

The Sizes of Large Hierarchical Long-Range Percolation Clusters

Yilun Shang

Abstract—We study a long-range percolation model in the hierarchical lattice Ω_N of order N where probability of connection between two nodes separated by distance k is of the form $\min\{\alpha\beta^{-k}, 1\}$, $\alpha \geq 0$ and $\beta > 0$. The parameter α is the percolation parameter, while β describes the long-range nature of the model. The Ω_N is an example of so called ultrametric space, which has remarkable qualitative difference between Euclidean-type lattices. In this paper, we characterize the sizes of large clusters for this model along the line of some prior work. The proof involves a stationary embedding of Ω_N into \mathbb{Z} . The phase diagram of this long-range percolation is well understood.

Keywords—percolation, component, hierarchical lattice, phase transition.

I. INTRODUCTION

PERCOLATION theory in the Euclidean lattice \mathbb{Z}^d started with the work of Broadbent and Hammersley in 1957. The infinity of the space of vertices and its geometry are principal features of this model; see e.g. [11] and references therein. Some questions of percolation in other non-Euclidean infinite systems is formulated in [4]. The study of long-range percolation on \mathbb{Z}^d traces back to [15] and leads to a range of interesting results in probability theory and statistical physics [1], [5], [6], [8], [18], [21]. On the other hand, hierarchical structures have been used in applications in the physics, genetics and social sciences thanks to the multi-scale organization of many natural objects [3], [13], [19], [20].

Recently, long-range percolation is studied on the hierarchical lattice Ω_N of order N (to be defined below), where classical methods for the usual lattice break down. The asymptotic long-range percolation on Ω_N is addressed in [10] for $N \rightarrow \infty$. The work [9], [12], [16] and [17] analyze the phase transition of long-range percolation on Ω_N for finite N using different connection probabilities and methodologies. The contact process on Ω_N for fixed N has been investigated in [2]. In this paper, we investigate the sizes of large connected components (or clusters) in the resulting percolation graph on Ω_N for fixed N . The form of the connection probabilities used here follow from a prior work [16].

For an integer $N \geq 2$, we define the set

$$\Omega_N := \left\{ \mathbf{x} = (x_1, x_2, \dots) : x_i \in \{0, 1, \dots, N-1\}, \right. \\ \left. i = 1, 2, \dots, \sum_{i=1}^{\infty} x_i < \infty \right\}, \quad (1)$$

Y. Shang is with the Department of Mathematics and Institute of Complex Systems, Shanghai Jiao Tong University, Shanghai 200240, China e-mail: shylmath@hotmail.com, shyl@sjtu.edu.cn

Manuscript received May 10, 2009; revised July 21, 2010.

and define a metric d on it:

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \mathbf{x} = \mathbf{y}, \\ \max\{i : x_i \neq y_i\}, & \mathbf{x} \neq \mathbf{y}. \end{cases} \quad (2)$$

The pair (Ω_N, d) is referred to as the hierarchical lattice of order N , which may be thought of as the set of leaves at the bottom of an infinite regular tree without a root, where the distance between two vertices is the number of levels (generations) from the bottom to their most recent common ancestor. Figure 1 shows the lattice Ω_2 along with its metric generating tree.

Such a distance d satisfies the strong triangle inequality

$$d(\mathbf{x}, \mathbf{y}) \leq \max\{d(\mathbf{x}, \mathbf{z}), d(\mathbf{z}, \mathbf{y})\}, \quad (3)$$

for any triple $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Omega_N$. Hence, (Ω_N, d) is an ultrametric (or non-Archimedean) space [14]. From its ultrametricity, it is clear that for every $\mathbf{x} \in \Omega_N$ there are $(N-1)N^{k-1}$ vertices at distance k from it.

Now consider a long-range percolation on Ω_N . For each $k \geq 1$, the probability of connection between \mathbf{x} and \mathbf{y} such that $d(\mathbf{x}, \mathbf{y}) = k$ is given by

$$p_k = \min \left\{ \frac{\alpha}{\beta^k}, 1 \right\}, \quad (4)$$

where $0 \leq \alpha < \infty$ and $0 < \beta < \infty$, all connections being independent. Two vertices $\mathbf{x}, \mathbf{y} \in \Omega_N$ are in the same cluster if there exists a finite sequence $\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n = \mathbf{y}$ of vertices such that each pair $(\mathbf{x}_{i-1}, \mathbf{x}_i)$, $i = 1, \dots, n$, of vertices presents an edge.

The rest of the paper is organized as follows. In Section 2, we provide the main results and Section 3 is devoted to the proofs.

II. MAIN RESULTS

Let \mathbb{N} be the non-negative integers including 0, and denote by $\ell := \min\{k \in \mathbb{N} : \alpha \leq \beta^{k+1}\}$. Let $|S|$ be the size of a set S . The connected component containing the node $\mathbf{x} \in \Omega_N$ is denoted by $C(\mathbf{x})$. Since, for every node \mathbf{x} , $|C(\mathbf{x})|$ has the same distribution, it suffices to consider only $|C(\mathbf{0})|$. The percolation probability is defined as

$$\theta(\alpha, \beta) := P(|C(\mathbf{0})| = \infty), \quad (5)$$

and the critical percolation value is defined as

$$\alpha_c(\beta) := \inf\{\alpha \geq 0 : \theta(\alpha, \beta) > 0\}. \quad (6)$$

The following theorem characterizes the phase transition for this model.

events. If there are infinitely many different n_i , then there must be some n_i for which $\{C_{n_i}(\mathbf{0}) \not\subseteq \overline{B_{n_i}(\mathbf{0})}\}$ holds. If there are only finitely many different n_i , then by definition the same thing holds. The above comments clearly yield (9) for any $\beta > N$. By monotonicity, we know that (9) holds for any $0 < \beta < N^2$. \square

Lemma 2. For any constant $K > 0$. The fraction of the vertices in $B_n(\mathbf{0})$ which are in a cluster of size at least $K(\beta/N)^n$, converges to θ almost surely as $n \rightarrow \infty$.

Proof. First assume that $\beta > N$. We will use the random embedding of the hierarchical lattice in \mathbb{Z} [17]. From the ergodic theorem we obtain for any $k > 0$,

$$\frac{1}{2N^n + 1} \sum_{\mathbf{x}=-N^n}^{N^n} 1_{\{\cap_{j=k}^{\infty} \{|C_j(\mathbf{x})| > K(\beta/N)^j\}\}} \rightarrow P(\cap_{j=k}^{\infty} \{|C_j(\mathbf{x})| > K(\beta/N)^j\}), \quad (12)$$

almost surely as $n \rightarrow \infty$.

By virtue of Lemma 1, the right-hand side of (12) increases to θ as $k \rightarrow \infty$. Hence, we have

$$A(n) := \frac{1}{2N^n + 1} \sum_{\mathbf{x}=-N^n}^{N^n} 1_{\{|C_n(\mathbf{x})| > K(\beta/N)^n\}} \rightarrow \theta, \quad (13)$$

almost surely as $n \rightarrow \infty$. By our construction in [17], the collection vertices $\{-N^n, -N^n + 1, -N^n + 2, \dots, N^n\}$ contains the image under the embedding of the ball $B_n(\mathbf{0})$ and this image contains a fraction $N^n/(2N^n + 1)$ of those vertices. The events $\{|C_n(\mathbf{x})| > K(\beta/N)^n\}$ are independent for vertices in different n -balls, and then

$$A_1(n) := \frac{1}{2N^n + 1} \sum_{\mathbf{x} \in B_n(\mathbf{0})} 1_{\{|C_n(\mathbf{x})| > K(\beta/N)^n\}} \quad (14)$$

and $A_2(n) := A(n) - A_1(n)$ are independent.

It is easy to see that $A_1(n)$ and $A_2(n)$ are bounded above by 1 and have asymptotically the same mean. By (13) we obtain that

$$\frac{1}{N^n} \sum_{\mathbf{x} \in B_n(\mathbf{0})} 1_{\{|C_n(\mathbf{x})| > K(\beta/N)^n\}} \rightarrow \theta, \quad (15)$$

almost surely as $n \rightarrow \infty$ for $\beta > N$. When $\beta \leq N$, we have $\theta = 1$ by Theorem 1. It is direct to check that the above derivations still hold. \square

Proof of Theorem 3. From Lemma 2 we have for every $K > 0$ and $\varepsilon > 0$

$$P\left(\left|\left\{\mathbf{x} \in B_n(\mathbf{0}) : |C_n(\mathbf{x})| > K\left(\frac{\beta}{N}\right)^n\right\}\right| > (\theta - \varepsilon)N^n\right) > 1 - \varepsilon, \quad (16)$$

for n large enough. A ball $B_n(\mathbf{y})$ is said to be good if and only if

$$\left|\left\{\mathbf{x} \in B_n(\mathbf{y}) : |C_n(\mathbf{x})| > K\left(\frac{\beta}{N}\right)^n\right\}\right| > (\theta - \varepsilon)N^n. \quad (17)$$

In what follows, we condition on the event that all n -balls in $B_{n+1}(\mathbf{0})$ are good. The probability of this event is bounded below by $(1 - \varepsilon)^N \geq 1 - N\varepsilon$.

For each good ball $B_n(\mathbf{y})$, $\mathbf{y} \in \Omega_N$, we make a partition of the set

$$B'_n(\mathbf{y}) := \left\{\mathbf{x} \in B_n(\mathbf{y}) : |C_n(\mathbf{x})| > K\left(\frac{\beta}{N}\right)^n\right\} \quad (18)$$

into super vertices. For $\mathbf{x} \in B'_n(\mathbf{y})$ we make a partition of $C_n(\mathbf{x})$ into $\lfloor |C_n(\mathbf{x})|/[K(\beta/N)^n] \rfloor$ super vertices, all of which have size at least $\lceil K(\beta/N)^n \rceil$. Denote by V_n the collection of super vertices that contain vertices in $B_{n+1}(\mathbf{0})$. For K large enough, if $B_n(\mathbf{y})$ is good, then V_n contains at least $(\theta - \varepsilon)N^n/[2K(\beta/N)^n] \geq (\theta - \varepsilon)N^n/(3K(\beta/N)^n)$ super vertices.

As in [12], we construct a new N -partite graph on V_n as follows. Let V_n be the vertex set and let E_n be the edge sets. Choose $\lceil K(\beta/N)^n \rceil$ original vertices from every super vertex in V_n . Choosing those vertices may be done in any way that is independent of the presence of edges of length $\geq n+1$. Denote these sets by A_n . The super vertices $x, y \in V_n$ are connected by an edge if there is at least one edge in the original graph which is present between vertices that make up the sets in A_n corresponding to x and y , respectively, and if the original vertices that make up x and y are at distance $n+1$ of each other. Otherwise, there is no edge between the super vertices. Since $\beta < N^2$, $(\theta - \varepsilon)N^n/(3K(\beta/N)^n)$ tends to infinity as $n \rightarrow \infty$. Hence, the expected degree of a vertex in V_n is larger than

$$\begin{aligned} & \frac{(N-1)(\theta - \varepsilon)N^n}{3K(\beta/N)^n} \left(1 - \left(1 - \frac{\alpha}{\beta^{n+1}}\right)^{K^2(\beta/N)^{2n}}\right) \\ & > \frac{(N-1)(\theta - \varepsilon)N^n}{3K(\beta/N)^n} \cdot \left(1 - \exp\left\{-\frac{\alpha}{\beta^{n+1}}K^2\left(\frac{\beta}{N}\right)^{2n}\right\}\right), \end{aligned} \quad (19)$$

which exceeds $\lambda := (N-1)(\theta - \varepsilon)\alpha K/(6\beta)$ for large n . Clearly, the parameter λ can be made large enough by choosing K large enough.

The N -partite graph (V_n, E_n) is an inhomogeneous random graphs; see [7] for backgrounds. The degree of every super vertex is asymptotically Poisson distributed, with mean bounded below by λ . The unique largest cluster of such an N -partite graph contains a fraction η of the super vertices almost surely as $n \rightarrow \infty$, where η is the largest solution of the equation

$$1 - \eta = e^{-\lambda\eta}. \quad (20)$$

We can choose λ sufficiently large and $\eta > 1 - \varepsilon$. Hence, for each $\varepsilon > 0$ and large n , the graph (V_n, E_n) contains a unique giant cluster containing a fraction $(1 - \varepsilon)N$ of the vertices in V_n with probability at least $1 - \varepsilon$.

Since we have conditioned on the event that all n -balls in $B_{n+1}(\mathbf{0})$ are good, the fraction of vertices in $B_{n+1}(\mathbf{0})$ that are part of vertices in V_n is larger than $\theta - 2\varepsilon$. Accordingly, conditional on the same event, the largest cluster in $B_{n+1}(\mathbf{0})$ is at least of size $(\eta - \varepsilon)(\theta - 2\varepsilon)N^n > (1 - 2\varepsilon)(\theta - 2\varepsilon)N^n$ with probability at least $1 - \varepsilon$. By the multiplication principle, we have the probability that the largest cluster in $B_{n+1}(\mathbf{0})$ is at least of size $(1 - 2\varepsilon)(\theta - 2\varepsilon)N^n$ is bounded below by

$(1 - \varepsilon)(1 - N\varepsilon)$. Now, choosing $\varepsilon' < \varepsilon / \max\{4, N + 1\}$, we finally obtain that

$$P(|C_n^m(\mathbf{0})| > (\theta - \varepsilon')N^n) \geq 1 - \varepsilon'. \quad (21)$$

The proof then readily follows. \square

ACKNOWLEDGMENT

The author would like to thank two anonymous referees for their helpful comments.

REFERENCES

- [1] M. Aizenman and C. M. Newman, Discontinuity of the percolation density in one-dimensional $1/|x - y|^2$ percolation models. *Comm. Math. Phys.*, 107(1986) 611–647.
- [2] S. R. Athreya and J. M. Swart, Survival of contact processes on the hierarchical group. *Probab. Theory Relat. Fields*, 147(2010) 529–563.
- [3] A. L. Barabási and E. Ravaz, Hierarchical organization in complex networks. *Phys. Rev. E*, 67(2003) 026112.
- [4] I. Benjamini and O. Schramm, Percolation beyond \mathbb{Z}^d , many questions and a few answers. *Elect. Commun. Probab.*, 1(1996) 71–82.
- [5] N. Berger, Transience, recurrence and critical behavior for long-range percolation. *Comm. Math. Phys.*, 226(2002) 531–558.
- [6] M. Biskup, On the scaling of chemical distance in long-range percolation models. *Ann. Probab.*, 32(2004) 2938–2977.
- [7] B. Bollobás, S. Janson and O. Riordan, The phase transition in inhomogeneous random graphs. *Random Struct. Algorithms*, 31(2007) 3–122.
- [8] D. Coppersmith, D. Gamarnik and M. Sviridenko, The diameter of a long-range percolation graph. *Random Struct. Algorithms*, 21(2002) 1–13.
- [9] D. A. Dawson and L. G. Gorostiza, Percolation in an ultrametric space. *arXiv:1006.4400v2*, 2011.
- [10] D. A. Dawson and L. G. Gorostiza, Percolation in a hierarchical random graph. *Comm. Stochastic Analysis*, 1(2007) 29–47.
- [11] G. Grimmett, *Percolation*. New York: Springer, 1999.
- [12] V. Koval, R. Meester and P. Trapman, Long-range percolation on the hierarchical lattice. *arXiv:1004.1251v1*, 2010.
- [13] R. Rammal, G. Toulouse and M. A. Virasoro, Ultrametricity for physicists. *Rev. Mod. Phys.*, 58(1986) 765–788.
- [14] P. Schneider, *Nonarchimedean Functional Analysis*. New York: Springer, 2002.
- [15] L. S. Schulman, Long range percolation in one dimension. *J. Phys. A: Math. Gen.*, 16(1983) L639–L641.
- [16] Y. Shang, Percolation in a hierarchical lattice. Submitted to *Ann. Probab.*
- [17] Y. Shang, Uniqueness of the infinite component for percolation on a hierarchical lattice. Submitted to *Random Struct. Algorithms*.
- [18] Y. Shang, The giant component in a random subgraph of a weak expander. *Int. J. Math. Comput. Sci.*, 7(2011) 95–99.
- [19] Y. Shang, Leader-following consensus problems with a time-varying leader under measurement noises. *arXiv:0909.4349*, to appear in *Adv. Dyn. Syst. Appl.*
- [20] J. Shen, Cucker-Smale flocking under hierarchical leadership. *SIAM J. Appl. Math.*, 68(2007) 694–719.
- [21] P. Trapman, The growth of the infinite long-range percolation cluster. *Ann. Probab.*, 38(2010) 1583–1608.



Yilun Shang was born in Shanghai, CHINA, in 1983. He received the B.S. and Ph.D. degrees from the Department of Mathematics, Shanghai Jiao Tong University, China, in 2005 and 2010, respectively. From 2010, he joined the Institute for Cyber Security and Department of Computer Science, University of Texas at San Antonio, USA, as a research fellow. He is a member of the editorial board of *Journal of Next Generation Information Technology* and *Pioneer Journal of Advances in Applied Mathematics*. His current research interests include random graph

theory, structure and dynamics of complex systems, probability and combinatorics.