' + 1$   $k$ -ary  $n-1$ -cubes,  $Q_{n-1}^{k,j}, Q_{n-1}^{k,j+1}, Q_{n-1}^{k,j+2}, \dots, Q_{n-1}^{k,j'-1}$  and  $Q_{n-1}^{k,j'}$ . There are  $j' - j$  pairs of adjacent vertices  $x^r \in Q_{n-1}^{k,r}$  and  $y^{r+1} \in Q_{n-1}^{k,r+1}$  where  $x^r$  is a black vertex and  $y^{r+1}$  is a white vertex for  $j \leq r \leq j' - 1$ . By Theorem 1, there is a hamiltonian path  $R_r$  of  $Q_{n-1}^{k,r}$  joining  $y^r$  to  $x^r$ , where  $j+1 \leq r \leq j'-1$ . Again, with Theorem 1, there exists a hamiltonian path  $T$  of  $Q_{n-1}^{k,j}$  joining  $w$  to  $x^j$ , and a hamiltonian path  $U$  of  $Q_{n-1}^{k,j'}$  joining  $y^{j'}$  to  $b$ . Let  $P = \langle w, T, x^j, y^{j+1}, R_{j+1}, x^{j+1}, y^{j+2}, R_{j+2}, x^{j+2}, \dots, y^{j'-1}, R_{j'-1}, x^{j'-1}, y^{j'}, U, b \rangle$ . Therefore,  $P$  is a path covering all the vertices of  $Q_{n-1}^{k,j}, Q_{n-1}^{k,j+1}, Q_{n-1}^{k,j+2}, \dots, Q_{n-1}^{k,j'}$  for  $0 \leq j \leq j' \leq k-1$  between  $w$  and  $b$ . Please see Figure 1 for an illustration.

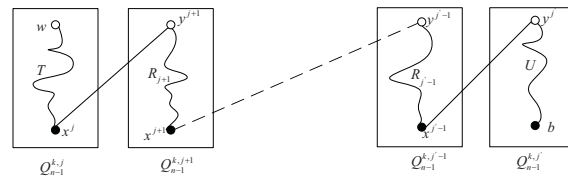


Fig. 1. The illustration for Case 3 of Lemma 1.

**Lemma 2.** Given  $Q_n^k$  and its  $k$  subcubes  $Q_{n-1}^{k,i}$  for  $0 \leq i \leq k-1$ . Let  $w$  be a white vertex,  $b$  a black vertex in  $Q_{n-1}^{k,i}$ , and  $j$  an integer with  $0 \leq i \leq j \leq k-1$ . There exists a path between  $w$  and  $b$  that covers all the vertices of  $Q_{n-1}^{k,i}, Q_{n-1}^{k,i+1}, \dots$ , and  $Q_{n-1}^{k,j}$ .

*Proof:* We consider the following two cases.

**Case 1.**  $j = i$ . There is only one  $k$ -ary  $(n - 1)$ -cube  $Q_{n-1}^{k,i}$ . By Theorem 1, the lemma holds in this case.

**Case 2.**  $j \neq i$ . There are  $j - i + 1$   $k$ -ary  $(n - 1)$ -cubes. According to Theorem 1, there is hamiltonian path  $P_i$  that covers all the vertices of  $Q_{n-1}^{k,i}$  between  $w$  and  $b$  of the form  $\langle w, S_i, x^i, y^i, T_i, b \rangle$ , where  $\{x^i, y^i\}$  is an edge of  $Q_{n-1}^{k,i}$  with  $\{x^i, y^i\} \cap \{w, b\} = \emptyset$ . Notice that by Theorem 1,  $Q_{n-1}^{k,r}$  is hamiltonian laceable and hence there exists a hamiltonian path  $P_r$  between  $x^r$  and  $y^r$  of the form  $\langle x^r, S_r, z^r, w^r, T_r, y^r \rangle$  for  $i + 1 \leq r \leq j$ . Let the required path between  $w$  and  $b$  be  $R$ , we have the following two subcases.

**Case 2.1.** If  $j - i + 1$  is even, then

$R = \langle w, S_i, x^i, x^{i+1}, S_{i+1}, z^{i+1}, z^{i+2}, (S_{i+2})^{-1}, x^{i+2}, x^{i+3}, S_{i+3}, z^{i+3}, z^{i+4}, (S_{i+4})^{-1}, x^{i+4}, \dots, x^j, S_j, z^j, w^j, T_j, y^j, y^{j-1}, (T_{j-1})^{-1}, w^{j-1}, w^{j-2}, T_{j-2}, y^{j-2}, y^{j-3}, (T_{j-3})^{-1}, w^{j-3}, \dots, y^{i+1}, y^i, T_i, b \rangle$ . Please see Figure 2 for an illustration.

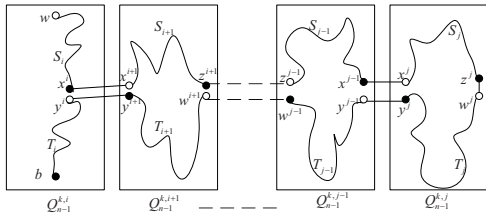


Fig. 2. The illustration for Lemma 2 when  $j - i + 1$  is even.

**Case 2.2.** If  $j - i + 1$  is odd, then

$R = \langle w, S_i, x^i, x^{i+1}, S_{i+1}, z^{i+1}, z^{i+2}, (S_{i+2})^{-1}, x^{i+2}, x^{i+3}, S_{i+3}, z^{i+3}, z^{i+4}, (S_{i+4})^{-1}, x^{i+4}, \dots, z^j, (S_j)^{-1}, x^j, y^j, (T_j)^{-1}, w^j, w^{j-1}, T_{j-1}, y^{j-1}, y^{j-2}, (T_{j-2})^{-1}, w^{j-2}, w^{j-3}, T_{j-3}, y^{j-3}, \dots, y^{i+1}, y^i, T_i, b \rangle$ . Please see Figure 2 for an illustration. ■

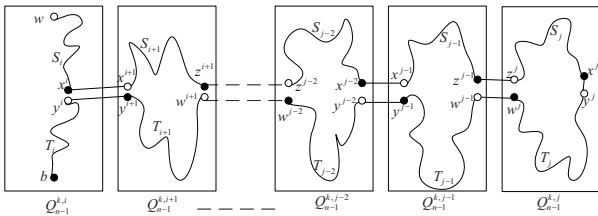


Fig. 3. The illustration for Lemma 2 when  $j - i + 1$  is odd.

**Lemma 3.** The graph  $Q_2^4$  is 3\*-laceable and 4\*-laceable.

*Proof:* The proof is by brute force. Reader can refer to Appendix A. ■

**Lemma 4.** The graph  $Q_2^6$  is 3\*-laceable and 4\*-laceable.

*Proof:* By brute force, we constructed all spanning containers. Please see Appendix B. ■

**Lemma 5.** The graph  $Q_2^k$  is 3\*-laceable and 4\*-laceable for any even integer  $k \geq 6$ .

*Proof:* With Lemma 4, we have shown that  $Q_2^6$  is 3\*-laceable and 4\*-laceable. Now we will present a recursive

algorithm that uses a 3\*-container (resp. 4\*-container) of  $Q_2^k$  to construct a 3\*-container (resp. 4\*-container) of  $Q_2^{k+2}$ . Let  $R$  be a subset of  $V(Q_2^k) \cup E(Q_2^k)$ . We define a function,  $f$ , which maps  $R$  from  $Q_2^k$  into  $Q_2^{k+2}$  in the following way:

(1) If  $(i, j) \in R \cap V(Q_2^k)$ , where  $0 \leq i, j \leq k - 1$ , then

$$f((i, j)) = \begin{cases} (i, j) & \text{if } 0 \leq i, j \leq k - 2; \\ (i + 2, j) & \text{if } i = k - 1, 0 \leq j \leq k - 2; \\ (i, j + 2) & \text{if } j = k - 1, 0 \leq i \leq k - 2; \\ (i + 2, j + 2) & \text{if } i = k - 1 = j. \end{cases}$$

(2) If  $((i, j), (i', j')) \in R \cap E(Q_2^k)$ , where  $i \leq i', j \leq j'$ , then  $f(((i, j), (i', j')))$

$$= \begin{cases} ((i, j), (i', j')) & \text{if } 0 \leq i, j \leq k - 3, \\ & 1 \leq i', j' \leq k - 2; \\ ((i + 2, j), (i' + 2, j')) & \text{if } i = i' = k - 1, \\ & 0 \leq j \leq k - 3, \\ & 1 \leq j' \leq k - 2; \\ ((i, j + 2), (i', j' + 2)) & \text{if } j = j' = k - 1, \\ & 0 \leq i \leq k - 3, \\ & 1 \leq i' \leq k - 2; \\ ((i, j), (i', j' + 2)) & \text{if } 0 \leq i = i' \leq k - 2, \\ & j = 0, j' = k - 1; \\ ((i, j), (i' + 2, j')) & \text{if } 0 \leq j = j' \leq k - 2, \\ & i = 0, i' = k - 1; \\ ((i, j + 2), (i' + 2, j' + 2)) & \text{if } i = 0, i' = k - 1, \\ & j = j' = k - 1; \\ ((i + 2, j), (i' + 2, j' + 2)) & \text{if } j = 0, j' = k - 1, \\ & i = i' = k - 1. \end{cases}$$

Let  $w$  be a white vertex and  $b$  be a black vertex of  $Q_2^k$ . We say that a 3\*-container (resp. 4\*-container)  $C(u, v)$  of  $Q_2^k$  is regular if  $C(w, b)$  contains some edges in  $\{((\alpha, k - 2), (\alpha, k - 1)) \mid 0 \leq \alpha \leq k - 1\}$  and  $\{((k - 2, \beta), (k - 1, \beta)) \mid 0 \leq \beta \leq k - 1\}$ . For example, all 3\*-containers and 4\*-containers of  $Q_2^6$  constructed in Lemma 4 are regular. Let  $C(w, b)$  be a regular 3\*-container (resp. 4\*-container) of  $Q_2^k$  with the endvertex set  $P = \{w = (0, 0), b = (x, y)\}$ . We construct a regular 3\*-container (resp. 4\*-container) of  $Q_2^{k+2}$  with the endvertex set  $f(P)$  using the following algorithm. Please see Figure 4 for an illustration.

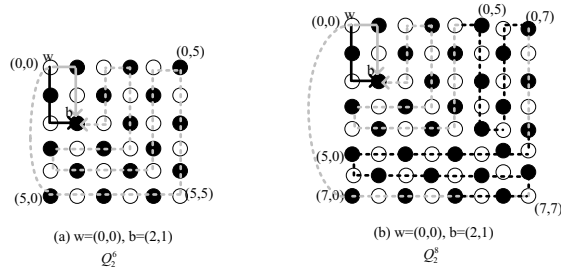


Fig. 4. Using the 4\*-container of  $Q_2^6$  to construct the 4\*-container of  $Q_2^8$ .

**Step 1.** In  $Q_2^k$ , let  $\{v_0, v_1, \dots, v_{t-1}\}$  and  $\{h_0, h_1, \dots, h_{s-1}\}$  be finite sequences of indices satisfying the following requirements:

- (1)  $0 \leq v_0 < v_1 < \dots < v_{t-1} \leq k - 1$  and  $k - 1 \geq h_0 > h_1 > \dots > h_{s-1} \geq 0$ ;

(2) for  $0 \leq i \leq k-1$ ,  $((v_i, k-2), (v_i, k-1))$  is an edge of  $C(w, b)$ ; for  $0 \leq j \leq k-1$ ,  $((k-2, h_j), (k-1, h_j))$  is an edge of  $C(w, b)$ .

**Step 2.** Let  $\overline{C}(w, b)$  be the image in  $Q_2^{k+2}$  of  $C(w, b) - (\{(v_i, k-2), (v_i, k-1) \mid 0 \leq i \leq k-1\} \cup \{(k-2, h_j), (k-1, h_j) \mid 0 \leq j \leq k-1\})$  under the function  $f$ . Please see Figure 5 for an illustration.

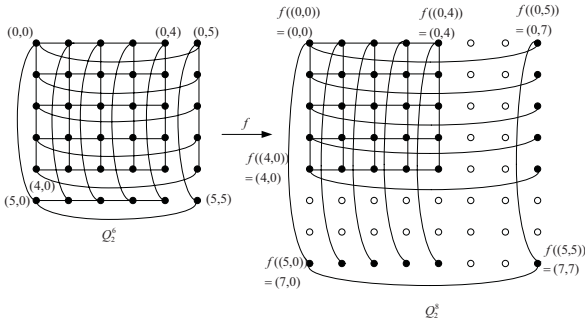


Fig. 5. Using function  $f$  to map a subset of edges and vertices of  $Q_2^k$  into  $Q_2^{k+2}$ .

**Step 3.** For any two positive integers  $r$  and  $d$ , we use  $[r]_d$  to denote  $r \pmod{d}$ . In  $Q_2^{k+2}$ , define the following path patterns, where  $r_1, r_2$  are integers:

$$\begin{aligned} I_\alpha(r_1, r_2) &= \langle (r_1, \alpha), ([r_1 + 1]_{k+2}, \alpha), \dots, (r_2, \alpha) \rangle; \\ I_\alpha^{-1}(r_2, r_1) &= \langle (r_2, \alpha), ([r_2 - 1]_{k+2}, \alpha), \dots, (r_1, \alpha) \rangle; \\ H_\beta(r_1, r_2) &= \langle (\beta, r_1), (\beta, [r_1 + 1]_{k+2}), \dots, (\beta, r_2) \rangle; \\ H_\beta^{-1}(r_2, r_1) &= \langle (\beta, r_2), (\beta, [r_2 - 1]_{k+2}), \dots, (\beta, r_1) \rangle. \end{aligned}$$

Let  $\bar{v}_i = v_i + 2$  if  $v_i = k-1$  and  $\bar{v}_i = v_i$  if  $0 \leq v_i \leq k-2$ , and  $\bar{h}_j = h_j + 2$  if  $h_j = k-1$  and  $\bar{h}_j = h_j$  if  $0 \leq h_j \leq k-2$ .

**Case 1.**  $v_0 = k-1$ .

Let  $P_0 = \langle (k+1, k-2), (k+1, k-1), (0, k-1), I_{k-1}(0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, 0), (0, k), (k+1, k), (k+1, k+1) \rangle$ .

**Case 1.1.**  $s = 1$ .

Let  $\bar{P}_0 = \langle (k-2, \bar{h}_0), (k-1, \bar{h}_0), H_{k-1}^{-1}(\bar{h}_0, [\bar{h}_0 + 1]_{k+2}), (k-1, [\bar{h}_0 + 1]_{k+2}), (k, [\bar{h}_0 + 1]_{k+2}), H_k([\bar{h}_0 + 1]_{k+2}, \bar{h}_0), (k, \bar{h}_0), (k+1, \bar{h}_0) \rangle$ . Then  $\overline{C}(w, b) \cup P_0 \cup \bar{P}_0$  is the 3\*-container (or 4\*-container) of  $Q_2^{k+2}$ .

**Case 1.2.**  $s \geq 2$ .

Let  $\bar{P}_i = \langle (k-2, \bar{h}_i), (k-1, \bar{h}_i), H_{k-1}^{-1}(\bar{h}_i, \bar{h}_{i+1} + 1), (k-1, \bar{h}_{i+1} + 1), (k, \bar{h}_{i+1} + 1), H_k(\bar{h}_{i+1} + 1, \bar{h}_i), (k, \bar{h}_i), (k+1, \bar{h}_i) \rangle$  for  $0 \leq i \leq s-2$ , and  $\bar{P}_{s-1} = \langle (k-2, \bar{h}_{s-1}), (k-1, \bar{h}_{s-1}), H_{k-1}^{-1}(\bar{h}_{s-1}, [\bar{h}_0 + 1]_{k+2}), (k-1, [\bar{h}_0 + 1]_{k+2}), (k, [\bar{h}_0 + 1]_{k+2}), H_k([\bar{h}_0 + 1]_{k+2}, \bar{h}_{s-1}), (k, \bar{h}_{s-1}), (k+1, \bar{h}_{s-1}) \rangle$ . Then  $\overline{C}(w, b) \cup P_0 \cup \{\bar{P}_i \mid 0 \leq i \leq s-1\}$  is the 3\*-container (or 4\*-container) of  $Q_2^{k+2}$ .

**Case 2.**  $v_{t-1} \leq k-2$  and  $((k-2, k-1), (k-1, k-1)) \in E(C(w, b))$  in  $Q_2^k$ .

**Case 2.1.**  $t = 1$ .

Let  $P_0 = \langle (\bar{v}_0, k-2), (\bar{v}_0, k-1), I_{k-1}(\bar{v}_0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \bar{v}_0), (\bar{v}_0, k), (\bar{v}_0, k+1) \rangle$ .

**Case 2.1.1**  $s = 1$ .

Let  $\bar{P}_0 = \langle (k-2, \bar{h}_0), (k-1, \bar{h}_0), H_{k-1}^{-1}(\bar{h}_0, 0), (k-1, 0), (k, 0), H_k(0, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+1, \bar{v}_0 - 1, k-1), (\bar{v}_0 - 1, k-1), (\bar{v}_0 - 1, k), I_k^{-1}(\bar{v}_0 - 1, 0), (0, k), (k+1, k), (k, k), (k, k+1), (k-1, k+1), (k-1, k), I_k^{-1}(k-1, \bar{v}_0), (\bar{v}_0, k), (\bar{v}_0, k+1) \rangle$ .

$1, [\bar{v}_0 - 1]_{k+2}), ([\bar{v}_0 - 1]_{k+2}, k-1), ([\bar{v}_0 - 1]_{k+2}, k), I_k^{-1}([\bar{v}_0 - 1]_{k+2}, k+1), (k+1, k), (k, k), (k, \bar{h}_0), (k+1, \bar{h}_0) \rangle$ . Then  $\overline{C}(w, b) \cup P_0 \cup \bar{P}_0$  is the 3\*-container (or 4\*-container) of  $Q_2^{k+2}$ .

**Case 2.1.2**  $s = 2$ .

Let  $\bar{P}_0 = \langle (k-2, \bar{h}_0), (k-1, \bar{h}_0), H_{k-1}^{-1}(\bar{h}_0, \bar{h}_1 + 1), (k-1, \bar{h}_1 + 1), (k, \bar{h}_1 + 1), H_k(\bar{h}_1 + 1, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+1, [\bar{v}_0 - 1]_{k+2}), ([\bar{v}_0 - 1]_{k+2}, k-1), ([\bar{v}_0 - 1]_{k+2}, k), I_k^{-1}([\bar{v}_0 - 1]_{k+2}, k+1), (k+1, k), (k, k), (k, \bar{h}_0), (k+1, \bar{h}_0) \rangle$ , and  $\bar{P}_1 = \langle (k-2, \bar{h}_1), (k-1, \bar{h}_1), H_{k-1}^{-1}(\bar{h}_1, 0), (k-1, 0), (k, 0), H_k(0, \bar{h}_1), (k, \bar{h}_1), (k+1, \bar{h}_1) \rangle$ . Then  $\overline{C}(w, b) \cup P_0 \cup \bar{P}_0 \cup \bar{P}_1$  is the 3\*-container (or 4\*-container) of  $Q_2^{k+2}$ .

**Case 2.1.3**  $s \geq 3$ .

Let  $\bar{P}_0 = \langle (k-2, \bar{h}_0), (k-1, \bar{h}_0), H_{k-1}^{-1}(\bar{h}_0, \bar{h}_1 + 1), (k-1, \bar{h}_1 + 1), (k, \bar{h}_1 + 1), H_k(\bar{h}_1 + 1, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+1, [\bar{v}_0 - 1]_{k+2}), ([\bar{v}_0 - 1]_{k+2}, k-1), ([\bar{v}_0 - 1]_{k+2}, k), I_k^{-1}([\bar{v}_0 - 1]_{k+2}, k+1), (k+1, k), (k, k), (k, \bar{h}_0), (k+1, \bar{h}_0) \rangle$ ,  $\bar{P}_i = \langle (k-2, \bar{h}_i), (k-1, \bar{h}_i), H_{k-1}^{-1}(\bar{h}_i, \bar{h}_{i+1} + 1), (k-1, \bar{h}_{i+1} + 1), (k, \bar{h}_{i+1} + 1), H_k(\bar{h}_{i+1} + 1, \bar{h}_i), (k, \bar{h}_i), (k+1, \bar{h}_i) \rangle$  for  $1 \leq i \leq s-2$ , and  $\bar{P}_{s-1} = \langle (k-2, \bar{h}_{s-1}), (k-1, \bar{h}_{s-1}), H_{k-1}^{-1}(\bar{h}_{s-1}, 0), (k-1, 0), (k, 0), H_k(0, \bar{h}_{s-1}), (k, \bar{h}_{s-1}), (k+1, \bar{h}_{s-1}) \rangle$ . Then  $\overline{C}(w, b) \cup P_0 \cup \{\bar{P}_i \mid 0 \leq i \leq s-1\}$  is the 3\*-container (or 4\*-container) of  $Q_2^{k+2}$ .

**Case 2.2.**  $t \geq 2$ .

Let  $P_i = \langle (\bar{v}_i, k-2), (\bar{v}_i, k-1), I_{k-1}(\bar{v}_i, \bar{v}_{i+1} - 1), (\bar{v}_{i+1} - 1, k-1), (\bar{v}_{i+1} - 1, k), I_k^{-1}(\bar{v}_{i+1} - 1, \bar{v}_i), (\bar{v}_i, k), (\bar{v}_i, k+1) \rangle$  for  $0 \leq i \leq t-2$ , and  $P_{t-1} = \langle (\bar{v}_{t-1}, k-2), (\bar{v}_{t-1}, k-1), I_{k-1}(\bar{v}_{t-1}, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \bar{v}_{t-1}), (\bar{v}_{t-1}, k), (\bar{v}_{t-1}, k+1) \rangle$ .

**Case 2.2.1**  $s = 1$ .

Using the same  $\bar{P}_0$  as in Case 2.1.1, then  $\overline{C}(w, b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \bar{P}_0$  is the 3\*-container (or 4\*-container) of  $Q_2^{k+2}$ .

**Case 2.2.2**  $s = 2$ .

Using the same  $\bar{P}_0$  and  $\bar{P}_1$  as in Case 2.1.2., then  $\overline{C}(w, b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \bar{P}_0 \cup \bar{P}_1$  is the 3\*-container (or 4\*-container) of  $Q_2^{k+2}$ .

**Case 2.2.3**  $s \geq 3$ .

Using the same  $\{\bar{P}_i \mid 0 \leq i \leq s-1\}$  as in Case 2.1.3., then  $\overline{C}(w, b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \{\bar{P}_i \mid 0 \leq i \leq s-1\}$  is the 3\*-container (or 4\*-container) of  $Q_2^{k+2}$ .

**Case 3.**  $v_{t-1} \leq k-2$  and  $((k-2, k-1), (k-1, k-1)) \notin E(C(w, b))$  in  $Q_2^k$ .

**Case 3.1.**  $t = 1$ .

Let  $P_0 = \langle (\bar{v}_0, k-2), (\bar{v}_0, k-1), I_{k-1}(\bar{v}_0, k-1), (k-1, k-1), H_{k-1}^{-1}(k-1, \bar{h}_0 + 1), (k-1, \bar{h}_0 + 1), (k, \bar{h}_0 + 1), H_k(\bar{h}_0 + 1, k-1), (k, k-1), (k+1, k-1), (0, k-1), I_{k-1}(0, \bar{v}_0 - 1), (\bar{v}_0 - 1, k-1), (\bar{v}_0 - 1, k), I_k^{-1}(\bar{v}_0 - 1, 0), (0, k), (k+1, k), (k, k), (k, k+1), (k-1, k+1), (k-1, k), I_k^{-1}(k-1, \bar{v}_0), (\bar{v}_0, k), (\bar{v}_0, k+1) \rangle$ .

**Case 3.1.1**  $s = 1$ .

Let  $\bar{P}_0 = \langle (k-2, \bar{h}_0), (k-1, \bar{h}_0), H_{k-1}^{-1}(\bar{h}_0, 0), (k-1, 0), (k, 0), H_k(0, \bar{h}_0), (k, \bar{h}_0), (k+1, \bar{h}_0) \rangle$ . Then  $\overline{C}(w, b) \cup P_0 \cup \bar{P}_0$  is the 3\*-container (or 4\*-container) of  $Q_2^{k+2}$ .

**Case 3.1.2**  $s \geq 2$ .

Let  $\bar{P}_i = \langle (k-2, \bar{h}_i), (k-1, \bar{h}_i), H_{k-1}^{-1}(\bar{h}_i, \bar{h}_{i+1} + 1), (k-1, \bar{h}_{i+1} + 1), (k, \bar{h}_{i+1} + 1), H_k(\bar{h}_{i+1} + 1, \bar{h}_i), (k, \bar{h}_i), (k+1, \bar{h}_i) \rangle$

for  $0 \leq i \leq s-2$ , and  $\bar{P}_{s-1} = \langle (k-2, \bar{h}_{s-1}), (k-1, \bar{h}_{s-1}), H_{k-1}^{-1}(\bar{h}_{s-1}, 0), (k-1, 0), (k, 0), H_k(0, \bar{h}_{s-1}), (k, \bar{h}_{s-1}), (k+1, \bar{h}_{s-1}) \rangle$ . Then  $\bar{C}(w, b) \cup P_0 \cup \{\bar{P}_i \mid 0 \leq i \leq s-1\}$  is the  $3^*$ -container (or  $4^*$ -container) of  $Q_2^{k+2}$ .

**Case 3.2.**  $t \geq 2$ .

Let  $P_i = \langle (\bar{v}_i, k-2), (\bar{v}_i, k-1), I_{k-1}(\bar{v}_i, \bar{v}_{i+1}-1), (\bar{v}_{i+1}-1, k-1), (\bar{v}_{i+1}-1, k), I_k^{-1}(\bar{v}_{i+1}-1, \bar{v}_i), (\bar{v}_i, k), (\bar{v}_i, k+1) \rangle$  for  $0 \leq i \leq t-2$ , and  $P_{t-1} = \langle (\bar{v}_{t-1}, k-2), (\bar{v}_{t-1}, k-1), I_{k-1}(\bar{v}_{t-1}, k-1), (k-1, k-1), H_{k-1}^{-1}(k-1, \bar{h}_0+1), (k-1, \bar{h}_0+1), (k, \bar{h}_0+1), H_k(\bar{h}_0+1, k-1), (k, k-1), (k+1, k-1), (0, k-1), I_{k-1}(0, \bar{v}_0-1), (\bar{v}_0-1, k-1), (\bar{v}_0-1, k), I_k^{-1}(\bar{v}_0-1, 0), (0, k), (k+1, k), (k, k), (k, k+1), (k-1, k+1), (k-1, k), I_k^{-1}(k-1, \bar{v}_{t-1}), (\bar{v}_{t-1}, k), (\bar{v}_{t-1}, k+1) \rangle$ .

**Case 3.2.1**  $s = 1$ .

Using the same  $\bar{P}_0$  as in Case 3.1.1, then  $\bar{C}(w, b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \bar{P}_0$  is the  $3^*$ -container (or  $4^*$ -container) of  $Q_2^{k+2}$ .

**Case 3.2.2**  $s \geq 2$ .

Using the same  $\{\bar{P}_i \mid 0 \leq i \leq s-1\}$  as in Case 3.1.2., then  $\bar{C}(w, b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \{\bar{P}_i \mid 0 \leq i \leq s-1\}$  is the  $3^*$ -container (or  $4^*$ -container) of  $Q_2^{k+2}$ .

**Case 4.**  $v_{t-1} = k-1$  for some  $t \geq 2$  and  $v_0 = 0$ .

**Case 4.1.**  $t = 2$ .

Let  $P_0 = \langle (\bar{v}_0, k-2), (\bar{v}_0, k-1), I_{k-1}(\bar{v}_0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \bar{v}_0), (\bar{v}_0, k), (\bar{v}_0, k+1) \rangle$ , and  $P_1 = \langle (k+1, k-2), (k+1, k-1), (k+1, k), (k+1, k+1) \rangle$ .

**Case 4.1.1.**  $s = 1$ .

Using the same  $\bar{P}_0$  as in Case 1.1., then  $\bar{C}(w, b) \cup P_0 \cup P_1 \cup \bar{P}_0$  is the  $3^*$ -container (or  $4^*$ -container) of  $Q_2^{k+2}$ .

**Case 4.1.2.**  $s \geq 2$ .

Using the same  $\{\bar{P}_i \mid 0 \leq i \leq s-1\}$  as in Case 1.2., then  $\bar{C}(w, b) \cup P_0 \cup P_1 \cup \{\bar{P}_i \mid 0 \leq i \leq s-1\}$  is the  $3^*$ -container (or  $4^*$ -container) of  $Q_2^{k+2}$ .

**Case 4.2.**  $t \geq 3$ .

Let  $P_i = \langle (\bar{v}_i, k-2), (\bar{v}_i, k-1), I_{k-1}(\bar{v}_i, \bar{v}_{i+1}-1), (\bar{v}_{i+1}-1, k-1), (\bar{v}_{i+1}-1, k), I_k^{-1}(\bar{v}_{i+1}-1, \bar{v}_i), (\bar{v}_i, k), (\bar{v}_i, k+1) \rangle$  for  $0 \leq i \leq t-3$ ,  $P_{t-2} = \langle (\bar{v}_{t-2}, k-2), (\bar{v}_{t-2}, k-1), I_{k-1}(\bar{v}_{t-2}, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \bar{v}_{t-2}), (\bar{v}_{t-2}, k), (\bar{v}_{t-2}, k+1) \rangle$ , and  $P_{t-1} = \langle (k+1, k-2), (k+1, k-1), (k+1, k), (k+1, k+1) \rangle$ .

**Case 4.2.1.**  $s = 1$ .

Using the same  $\bar{P}_0$  as in Case 1.1., then  $\bar{C}(w, b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \bar{P}_0$  is the  $3^*$ -container (or  $4^*$ -container) of  $Q_2^{k+2}$ .

**Case 4.2.2.**  $s \geq 2$ .

Using the same  $\{\bar{P}_i \mid 0 \leq i \leq s-1\}$  as in Case 1.2., then  $\bar{C}(w, b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \{\bar{P}_i \mid 0 \leq i \leq s-1\}$  is the  $3^*$ -container (or  $4^*$ -container) of  $Q_2^{k+2}$ . ■

**Theorem 3.** For any integer  $n \geq 2$  and any even integer  $k \geq 4$ , the graph  $Q_n^k$  is  $m^*$ -laceable where  $1 \leq m \leq 2n$ .

*Proof:* According to Theorem 2-3 and Lemma 3-5, the theorem holds for any even integer  $k \geq 4$  when  $n = 2$ . We will give the proof of the theorem by mathematical induction on  $n$ . By induction hypothesis, assume that  $Q_{n-1}^{k,i}$  is  $m^*$ -laceable for  $1 \leq m \leq 2n-2$ , where  $0 \leq i \leq k-1$ . Given a white vertex  $w \in V(Q_{n-1}^{k,j})$  and a black vertex  $b \in V(Q_{n-1}^{k,i})$ . We will show that we can use the  $m^*$ -containers of  $Q_{n-1}^{k,j}$  to construct a  $(m+2)^*$ -container of  $Q_n^k$  between  $w$  and  $b$ .

**Case 1.** For  $j = j'$ . Without loss of generality, we let  $j = j' = 0$ .

In this case, we have  $\{w, b\} \in Q_{n-1}^{k,0}$ . By induction hypothesis, there are  $m$  internal disjoint paths  $\{P_i\}_{i=0}^{m-1}$  whose union covers all vertices of  $Q_{n-1}^{k,0}$  between  $w$  and  $b$  for  $1 \leq m \leq 2n-2$ . By Lemma 2, there exists a path  $S$  covering all vertices of  $Q_{n-1}^{k,i}$  for  $1 \leq i \leq k-2$  between  $w^1$  and  $b^1$ . We can let  $P_m = \langle w, w^1, S, b^1, b \rangle$ . In  $Q_{n-1}^{k,k-1}$ , there exist a hamiltonian path  $R$  joining from  $w^{k-1}$  to  $b^{k-1}$  by Theorem 1. Also, we can let  $P_{m+1} = \langle w, w^{k-1}, R, b^{k-1}, b \rangle$ . Therefore, there are  $m+2$  internal disjoint paths  $\{P_i\}_{i=0}^{m+1}$  whose union covers all vertices of  $Q_n^k$  between  $w$  and  $b$ . Please see Figure 6 for an illustration.

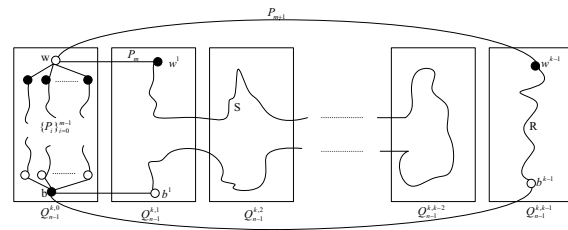


Fig. 6. The illustration for Case 1 of Theorem 3.

**Case 2.** For  $|j' - j| = 1$ . Without loss of generality, we let  $j = 0$  and  $j' = 1$ .

We have the following two cases.

**Case 2.1.** Suppose that  $d(w, b) = 1$ . It is easy to see that we can let  $P_{m+1} = \langle w, b \rangle$ .

**Case 2.1.1.** If  $m = 1$ .

Let  $z$  be any black vertex of  $Q_{n-1}^{k,0}$ . By Theorem 1, there exist a hamiltonian path  $S$  of  $Q_{n-1}^{k,0}$  from  $w$  to  $z$ , and a hamiltonian path  $T$  of  $Q_{n-1}^{k,1}$  from  $z^1$  to  $b$ . So we set  $P_0 = \langle w, S, z, z^1, T, b \rangle$ . According to Lemma 1, a hamiltonian path  $R$  between  $w^{k-1} \in Q_{n-1}^{k,k-1}$  and  $b^2 \in Q_{n-1}^{k,2}$  covers all vertices of  $Q_{n-1}^{k,i}$  for  $2 \leq i \leq k-1$ . We can write  $P_1$  as  $\langle w, w^{k-1}, R, b^2, b \rangle$ . Hence, there are 3 internal disjoint paths  $\{P_0, P_1, P_2\}$  whose union covers all vertices of  $Q_n^k$  between  $w$  and  $b$ . Please see Figure 7 for an illustration.

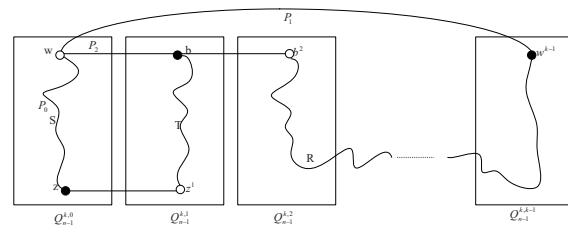


Fig. 7. The illustration for Case 2.1.1 of Theorem 3.

**Case 2.1.2.** If  $m \geq 2$ .

According to the induction hypothesis, given any black vertex  $z \in V(Q_{n-1}^{k,0} - N(w))$ , there exist  $m$  internal disjoint paths  $\{R_i\}_{i=0}^{m-1}$  whose union covers all vertices of  $Q_{n-1}^{k,0}$  between  $w$  and  $z$  for  $2 \leq m \leq 2n-2$ . Let  $R_i = \langle w, S_i, y_i, z \rangle$  for  $0 \leq i \leq m-1$ . We set  $P_0 = \langle w, S_0, y_0, z, z^1, y_0^1, (S_0^1)^{-1}, b \rangle$  and  $P_i = \langle w, S_i, y_i, y_i^1, (S_i^1)^{-1}, b \rangle$  for  $1 \leq i \leq m-1$ . By

Lemma 1, there is a hamiltonian path  $T$  between  $w^{k-1} \in Q_{n-1}^{k,k-1}$  and  $b^2 \in Q_{n-1}^{k,2}$  covering all vertices of  $Q_{n-1}^{k,i}$  for  $2 \leq i \leq k-1$ . Set  $P_m = \langle w, w^{k-1}, T, b^2, b \rangle$ . Consequently, there are  $m+2$  internal disjoint paths  $\{P_i\}_{i=0}^{m+1}$  whose union covers all vertices of  $Q_n^k$  between  $w$  and  $b$ . Please see Figure 8 for an illustration.

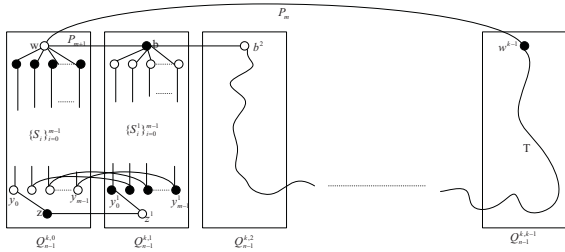


Fig. 8. The illustration for Case 2.1.2 of Theorem 3.

**Case 2.2.** Suppose that  $d(w, b) \geq 3$ .

**Case 2.2.1.** If  $m = 1$ .

Given any black vertex  $z$  in  $Q_{n-1}^{k,0}$ , by Theorem 1, there is a hamiltonian path  $R$  of  $Q_{n-1}^{k,0}$  joining from  $w$  to  $z$ . So there is also a hamiltonian path  $S$  of  $Q_{n-1}^{k,1}$  between  $w^1$  to  $z^1$ . We can set  $S = \langle w^1, S'_1, b, S'_2, z^1 \rangle$ . By Lemma 1, there exists a hamiltonian path  $T$  between  $w^{k-1} \in Q_{n-1}^{k,k-1}$  and  $b^2 \in Q_{n-1}^{k,2}$  covering all vertices of  $Q_{n-1}^{k,i}$  for  $2 \leq i \leq k-1$ . We let  $P_0 = \langle w, R, z, z^1, (S'_2)^{-1}, b \rangle$ ,  $P_1 = \langle w, w^1, S'_1, b \rangle$ , and  $P_2 = \langle w, w^{k-1}, T, b^2, b \rangle$ . Therefore, there are 3 internal disjoint paths  $\{P_0, P_1, P_2\}$  whose union covers all vertices of  $Q_n^k$  between  $w$  and  $b$ . Please see Figure 9 for an illustration.

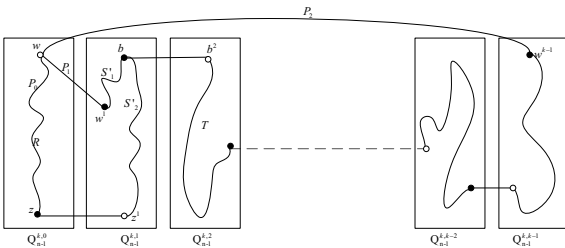


Fig. 9. The illustration for Case 2.2.1 of Theorem 3.

**Case 2.2.2.** If  $m \geq 2$ .

Let  $z$  be a black vertex of  $V(Q_{n-1}^{k,0} - N(w))$ . In  $Q_{n-1}^{k,0}$ , according to the induction hypothesis, there exist  $m$  internal disjoint paths  $\{S_i\}_{i=0}^{m-1}$  whose union covers all vertices of  $Q_{n-1}^{k,0}$  between  $w$  and  $z$  for  $2 \leq m \leq 2n-2$ . So as in  $Q_{n-1}^{k,1}$ , there exist  $m$  internal disjoint paths  $\{T_i\}_{i=0}^{m-1}$  whose union covers all vertices of  $Q_{n-1}^{k,1}$  between  $z^1$  and  $b$  for  $2 \leq m \leq 2n-2$ . Let  $T_0 = \langle z^1, y_0, T'_0, x_0, w^1, T''_0, b \rangle$  and  $T_i = \langle z^1, y_i, T'_i, b \rangle$  for  $1 \leq i \leq m-1$  in  $Q_{n-1}^{k,1}$ .

**Case 2.2.2.1.** If  $b^0 \notin V(S_0)$ .

Without loss of generality, let  $b^0 \in V(S_{m-1})$ . In  $Q_{n-1}^{k,0}$ , we also let  $S_0 = \langle w, x_0, e, S'_0, y_0, z \rangle$ ,  $S_i = \langle w, S'_i, y_i, z \rangle$  for  $1 \leq i \leq m-2$ , and  $S_{m-1} = \langle w, S'_{m-1}, b^0, f, S''_{m-1}, y_{m-1}, z \rangle$ . A hamiltonian path  $R$  is embedded in  $Q_{n-1}^{k,k-1}$  between  $w^{k-1}$  and  $f^{k-1}$  by Theorem 1. Write  $R$  as  $\langle w^{k-1}, R', e^{k-1}, g, R'', f^{k-1} \rangle$ . Notice that  $g^{k-2}$  is a black vertex and  $b^2$  is a white vertex.

According to Lemma 1, there is a hamiltonian path  $U$  between  $g^{k-2}$  and  $b^2$  covering all vertices of  $Q_{n-1}^{k,i}$  for  $2 \leq i \leq k-2$ . We can set  $P_0 = \langle w, x_0, x_0, (T'_0)^{-1}, y_0, z^1, y_{m-1}, T_{m-1}, b \rangle$ ,  $P_1 = \langle w, w^1, T''_0, b \rangle$ ,  $P_2 = \langle w, w^{k-1}, R', e^{k-1}, e, S'_0, y_0, z, y_{m-1}, (S''_{m-1})^{-1}, f, f^{k-1}, (R'')^{-1}, g, g^{k-2}, U, b^2, b \rangle$ ,  $P_3 = \langle w, S'_{m-1}, b^0, b \rangle$ , and  $P_i = \langle w, S'_{i-3}, y_{i-3}^0, y_{i-3}, T'_{i-3}, b \rangle$  for  $4 \leq i \leq m+1$ . So, there are  $m+2$  internal disjoint paths  $\{P_i\}_{i=0}^{m+1}$  whose union covers all vertices of  $Q_n^k$  between  $w$  and  $b$ . Please see Figure 10 for an illustration.

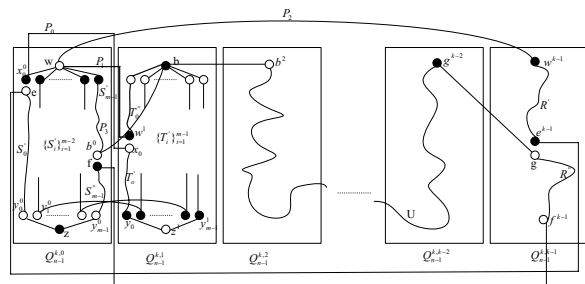


Fig. 10. The illustration for Case 2.2.2.1 of Theorem 3.

**Case 2.2.2.2.** If  $b^0 \in V(S_0)$ .

Let  $S_0 = \langle w, x_0, e, S'_0, b^0, f, S''_0, y_0, z \rangle$ , and  $S_i = \langle w, S'_i, y_i, z \rangle$  for  $1 \leq i \leq m-1$ . A hamiltonian path  $R$  is embedded in  $Q_{n-1}^{k,k-1}$  between  $w^{k-1}$  and  $f^{k-1}$  by Theorem 1.  $R$  is written as  $\langle w^{k-1}, R', e^{k-1}, g, R'', f^{k-1} \rangle$ . Notice that  $g^{k-2}$  is a black vertex and  $b^2$  is a white vertex. According to Lemma 1, there is a hamiltonian path  $U$  between  $g^{k-2}$  and  $b^2$  covering all vertices of  $Q_{n-1}^{k,i}$  for  $2 \leq i \leq k-2$ . We let  $P_0 = \langle w, x_0, x_0, (T'_0)^{-1}, y_0, z^1, y_{m-1}, T_{m-1}, b \rangle$ ,  $P_1 = \langle w, w^1, T''_0, b \rangle$ ,  $P_2 = \langle w, w^{k-1}, R', e^{k-1}, e, S'_0, b^0, b \rangle$ ,  $P_3 = \langle w, S'_{m-1}, y_{m-1}, z, y_0, (S''_0)^{-1}, f, f^{k-1}, (R'')^{-1}, g, g^{k-2}, U, b^2, b \rangle$ , and  $P_i = \langle w, S'_{i-3}, y_{i-3}^0, y_{i-3}, T'_{i-3}, b \rangle$  for  $4 \leq i \leq m+1$ . Hence, there are  $m+2$  internal disjoint paths  $\{P_i\}_{i=0}^{m+1}$  whose union covers all vertices of  $Q_n^k$  between  $w$  and  $b$ . Please see Figure 11 for an illustration.

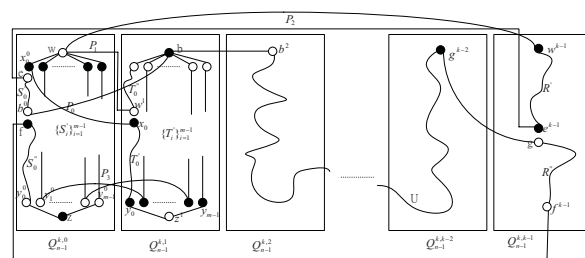


Fig. 11. The illustration for Case 2.2.2.2 of Theorem 3.

**Case 3.** For  $|j' - j| \geq 2$ . Without loss of generality, we let  $j = 0$  and  $2 \leq j' \leq \frac{k}{2}$  be even.

Because  $b \in Q_{n-1}^{k,j'}$  where  $j'$  is even,  $b^i$  is a white (resp. black) vertex in  $Q_{n-1}^{k,i}$  for  $0 \leq i \leq k-1$  when  $i$  is odd (resp. even). It is easy to see that  $w^i$  is a black (resp. white) vertex in  $Q_{n-1}^{k,i}$  for  $0 \leq i \leq k-1$  when  $i$  is odd (resp. even). By the induction hypothesis, there exist  $m$  internal disjoint paths  $\{R_p^i\}_{p=0}^{m-1}$  of  $Q_{n-1}^{k,i}$  between  $w^i$  and  $b^i$  for  $0 \leq i \leq j'$ .

Let  $R_p^i = \langle w^i, x_p^i, U_p^i, y_p^i, b^i \rangle$  for  $0 \leq p \leq m - 1$  and  $0 \leq i \leq j'$ . According to Lemma 2, a hamiltonian path  $S$  covers all vertices of  $Q_{n-1}^{k,i}$  for  $j'+1 \leq i \leq k-2$  joining from  $w^{j'+1}$  to  $b^{j'+1}$ . There is a hamiltonian path  $T$  of  $Q_{n-1}^{k,k-1}$  from  $w^{k-1}$  to  $b^{k-1}$  by Theorem 1. Hence, we can write  $P_p = \langle w = w^0, x_p^0, U_p^0, y_p^0, y_p^1, (U_p^1)^{-1}, x_p^1, x_p^2, U_p^2, \dots, (U_p^{j'-1})^{-1}, x_p^{j'-1}, x_p^{j'}, U_p^{j'}, y_p^{j'}, b^{j'} = b \rangle$  for  $0 \leq p \leq m-1$ ,  $P_m = \langle w = w^0, w^1, w^2, \dots, w^{j'}, w^{j'+1}, S, b^{j'+1}, b^{j'} = b \rangle$ , and  $P_{m+1} = \langle w = w^0, w^{k-1}, T, b^{k-1}, b^0, b^1, \dots, b^{j'-1}, b^{j'} = b \rangle$ . Therefore, there are  $m+2$  internal disjoint paths  $\{P_i\}_{i=0}^{m+1}$  whose union covers all vertices of  $Q_n^k$  between  $w$  and  $b$ . Please see Figure 12 for an illustration.

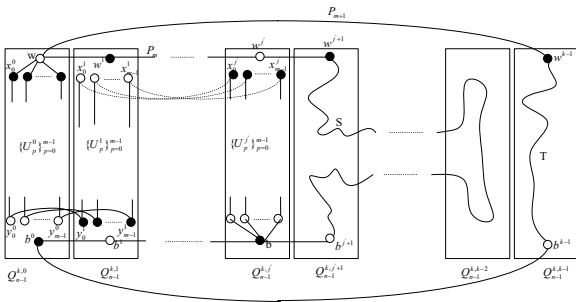


Fig. 12. The illustration for Case 3 of Theorem 3.

**Case 4.** For  $|j' - j| \geq 2$ . Without loss of generality, we let  $j = 0$  and  $3 \leq j' \leq \frac{k}{2} + 1$  be odd.

**Case 4.1.** If  $m = 1$ .

Choosing a black vertex  $z$  of  $Q_{n-1}^{k,0}$ , by Theorem 1, there is a hamiltonian path  $R$  of  $Q_{n-1}^{k,0}$  joining from  $w$  to  $z$ . In  $Q_{n-1}^{k,k-1}$ , there exists a hamiltonian path  $S$  of  $Q_{n-1}^{k,k-1}$  between  $w^{k-1}$  and  $z^{k-1}$ . We can let  $S = \langle w^{k-1}, S', e, b^{k-1}, S'', z^{k-1} \rangle$ , where  $b^{k-1}$  is a black vertex of  $Q_{n-1}^{k,k-1}$ , so  $e$  is a white vertex of  $Q_{n-1}^{k,k-1}$ . By Theorem 1, there is a hamiltonian path  $T$  of  $Q_{n-1}^{k,k-2}$  joining from  $e^{k-2}$  to  $b^{k-2}$ . Let  $T = \langle e^{k-2}, W, f^{k-2}, b^{k-2} \rangle$ . In  $Q_{n-1}^{k,i}$ , we also have a hamiltonian path  $T^i$  between  $e^i$  and  $b^i$  for  $j' \leq i \leq k-3$ , so we let  $T^i = \langle e^i, W^i, f^i, b^i \rangle$ . According to Lemma 1, there is a hamiltonian path  $U$  between a black vertex  $w^1 \in Q_{n-1}^{k,1}$  and a white vertex  $b^{j'-1} \in Q_{n-1}^{k,j'-1}$  covering all vertices of  $Q_{n-1}^{k,i}$  for  $2 \leq i \leq j' - 1$ . We set  $P_0 = \langle w, w^1, U, b^{j'-1}, b \rangle$ ,  $P_1 = \langle w, R, z, z^{k-1}, (S'')^{-1}, b^{k-1}, b^{k-2}, \dots, b^{j'+1}, b^{j'} = b \rangle$ , and  $P_2 = \langle w, w^{k-1}, S', e, e^{k-2}, W, f^{k-2}, f^{k-3}, (W^{k-3})^{-1}, e^{k-3}, e^{k-4}, W^{k-4}, f^{k-4}, \dots, e^{j'+1}, W^{j'+1}, f^{j'+1}, f^{j'}, W^{j'}, b^{j'} = b \rangle$ . Hence, there are 3 internal disjoint paths  $\{P_0, P_1, P_2\}$  whose union covers all vertices of  $Q_n^k$  between  $w$  and  $b$ . Please see Figure 13 for an illustration.

**Case 4.2.** If  $m \geq 2$ .

Given a white vertex  $z$  in  $Q_{n-1}^{k,j'}$  such that  $z$  is adjacent to  $b$ . So  $z^i$  is a black (resp. white) vertex and  $w^i$  is a white (reps. black) vertex of  $Q_{n-1}^{k,i}$  if  $0 \leq i \leq j' - 1$  when  $i$  is even (resp. odd). By the induction hypothesis, there exist  $m$  internal disjoint paths  $\{R_i\}_{i=0}^{m-1}$  of  $Q_{n-1}^{k,0}$  between  $w$  and  $z^0$ . We write  $R_0 = \langle w, x_0(1), x_0(2), \dots, x_0(\alpha), z^0 \rangle$ , and  $R_p = \langle w, x_p, S_p, y_p, z^0 \rangle$  for  $1 \leq p \leq m - 1$ . Again, by the induction hypothesis, there exist  $m$  internal disjoint paths

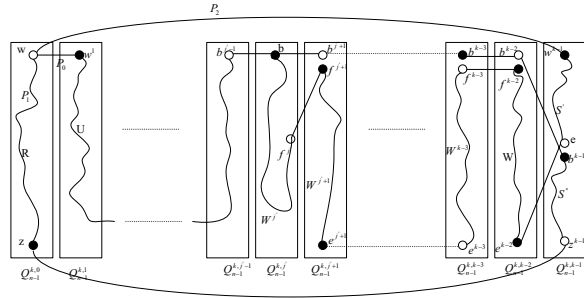


Fig. 13. The illustration for Case 4.1 of Theorem 3.

$\{T_p^i\}_{p=0}^{m-1}$  of  $Q_{n-1}^{k,i}$  between  $w^i$  and  $z^i$  for  $2 \leq i \leq j' - 1$ . We let  $T_p^i = \langle w^i, x_p^i, U_p^i, t_p^i, z^i \rangle$  for  $0 \leq p \leq m - 1$  and  $2 \leq i \leq j' - 1$ . Notice that  $b^{j'-1}$  is adjacent to  $z^{j'-1}$ , without loss of generality, we let  $t_{m-1}^{j'-1} = b^{j'-1}$ . In  $Q_{n-1}^{k,j'}$ , there are  $m$  internal disjoint paths  $\{W_i\}_{i=0}^{m-1}$  from  $b$  to  $z$  by the induction hypothesis. We can write  $W_p = \langle z, t_p^{j'}, Y_p, b \rangle$  for  $0 \leq p \leq m - 2$  and  $W_{m-1} = \langle z, b \rangle$ . According to Lemma 1, there is a hamiltonian path  $V$  between  $w^{k-1} \in Q_{n-1}^{k,k-1}$  and  $b^{j'+1} \in Q_{n-1}^{k,j'+1}$  covering all vertices of  $Q_{n-1}^{k,i}$  for  $j' + 1 \leq i \leq k - 1$ . Set  $P_0 = \langle w, w^{k-1}, V, b^{j'+1}, b \rangle$ ,  $P_1 = \langle w, w^1, w^2, x_0^2, U_0^2, t_0^2, t_0^3, (U_0^3)^{-1}, x_0^3, w^3, w^4, \dots, w^{j'-1}, x_0^{j'-1}, U_0^{j'-1}, t_0^{j'-1}, t_0^{j'}, Y_0, b \rangle$ ,  $P_2 = \langle w, x_0(1), x_0^1(1), x_0^1(2), x_0(2), \dots, x_0(\alpha - 1), x_0^1(\alpha - 1), x_0^1(\alpha), x_0(\alpha), z^0, z^1, \dots, z^{j'}, b \rangle$ ,  $P_3 = \langle w, x_{m-1}, S_{m-1}, y_{m-1}, y_{m-1}^1, (S_{m-1}^1)^{-1}, x_{m-1}^1, x_{m-1}^2, U_{m-1}^2, t_{m-1}^2, t_{m-1}^3, (U_{m-1}^3)^{-1}, x_{m-1}^3, \dots, x_{m-1}^{j'-1}, U_{m-1}^{j'-1}, t_{m-1}^{j'-1} = b^{j'-1}, b \rangle$ , and  $P_i = \langle w, x_{i-3}, S_{i-3}, y_{i-3}, y_{i-3}^1, (S_{i-3}^1)^{-1}, x_{i-3}^1, x_{i-3}^2, U_{i-3}^2, t_{i-3}^2, t_{i-3}^3, (U_{i-3}^3)^{-1}, x_{i-3}^3, \dots, x_{i-3}^{j'-1}, U_{i-3}^{j'-1}, t_{i-3}^{j'-1}, t_{i-3}^{j'}, Y_{i-3}, b \rangle$  for  $4 \leq i \leq m + 1$ . So, there are  $m+2$  internal disjoint paths  $\{P_i\}_{i=0}^{m+1}$  whose union covers all vertices of  $Q_n^k$  between  $w$  and  $b$ . Please see Figure 14 for an illustration.

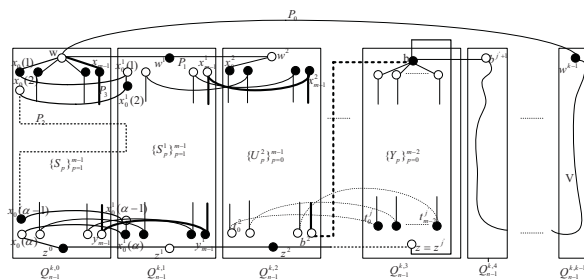


Fig. 14. The illustration for Case 4.2 of Theorem 3.

APPENDIX A  
PROOF OF LEMMA 3

Notice that  $Q_2^4$  is vertex symmetric. W.L.O.G, let  $w = (0, 0)$ . There are only two cases for  $b$ . That is,  $b \in \{(1, 0), (2, 1)\}$ .

**Case 1.** To prove that  $Q_2^4$  is 3\*-laceable.

Case 1.1. Let  $b = (1, 0)$ .

The three disjoint paths  $\{P_1, P_2, P_3\}$  between  $w$  and  $b$  whose

union covers all vertices of  $Q_2^4$  are  $P_1 = \langle(0, 0), (1, 0)\rangle$ ,  $P_2 = \langle(0, 0), (0, 1), (1, 1), (1, 0)\rangle$ , and  $P_3 = \langle(0, 0), (3, 0), (3, 1), (3, 2), (3, 3), (2, 3), (1, 3), (0, 3), (0, 2), (1, 2), (2, 2), (2, 1), (2, 0), (1, 0)\rangle$ .

Case 1.2. Let  $b = (2, 1)$ .

The three disjoint paths  $\{R_1, R_2, R_3\}$  between  $w$  and  $b$  whose union covers all vertices of  $Q_2^4$  are  $R_1 = \langle(0, 0), (1, 0), (2, 0), (2, 1)\rangle$ ,  $R_2 = \langle(0, 0), (0, 1), (1, 1), (2, 1)\rangle$ , and  $R_3 = \langle(0, 0), (3, 0), (3, 1), (3, 2), (3, 3), (2, 3), (1, 3), (0, 3), (0, 2), (1, 2), (2, 2), (2, 1)\rangle$ .

**Case 2.** To prove that  $Q_2^4$  is 4\*-laceable.

Case 2.1. Let  $b = (1, 0)$ .

The four disjoint paths  $\{P_1, P_2, P_3, P_4\}$  between  $w$  and  $b$  whose union covers all vertices of  $Q_2^4$  are  $P_1 = \langle(0, 0), (1, 0)\rangle$ ,  $P_2 = \langle(0, 0), (0, 1), (1, 1), (1, 0)\rangle$ ,  $P_3 = \langle(0, 0), (0, 3), (0, 2), (1, 2), (1, 3), (1, 0)\rangle$ , and  $P_4 = \langle(0, 0), (3, 0), (3, 1), (3, 2), (3, 3), (2, 3), (2, 2), (2, 1), (2, 0), (1, 0)\rangle$ .

Case 2.2. Let  $b = (2, 1)$ .

The four disjoint paths  $\{R_1, R_2, R_3, R_4\}$  between  $w$  and  $b$  whose union covers all vertices of  $Q_2^4$  are  $R_1 = \langle(0, 0), (3, 0), (3, 1), (2, 1)\rangle$ ,  $R_2 = \langle(0, 0), (1, 0), (2, 0), (2, 1)\rangle$ ,  $R_3 = \langle(0, 0), (0, 1), (1, 1), (2, 1)\rangle$ , and  $R_4 = \langle(0, 0), (0, 3), (0, 2), (1, 2), (1, 3), (2, 3), (3, 3), (3, 2), (2, 2), (2, 1)\rangle$ .

#### APPENDIX B PROOF OF LEMMA 4

Notice that  $Q_2^6$  is vertex symmetric. W.L.O.G, let  $w = (0, 0)$ . There are four cases for  $b$ . That is,  $b \in \{(1, 0), (2, 1), (3, 0), (3, 2)\}$ .

**Case 1.** To prove that  $Q_2^6$  is 3\*-laceable.

Case 1.1. Let  $b = (1, 0)$ .

The three disjoint paths  $\{P_1, P_2, P_3\}$  between  $w$  and  $b$  whose union covers all vertices of  $Q_2^6$  are  $P_1 = \langle(0, 0), (1, 0)\rangle$ ,  $P_2 = \langle(0, 0), (0, 1), (1, 1), (1, 0)\rangle$ , and  $P_3 = \langle(0, 0), (5, 0), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (3, 5), (2, 5), (1, 5), (0, 5), (0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (3, 0), (3, 1), (3, 2), (3, 3), (2, 3), (1, 3), (0, 3), (0, 2), (1, 2), (2, 2), (2, 1), (2, 0), (1, 0)\rangle$ .

Case 1.2. Let  $b = (2, 1)$ .

The three disjoint paths  $\{R_1, R_2, R_3\}$  between  $w$  and  $b$  whose union covers all vertices of  $Q_2^6$  are  $R_1 = \langle(0, 0), (1, 0), (2, 0), (2, 1)\rangle$ ,  $R_2 = \langle(0, 0), (0, 1), (1, 1), (2, 1)\rangle$ , and  $R_3 = \langle(0, 0), (5, 0), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (3, 5), (2, 5), (1, 5), (0, 5), (0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (3, 0), (3, 1), (3, 2), (3, 3), (2, 3), (1, 3), (0, 3), (0, 2), (1, 2), (2, 2), (2, 1)\rangle$ .

Case 1.3. Let  $b = (3, 0)$ .

The three disjoint paths  $\{S_1, S_2, S_3\}$  between  $w$  and  $b$  whose union covers all vertices of  $Q_2^6$  are  $S_1 = \langle(0, 0), (1, 0), (2, 0), (3, 0)\rangle$ ,  $S_2 = \langle(0, 0), (5, 0), (4, 0), (3, 0)\rangle$ , and  $S_3 = \langle(0, 0), (0, 5), (1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (5, 4), (4, 4), (3, 4), (2, 4), (1, 4), (0, 4), (0, 3), (1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (5, 2), (5, 1), (4, 1), (4, 2), (3, 2), (2, 2), (1, 2), (0, 2), (0, 1), (1, 1), (2, 1), (3, 1), (3, 0)\rangle$ .

Case 1.4. Let  $b = (3, 2)$ .

The three disjoint paths  $\{T_1, T_2, T_3\}$  between  $w$  and  $b$  whose union covers all vertices of  $Q_2^6$  are  $T_1 = \langle(0, 0), (1, 0), (2, 0),$

$(3, 0), (3, 1), (3, 2)\rangle$ ,  $T_2 = \langle(0, 0), (0, 1), (0, 2), (1, 2), (1, 1), (2, 1), (2, 2), (3, 2)\rangle$ , and  $T_3 = \langle(0, 0), (5, 0), (4, 0), (4, 1), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (3, 5), (2, 5), (1, 5), (0, 5), (0, 4), (0, 3), (1, 3), (1, 4), (2, 4), (2, 3), (3, 3), (3, 4), (4, 4), (4, 3), (4, 2), (3, 2)\rangle$ .

**Case 2.** To prove that  $Q_2^6$  is 4\*-laceable.

Case 2.1. Let  $b = (1, 0)$ .

The four disjoint paths  $\{P_1, P_2, P_3, P_4\}$  between  $w$  and  $b$  whose union covers all vertices of  $Q_2^6$  are  $P_1 = \langle(0, 0), (1, 0)\rangle$ ,  $P_2 = \langle(0, 0), (0, 1), (1, 1), (1, 0)\rangle$ ,  $P_3 = \langle(0, 0), (0, 5), (0, 4), (0, 3), (0, 2), (1, 2), (1, 3), (1, 4), (1, 5), (1, 0)\rangle$ , and  $P_4 = \langle(0, 0), (5, 0), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (3, 0), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (2, 5), (2, 4), (2, 3), (2, 2), (2, 1), (2, 0), (1, 0)\rangle$ .

Case 2.2. Let  $b = (2, 1)$ .

The four disjoint paths  $\{R_1, R_2, R_3, R_4\}$  between  $w$  and  $b$  whose union covers all vertices of  $Q_2^6$  are  $R_1 = \langle(0, 0), (1, 0), (2, 0), (2, 1)\rangle$ ,  $R_2 = \langle(0, 0), (0, 1), (1, 1), (2, 1)\rangle$ ,  $R_3 = \langle(0, 0), (5, 0), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (3, 0), (3, 1), (2, 1)\rangle$ , and  $R_4 = \langle(0, 0), (0, 5), (1, 5), (2, 5), (3, 5), (3, 4), (2, 4), (1, 4), (0, 4), (0, 3), (0, 2), (1, 2), (1, 3), (2, 3), (3, 3), (3, 2), (2, 2), (2, 1)\rangle$ .

Case 2.3. Let  $b = (3, 0)$ .

The four disjoint paths  $\{S_1, S_2, S_3, S_4\}$  between  $w$  and  $b$  whose union covers all vertices of  $Q_2^6$  are  $S_1 = \langle(0, 0), (1, 0), (2, 0), (3, 0)\rangle$ ,  $S_2 = \langle(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (3, 0)\rangle$ ,  $S_3 = \langle(0, 0), (5, 0), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (3, 0)\rangle$ , and  $S_4 = \langle(0, 0), (0, 5), (0, 4), (0, 3), (0, 2), (1, 2), (1, 3), (1, 4), (1, 5), (2, 5), (2, 4), (2, 3), (2, 2), (3, 2), (3, 3), (3, 4), (3, 5), (3, 0)\rangle$ .

Case 2.4. Let  $b = (3, 2)$ .

The four disjoint paths  $\{T_1, T_2, T_3, T_4\}$  between  $w$  and  $b$  whose union covers all vertices of  $Q_2^6$  are  $T_1 = \langle(0, 0), (1, 0), (2, 0), (3, 0), (3, 1), (3, 2)\rangle$ ,  $T_2 = \langle(0, 0), (0, 1), (0, 2), (1, 2), (1, 1), (2, 1), (2, 2), (3, 2)\rangle$ ,  $T_3 = \langle(0, 0), (5, 0), (4, 0), (4, 1), (5, 1), (5, 2), (4, 2), (3, 2)\rangle$ , and  $T_4 = \langle(0, 0), (0, 5), (1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (5, 4), (5, 3), (4, 3), (4, 4), (3, 4), (2, 4), (1, 4), (0, 4), (0, 3), (1, 3), (2, 3), (3, 3), (3, 2)\rangle$ .

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