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The Pell Equation $x^2-(k^2-k)y^2=2^t$

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***Abstract*—Let k,t,d be arbitrary integers with $k\geq 2$, $t\geq 0$ and $d=k^2-k$. In the first section we give some preliminaries from Pell equations $x^2-dy^2=1$ and $x^2-dy^2=N$, where N be any fixed positive integer. In the second section, we consider the integer solutions of Pell equations $x^2-dy^2=1$ and $x^2-dy^2=2^t$. We give a method for the solutions of these equations. Further we derive recurrence relations on the solutions of these equations.

***Keywords*—Pell equation, solutions of Pell equation.

I. Preliminary facts.

Let $d\neq 1$ be a positive non-square integer and N be any fixed positive integer. Then the equation

$$x^2-dy^2=\pm N \tag{1}$$

is known as Pell equation and is named after John Pell (1611-1685), a mathematician who searched for integer solutions to equations of this type in the seventeenth century. Ironically, Pell was not the first to work on this problem, nor did he contribute to our knowledge for solving it. Euler (1707-1783), who brought us the ψ-function, accidentally named the equation after Pell, and the name stuck.

For $N=1$, the Pell equation

$$x^2-dy^2=\pm 1 \tag{2}$$

is known as the classical Pell equation and was first studied by Brahmagupta (598-670) and Bhaskara (1114-1185), (see [1]). Its complete theory was worked out by Lagrange (1736-1813), not Pell. It is often said that Euler (1707-1783) mistakenly attributed Brouncker's (1620-1684) work on this equation to Pell. However the equation appears in a book by Rahn (1622-1676) which was certainly written with Pell's help: some say entirely written by Pell. Perhaps Euler knew what he was doing in naming the equation. Baltus [2], Kaplan and Williams [5], Lenstra [7], Matthews [8], Mollin, Poorten and Williams [9], Stevenhagen [10], Tekcan [12,13,14], and the others consider some specific Pell equations and their integer solutions. Further details on Pell equations can be found in [3,10].

The Pell equation in (2) has infinitely many integer solutions (x_n,y_n) for $n\geq 1$. The first non-trivial positive integer solution (x_1,y_1) (in this case x_1 or $x_1+y_1\sqrt{d}$ is minimum) of this equation is called the fundamental solution, because all other solutions can be (easily) derived from it. In fact, if (x_1,y_1) is the fundamental solution of $x^2-dy^2=1$, then the n-th positive solution of it, say (x_n,y_n), is defined by the equality

$$x_n+y_n\sqrt{d}=(x_1+y_1\sqrt{d})^n \tag{3}$$

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for integer $n\geq 2$. (Furthermore, all nontrivial solutions can be obtained considering the four cases $(\pm x_n,\pm y_n)$ for $n\geq 1$.)

There are several methods for finding the fundamental solution of Pell's equation $x^2-dy^2=1$ for a positive non-square integer d, e.g., the cyclic method [4, p. 30], known in India in the 12-th century, or the slightly less efficient but more regular English method (17-th century) which produce all solutions of $x^2-dy^2=1$ [4, p. 32]. But the most efficient method for finding the fundamental solution is based on the simple finite continued fraction expansion of $\sqrt{d}$. We can describe it as follows (see [2] and also [6, p.154]): Let

$$[a_0;\overline{a_1,a_2,\cdots,a_r,2a_0}]$$

be the simple continued fraction of $\sqrt{d}$, where $a_0=\lfloor\sqrt{d}\rfloor$. Let $p_0=a_0$, $p_1=1+a_0a_1$, $q_0=1$, $q_1=a_1$. In general

$$\begin{aligned} p_n &= a_np_{n-1}+p_{n-2} \\ q_n &= a_nq_{n-1}+q_{n-2} \end{aligned} \tag{4}$$

for $n\geq 2$. Then the fundamental solution of $x^2-dy^2=1$ is

$$(x_1,y_1)=\begin{cases} (p_r,q_r) & if\ r\ is\ odd \\ (p_{2r+1},q_{2r+1}) & if\ r\ is\ even. \end{cases} \tag{5}$$

On the other hand, in connection with (1) and (2), it is well known that if (X_1,Y_1) and (x_{n-1},y_{n-1}) are integer solutions of $x^2-dy^2=\pm N$ and $x^2-dy^2=1$, respectively, then (X_n,Y_n) is also a positive solution of $x^2-dy^2=\pm N$, where

$$X_n+dY_n=(x_{n-1}+dy_{n-1})(X_1+dY_1) \tag{6}$$

for $n\geq 2$.

In this work we will define by recurrence an infinite sequence of positive solutions of the Pell equation $x^2-dy^2=2^t$, where $d=k^2-k$ with $k\geq 2$ an integer and $t\geq 0$ is also an integer. We will also express the obtained solutions for $t\geq 1$ in terms of the "fundamental solution" of $x^2-dy^2=1$ in two cases $k=2$ or $k\geq 3$.

II. The Pell equation $x^2-(k^2-k)y^2=2^t$.

Let $d=k^2-k$ be a positive non-square integer for an integer $k\geq 2$ and let $t\geq 0$ be an arbitrary integer. In this section we consider the integer solutions of Pell equation $x^2-(k^2-k)y^2=2^t$. First we consider the case $t=0$, that is, the classical Pell equation

$$x^2-(k^2-k)y^2=1.$$

Theorem 2.1: Let $d=k^2-k$ with $k\geq 2$. Then

1) The continued fraction expansion of $\sqrt{d}$ is given by

$$\sqrt{d}=\begin{cases} [1;\overline{2}] & if\quad k=2 \\ [k-1;\overline{2,2k-2}] & otherwise. \end{cases}$$

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2) The fundamental solution of $x^2 - dy^2 = 1$ is

$$(x_1, y_1) = (2k-1, 2).$$

3) For $n \geq 4$,

$$\begin{aligned} x_n &= (4k-1)(x_{n-1} - x_{n-2}) + x_{n-3} \\ y_n &= (4k-1)(y_{n-1} - y_{n-2}) + y_{n-3}. \end{aligned}$$

Proof: 1) Let $k = 2$. Then it is easily seen that the continued fraction expansion of $\sqrt{2}$ is $[1; \overline{2}]$. Now let $k \geq 3$. Then

$$\begin{aligned} \sqrt{k^2-k} &= k-1+(\sqrt{k^2-k}-(k-1)) \\ &= k-1+\cfrac{1}{\frac{1}{\sqrt{k^2-k}-(k-1)}} \\ &= k-1+\cfrac{1}{\frac{\sqrt{k^2-k}+(k-1)}{k-1}} \\ &= k-1+\cfrac{1}{2+\frac{\sqrt{k^2-k}-(k-1)}{k-1}} \\ &= k-1+\cfrac{1}{2+\frac{1}{\frac{k-1}{\sqrt{k^2-k}-(k-1)}}} \\ &= k-1+\cfrac{1}{2+\frac{1}{\sqrt{k^2-k}+(k-1)}} \\ &= k-1+\cfrac{1}{2+\frac{1}{2k-2+\left(\sqrt{k^2-k}-(k-1)\right)}}. \end{aligned}$$

Therefore the continued fraction expansion of $\sqrt{d}$ is $[k-1; \overline{2, 2k-2}]$.

2) The case $k = 2$ is clear since $(x_1, y_1) = (3, 2)$ is clearly a minimum solution of $x^2 - 2y^2 = 1$. On the other hand, for $k \geq 3$, using the method described in the precedent section to find a fundamental solution, we get $r = 1$ with $a_0 = k-1$, $a_1 = 2$. Hence, $(x_1, y_1) = (p_1, q_1) = (2k-1, 2)$ is the fundamental solution since $p_0 = a_0 = k-1, p_1 = 1+a_0a_1 = 1+(k-1)2 = 2k-1$ and $q_0 = 1, q_1 = a_1 = 2$ by (4) and (5).

3) Note that by (3), if (x_1, y_1) is the fundamental solution of $x^2 - (k^2-k)y^2 = 1$, then the other solutions (x_n, y_n) of $x^2 - (k^2-k)y^2 = 1$ can be derived by using the equalities $x_n + \sqrt{d}y_n = (x_1 + \sqrt{d}y_1)^n$ for $n \geq 2$, in other words

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $n \geq 2$. Therefore it can be shown by induction on n that

$$x_n = (4k-1)(x_{n-1} - x_{n-2}) + x_{n-3}.$$

Similarly it can be proved that $y_n = (4k-1)(y_{n-1} - y_{n-2}) + y_{n-3}$. ∎

Next we consider the general case, that is the case

$$x^2 - (k^2-k)y^2 = 2^t$$

for $t \geq 1$. But we have to consider the problem in two cases: $k = 2$ and $k \geq 3$. Note that we denote the integer solutions of $x^2 - (k^2-k)y^2 = 2^t$ by (X_n, Y_n), and denote the integer solutions of $x^2 - (k^2-k)y^2 = 1$ by (x_n, y_n). Then we have the following theorem.

Theorem 2.2: Let $k = 2$ and let t be an arbitrary integer with $t \geq 1$. Define a sequence $\{(X_n, Y_n)\}$ of positive integers by

$$(X_1, Y_1) = \begin{cases} \left(2^{\frac{t+1}{2}}, 2^{\frac{t-1}{2}}\right) & if\ t\ is\ odd \\ \left(3.2^{\frac{t}{2}}, 2^{\frac{t}{2}+1}\right) & if\ t\ is\ even \end{cases}$$

and

$$X_n = \begin{cases} 2^{\frac{t+1}{2}}x_{n-1} + 2^{\frac{t+1}{2}}y_{n-1} & if\ t\ is\ odd \\ 3.2^{\frac{t}{2}}x_{n-1} + 2^{\frac{t}{2}+2}y_{n-1} & if\ t\ is\ even \end{cases}$$

$$Y_n = \begin{cases} 2^{\frac{t-1}{2}}x_{n-1} + 2^{\frac{t+1}{2}}y_{n-1} & if\ t\ is\ odd \\ 2^{\frac{t}{2}+1}x_{n-1} + 3.2^{\frac{t}{2}}y_{n-1} & if\ t\ is\ even, \end{cases}$$

where $\{(x_n, y_n)\}$ is the sequence of positive solutions of $x^2 - 2y^2 = 1$. Then

1) (X_n, Y_n) is a solution of $x^2 - 2y^2 = 2^t$ for any integer $n \geq 1$.
2) For $n \geq 2$,

$$\begin{aligned} X_{n+1} &= 3X_n + 4Y_n \\ Y_{n+1} &= 2X_n + 3Y_n. \end{aligned}$$

3) For $n \geq 4$,

$$\begin{aligned} X_n &= 7(X_{n-1} - X_{n-2}) + X_{n-3} \\ Y_n &= 7(Y_{n-1} - Y_{n-2}) + Y_{n-3}. \end{aligned}$$

Proof: 1) Let us assume t is odd. Then it is easily seen that $(X_1, Y_1) = \left(2^{\frac{t+1}{2}}, 2^{\frac{t-1}{2}}\right)$ is a solution of $x^2 - 2y^2 = 2^t$ since

$$\begin{aligned} X_1^2 - dY_1^2 &= \left(2^{\frac{t+1}{2}}\right)^2 - 2\left(2^{\frac{t-1}{2}}\right)^2 \\ &= 2^{t+1} - 2.2^{t-1} \\ &= 2^t(2-1) \\ &= 2^t. \end{aligned}$$

On the other hand, as it was said previously, (X_n, Y_n) is also a solution for $n \geq 2$. We can prove this as follows. Recall that (x_{n-1}, y_{n-1}) is a solution of $x^2 - 2y^2 = 1$, that is,

$$x_{n-1}^2 - 2y_{n-1}^2 = 1.$$

Further we see as above that (X_1, Y_1) is a solution of $x^2 - 2y^2 = 2^t$, that is,

$$X_1^2 - 2Y_1^2 = 2^t.$$

Combining these two results we find that

$$\begin{aligned}
X_n^2 - 2Y_n^2 &= \left(2^{\frac{t+1}{2}} x_{n-1} + 2^{\frac{t+1}{2}} y_{n-1}\right)^2 \\
&\quad -2\left(2^{\frac{t-1}{2}} x_{n-1} + 2^{\frac{t+1}{2}} y_{n-1}\right)^2 \\
&= 2^{t+1}x_{n-1}^2 + 2.2^{\frac{t+1}{2}}.2^{\frac{t+1}{2}} x_{n-1}y_{n-1} \\
&\quad +2^{t+1}y_{n-1}^2 \\
&\quad -2\left(\begin{array}{c} 2^{t-1}x_{n-1}^2 + \\ 2.2^{\frac{t-1}{2}}.2^{\frac{t+1}{2}} x_{n-1}y_{n-1} \\ +2^{t+1}y_{n-1}^2 \end{array}\right) \\
&= x_{n-1}^2\left(2^{t+1} - 2.2^{t-1}\right) \\
&\quad +x_{n-1}y_{n-1}\left(2^{t+2} - 2^{t+2}\right) \\
&\quad +y_{n-1}^2\left(2^{t+1} - 2.2^{t+1}\right) \\
&= 2^t\left(x_{n-1}^2 - 2y_{n-1}^2\right) \\
&= 2^t.
\end{aligned}$$

Therefore (X_n, Y_n) is a solution of $x^2 - 2y^2 = 2^t$.

2) Note that

$$\begin{aligned}
X_{n+1} + Y_{n+1}\sqrt{d} &= (x_n + y_n\sqrt{d})(X_1 + Y_1\sqrt{d}) \\
&= (x_1 + y_1\sqrt{d})^n(X_1 + Y_1\sqrt{d}) \\
&= (x_1 + y_1\sqrt{d}) \\
&\quad \times \left[\begin{array}{c} (x_1 + y_1\sqrt{d})^{n-1} \\ \times(X_1 + Y_1\sqrt{d}) \end{array}\right] \\
&= (x_1 + y_1\sqrt{d}) \\
&\quad \times \left[\begin{array}{c} (x_{n-1} + y_{n-1}\sqrt{d}) \\ \times(X_1 + Y_1\sqrt{d}) \end{array}\right] \\
&= (x_1 + y_1\sqrt{d})(X_n + Y_n\sqrt{d}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} &= \begin{pmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{pmatrix}\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \\
&= \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \\
&= \begin{pmatrix} 3X_n + 4Y_n \\ 2X_n + 3Y_n \end{pmatrix},
\end{aligned}$$

that is

$$\begin{aligned}
X_{n+1} &= 3X_n + 4Y_n \\
Y_{n+1} &= 2X_n + 3Y_n.
\end{aligned}$$

3) Recall that

$$X_n = \left(2^{\frac{t+1}{2}} x_{n-1} + 2^{\frac{t+1}{2}} y_{n-1}\right)$$

and also

$$X_{n+1} = 3X_n + 4Y_n.$$

Combining these two results we find by induction on n that

$$X_n = 7(X_{n-1} - X_{n-2}) + X_{n-3}.$$

Similarly we can show that $Y_n = 7(Y_{n-1} - Y_{n-2}) + Y_{n-3}$ for $n \geq 4$.

Now we consider the case t is even.

1) It is easily seen that $(X_1, Y_1) = \left(3.2^{\frac{t}{2}}, 2^{\frac{t}{2}+1}\right)$ is a solution of $x^2 - 2y^2 = 2^t$ since

$$\begin{aligned}
X_1^2 - dY_1^2 &= \left(3.2^{\frac{t}{2}}\right)^2 - 2\left(2^{\frac{t}{2}+1}\right)^2 \\
&= 9.2^t - 2.2^{t+2} \\
&= 2^t(9-8) \\
&= 2^t.
\end{aligned}$$

We know that (x_{n-1}, y_{n-1}) is a solution of $x^2 - 2y^2 = 1$, that is,

$$x_{n-1}^2 - 2y_{n-1}^2 = 1,$$

and also $(X_1, Y_1) = \left(3.2^{\frac{t}{2}}, 2^{\frac{t}{2}+1}\right)$ is a solution of $x^2 - 2y^2 = 2^t$, that is,

$$X_1^2 - 2Y_1^2 = 2^t.$$

Applying these two results we find that

$$\begin{aligned}
X_n^2 - 2Y_n^2 &= \left(3.2^{\frac{t}{2}} x_{n-1} + 2^{\frac{t}{2}+2} y_{n-1}\right)^2 \\
&\quad -2\left(2^{\frac{t}{2}+1} x_{n-1} + 3.2^{\frac{t}{2}} y_{n-1}\right)^2 \\
&= 9.2^t x_{n-1}^2 + 2.3.2^{\frac{t}{2}}.2^{\frac{t}{2}+2} x_{n-1}y_{n-1} \\
&\quad +2^{t+4}y_{n-1}^2 \\
&\quad -2\left(\begin{array}{c} 2^{t+2}x_{n-1}^2 + \\ 2.3.2^{\frac{t}{2}+1}.2^{\frac{t}{2}} x_{n-1}y_{n-1} \\ +9.2^t y_{n-1}^2 \end{array}\right) \\
&= x_{n-1}^2\left(9.2^t - 2.2^{t+2}\right) \\
&\quad +x_{n-1}y_{n-1}\left(\begin{array}{c} 2.3.2^{\frac{t}{2}}.2^{\frac{t}{2}+2} \\ -2.2.3.2^{\frac{t}{2}+1}.2^{\frac{t}{2}} \end{array}\right) \\
&\quad +y_{n-1}^2\left(2^{t+4} - 2.9.2^t\right) \\
&= 2^t\left(x_{n-1}^2 - 2y_{n-1}^2\right) \\
&= 2^t.
\end{aligned}$$

Therefore (X_n, Y_n) is a solution of $x^2 - 2y^2 = 2^t$.

2) It can be proved as in the same way that 2) was proved since $(x_1, y_1) = (3, 2)$ is the fundamental solution of $x^2 - 2y^2 = 1$ and

$$\begin{aligned}
X_{n+1} + Y_{n+1}\sqrt{d} &= (x_n + y_n\sqrt{d})(X_1 + Y_1\sqrt{d}) \\
&= (x_1 + y_1\sqrt{d})(X_n + Y_n\sqrt{d}).
\end{aligned}$$

3) Recall that

$$X_n = \left(3.2^{\frac{t}{2}} x_{n-1} + 2^{\frac{t}{2}+2} y_{n-1}\right)$$

and also

$$X_{n+1} = 3X_n + 4Y_n.$$

Combining these two results we find by induction on n that

$$X_n = 7(X_{n-1} - X_{n-2}) + X_{n-3}.$$

Similarly we can show that $Y_n = 7(Y_{n-1} - Y_{n-2}) + Y_{n-3}$ for $n \geq 4$. ■

Now we consider the case $k \geq 3$.

Theorem 2.3: Let k and t be arbitrary integers with $k \geq 3$ and $t \geq 1$ is even. Define a sequence $\{(X_n, Y_n)\}$ of positive integers by

$$(X_1, Y_1) = \left(2^{\frac{t}{2}}(2k-1), 2^{\frac{t}{2}+1}\right)$$

and

$$\begin{aligned} X_n &= \left(2^{\frac{t}{2}}(2k-1)x_{n-1} + 2^{\frac{t}{2}+1}(k^2-k)y_{n-1}\right) \\ Y_n &= \left(2^{\frac{t}{2}+1}x_{n-1} + 2^{\frac{t}{2}}(2k-1)y_{n-1}\right), \end{aligned}$$

where $\{(x_n, y_n)\}$ is the sequence of positive solutions of $x^2 - (k^2-k)y^2 = 1$. Then

1) (X_n, Y_n) is a solution of $x^2 - (k^2-k)y^2 = 2^t$ for any integer $n \geq 1$.
2) For $n \geq 2$,

$$\begin{aligned} X_{n+1} &= (2k-1)X_n + (2k^2-2k)Y_n \\ Y_{n+1} &= 2X_n + (2k-1)Y_n. \end{aligned}$$

3) For $n \geq 4$,

$$\begin{aligned} X_n &= (4k-1)(X_{n-1} - X_{n-2}) + X_{n-3} \\ Y_n &= (4k-1)(Y_{n-1} - Y_{n-2}) + Y_{n-3}. \end{aligned}$$

Proof: 1) Note that (X_1, Y_1) is a solution of $x^2 - (k^2-k)y^2 = 2^t$ since

$$\begin{aligned} X_1^2 - (k^2-k)Y_1^2 &= \left(2^{\frac{t}{2}}(2k-1)\right)^2 \\ &\quad -(k^2-k)\left(2^{\frac{t}{2}+1}\right)^2 \\ &= 2^t(4k^2-4k+1) - (k^2-k)(2^{t+2}) \\ &= 2^t(4k^2-4k+1-4k^2+4k) \\ &= 2^t. \end{aligned}$$

Note that (x_{n-1}, y_{n-1}) is a solution of $x^2 - (k^2-k)y^2 = 1$, that is,

$$x_{n-1}^2 - (k^2-k)y_{n-1}^2 = 1.$$

Also we see as above that (X_1, Y_1) is a solution of $x^2 - (k^2-k)y^2 = 2^t$, that is,

$$X_1^2 - (k^2-k)Y_1^2 = 2^t.$$

Applying these two results we find that

$$\begin{aligned} X_n^2 - (k^2-k)Y_n^2 &= \begin{pmatrix} 2^{\frac{t}{2}}(2k-1)x_{n-1} \\ +2^{\frac{t}{2}+1}(k^2-k)y_{n-1} \end{pmatrix}^2 \\ &\quad -(k^2-k)\begin{pmatrix} 2^{\frac{t}{2}+1}x_{n-1} \\ +2^{\frac{t}{2}}(2k-1)y_{n-1} \end{pmatrix}^2 \\ &= 2^t(2k-1)^2x_{n-1}^2 + 2.2^{\frac{t}{2}}.2^{\frac{t}{2}+1} \\ &\quad \times(2k-1)(k^2-k)x_{n-1}y_{n-1} \\ &\quad +2^{t+2}(k^2-k)^2y_{n-1}^2 \\ &\quad -(k^2-k) \\ &\quad \times\begin{pmatrix} 2^{t+2}x_{n-1}^2 \\ +2.2^{\frac{t}{2}+1}.2^{\frac{t}{2}}(2k-1) \\ \times x_{n-1}y_{n-1} \\ +2^t(2k-1)^2y_{n-1}^2 \end{pmatrix} \\ &= x_{n-1}^2\left(2^t(2k-1)^2 - 2^{t+2}(k^2-k)\right) \\ &\quad +x_{n-1}y_{n-1}\begin{pmatrix} 2.2^{\frac{t}{2}}.2^{\frac{t}{2}+1}(2k-1) \\ \times(k^2-k) \\ -(k^2-k)2.2^{\frac{t}{2}+1} \\ \times 2^{\frac{t}{2}}(2k-1) \end{pmatrix} \\ &\quad +y_{n-1}^2\begin{pmatrix} 2^{t+2}(k^2-k)^2 \\ -(k^2-k)2^t(2k-1)^2 \end{pmatrix} \\ &= x_{n-1}^2(2^t) - y_{n-1}^2\left(2^t(k^2-k)\right) \\ &= 2^t\left(x_{n-1}^2 - (k^2-k)y_{n-1}^2\right) \\ &= 2^t. \end{aligned}$$

Therefore (X_n, Y_n) is a solution of $x^2 - 2y^2 = 2^t$.

2) Recall that

$$\begin{aligned} \begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} &= \begin{pmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{pmatrix}\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \\ &= \begin{pmatrix} 2k-1 & 2k^2-2k \\ 2 & 2k-1 \end{pmatrix}\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \\ &= \begin{pmatrix} (2k-1)X_n + (2k^2-2k)Y_n \\ 2X_n + (2k-1)Y_n \end{pmatrix}. \end{aligned}$$

So

$$\begin{aligned} X_{n+1} &= (2k-1)X_n + (2k^2-2k)Y_n \\ Y_{n+1} &= 2X_n + (2k-1)Y_n. \end{aligned}$$

3) Applying the equalities

$$X_n = \left(2^{\frac{t}{2}}(2k-1)x_{n-1} + 2^{\frac{t}{2}+1}(k^2-k)y_{n-1}\right)$$

and

$$X_{n+1} = (2k-1)X_n + (2k^2-2k)Y_n$$

we find by induction on n that

$$X_n = (4k-1)(X_{n-1} - X_{n-2}) + X_{n-3}$$

for $n \geq 4$. Similarly it can be shown that

$$Y_n = (4k-1)(Y_{n-1} - Y_{n-2}) + Y_{n-3}.$$

■

Example 2.1: Let $k = 2$ and let $t = 2$. Then by Theorem 2.2, $(X_1, Y_1) = (6, 4)$ is a solution of $x^2 - 2y^2 = 4$, and some other solutions are

$$\begin{aligned} (X_2, Y_2) &= (34, 24) \\ (X_3, Y_3) &= (198, 140) \\ (X_4, Y_4) &= (1154, 816) \\ (X_5, Y_5) &= (6726, 4756) \\ (X_6, Y_6) &= (39202, 27720). \end{aligned}$$

Let $t = 5$. Then $(X_1, Y_1) = (8, 4)$ is a solution of $x^2 - 2y^2 = 32$, and some other solutions are

$$\begin{aligned} (X_2, Y_2) &= (40, 28) \\ (X_3, Y_3) &= (232, 164) \\ (X_4, Y_4) &= (1352, 956) \\ (X_5, Y_5) &= (7280, 5172) \\ (X_6, Y_6) &= (42528, 30076). \end{aligned}$$

Example 2.2: Let $k = 6$ and let $t = 4$. Then by Theorem 2.3, $(X_1, Y_1) = (44, 8)$ is a solution of $x^2 - 30y^2 = 16$, and some other solutions are

$$\begin{aligned} (X_2, Y_2) &= (964, 176) \\ (X_3, Y_3) &= (21164, 3864) \\ (X_4, Y_4) &= (464644, 84832) \\ (X_5, Y_5) &= (10201004, 1862440) \\ (X_6, Y_6) &= (223957444, 40888848). \end{aligned}$$

Concluding remark. Note that in Theorem 2.3, we only consider the case t is even. When we consider the case t is odd, then we find that there is no solution (X_1, Y_1) of $x^2 - (k^2 - k)y^2 = 2^t$ for some values of k, or there is a solution (X_1, Y_1) of $x^2 - (k^2 - k)y^2 = 2^t$ for some values of k. But we can not determine when $x^2 - (k^2 - k)y^2 = 2^t$ has a solution or not. For example for $k = 8$ and $t = 3$, we find that $(X_1, Y_1) = (8, 1)$ is a solution of $x^2 - 56y^2 = 8$. Similarly for $k = 17$ and $t = 9$, we find that $(X_1, Y_1) = (28, 1)$ is a solution of $x^2 - 272y^2 = 512$. But for $k = 10$ and for every odd t, there is no solution of $x^2 - 90y^2 = 2^t$.

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