

The Panpositionable Hamiltonicity of k -ary n -cubes

Chia-Jung Tsai and Shin-Shin Kao

Abstract—The hypercube Q_n is one of the most well-known and popular interconnection networks and the k -ary n -cube Q_n^k is an enlarged family from Q_n that keeps many pleasing properties from hypercubes. In this article, we study the panpositionable hamiltonicity of Q_n^k for $k \geq 3$ and $n \geq 2$. Let x, y of $V(Q_n^k)$ be two arbitrary vertices and C be a hamiltonian cycle of Q_n^k . We use $d_C(x, y)$ to denote the distance between x and y on the hamiltonian cycle C . Define l as an integer satisfying $d(x, y) \leq l \leq \frac{1}{2}|V(Q_n^k)|$. We prove the followings:

- When $k = 3$ and $n \geq 2$, there exists a hamiltonian cycle C of Q_n^k such that $d_C(x, y) = l$.
- When $k \geq 5$ is odd and $n \geq 2$, we request that $l \notin S$ where S is a set of specific integers. Then there exists a hamiltonian cycle C of Q_n^k such that $d_C(x, y) = l$.
- When $k \geq 4$ is even and $n \geq 2$, we request $l - d(x, y)$ to be even. Then there exists a hamiltonian cycle C of Q_n^k such that $d_C(x, y) = l$.

The result is optimal since the restrictions on l is due to the structure of Q_n^k by definition.

Index Terms—Hamiltonian, panpositionable, bipanpositionable, k -ary n -cube.

I. INTRODUCTION

THE n -dimensional hypercube Q_n is one of the most well-known and popular interconnection networks due to its excellent properties as the following: it is vertex-symmetric and edge-symmetric; it is hamiltonian; it allows cycle/path embedding when faults occur and so on. (See [1], [2] for these results and their references). Therefore, numerous studies have been devoted to the hypercube family [3]–[6], [11], [12].

The k -ary n -cube Q_n^k is an enlarged family from Q_n that keeps many pleasing properties from hypercubes. More precisely, each vertex of Q_n^k is labeled by a n -bit finite sequence $(u_0, u_1, \dots, u_{n-1})$, where $0 \leq u_i \leq k - 1$ for $0 \leq i \leq n - 1$, and every two vertices u and v are adjacent if and only if $|u_i - v_i| = 1$ or $k - 1$ for some i and $u_j = v_j$ for any $0 \leq j \leq n - 1$ with $j \neq i$. It is obviously that the hypercube Q_n is indeed a subclass of the k -ary n -cube when $k = 2$. Some properties of Q_n^k mentioned in [6] are listed here: it is known that Q_n^k is vertex-symmetric and edge-symmetric [3]; it is hamiltonian [4], [5]; it has diameter $n \lfloor \frac{k}{2} \rfloor$ [4], [5]; it has a recursive structure; and it contains many important interconnection networks such as cycles (of certain lengths) [3], meshes (of certain dimensions) [4], and even hypercubes

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(of certain dimensions) [5]. However, as opposed to Q_n , Q_n^k has not received enough attention. In this article, we want to prove the panpositionability of Q_n^k . Readers can refer to [7] for the concept of panpositionability. A hamiltonian graph G is panpositionable if for any two different vertices u and v of G and any integer l with $d_G(u, v) \leq l \leq \frac{|V(G)|}{2}$, there exists a hamiltonian cycle C of G with $d_C(u, v) = l$. Similar to the hamiltonicity for the communication between processors in an interconnection network, panpositionable hamiltonicity allows more flexible communication in a hamiltonian network. It is easy to see that the panpositionable hamiltonian property inherits the hamiltonian property and advances it further [8].

The article is organized as follows. In Section 2, we introduce the graph terminologies and notations used in this paper, the precise definition of Q_n^k , and two lemmas. In Section 3, we study the panpositionability of Q_n^k , where $k \geq 3$ is an odd integer and $n \geq 2$ is an integer. In Section 4, we study the panpositionability in the bipartite version of Q_n^k , where $k \geq 4$ is an even integer and $n \geq 2$ is an integer. Our conclusion is given in the last section.

II. PRELIMINARIES

For the graph definitions and notations we follow [9]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{\{u, v\} | \{u, v\} \text{ is an unordered pair of } V\}$. We say that V is the vertex set and E is the edge set of G . Two vertices u and v are adjacent if $\{u, v\} \in E$. A path is represented by a finite sequence of vertices $\langle v_0, v_1, v_2, \dots, v_n \rangle$, where every two consecutive vertices are adjacent. If P is a path represented by $\langle v_0, v_1, v_2, \dots, v_n \rangle$, then we define $\text{inv}(P) = \langle v_n, v_{n-1}, v_{n-2}, \dots, v_0 \rangle$. The length of a path P is the number of edges in P . We write the path $\langle v_0, v_1, \dots, v_n \rangle$ as $\langle v_0, v_1, \dots, v_{s-1}, P_1, v_{i+1}, \dots, v_{j-1}, P_2, v_{t+1}, \dots, v_n \rangle$, where $P_1 = \langle v_s, v_{s+1}, \dots, v_i \rangle$ and $P_2 = \langle v_j, v_{j+1}, \dots, v_t \rangle$. We use $d_G(u, v)$ to denote the distance between u and v in G , i.e., the length of the shortest path between u and v in G . A cycle is a path of at least three vertices such that the first vertex is the same as the last vertex. A hamiltonian cycle of G is a cycle that visits every vertex of G exactly once. We use $d_C(u, v)$ to denote the distance between u and v in a cycle C of G , i.e., the length of the shorter path between u and v in C . A hamiltonian graph is a graph with a hamiltonian cycle.

A hamiltonian path in a graph G is a path joining two distinct vertices u and v of G that visits every vertex of G exactly once. A graph G is hamiltonian-connected if there is a hamiltonian path joining any two distinct vertices of G . Note that any (nontrivial) bipartite graph cannot be hamiltonian-connected, whereas a bipartite graph is hamiltonian laceable if there exists a hamiltonian path joining every two vertices which are in distinct partite [10].

The concept of hamiltonian panpositionability was first proposed by S. Kao etc. [7]. A hamiltonian graph G is *panpositionable* if for any two different vertices u and v of G and any integer l with $d_G(u, v) \leq l \leq \frac{|V(G)|}{2}$, there exists a hamiltonian cycle C of G with $d_C(u, v) = l$. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $V(G) = V_0 \cup V_1$ and $E(G)$ is a subset of $\{\{u, v\} | u \in V_0, v \in V_1\}$. A hamiltonian bipartite graph G is *bipanpositionable* if for any two different vertices u and v of G and any integer l with $d_G(u, v) \leq l \leq \frac{|V(G)|}{2}$ and $(l - d_G(u, v))$ is even, there exists a hamiltonian cycle C of G with $d_C(u, v) = l$.

The k -ary n -cube, Q_n^k , is defined for all integers $k \geq 2$ and $n \geq 1$. The subclass Q_n^2 is the well-studied hypercube family. The subclass Q_1^k with $k \geq 3$ is defined as the cycle of length k . The k -ary n -cube, Q_n^k , for $k \geq 3$ and $n \geq 2$ is defined as follows. Let $u \in V(Q_n^k)$ be represented by $(u_0, u_1, \dots, u_{n-1})$, where $0 \leq u_i \leq k - 1$. u and v are adjacent if and only if $|u_i - v_i| = 1$ or $k - 1$ for some i and $u_j = v_j$ for any $0 \leq j \leq n - 1$ with $j \neq i$. It is shown that Q_n^k is bipartite if k is even [11]. Here we mention some properties of Q_n^k that will be used in this article.

It is known that Q_n^k is *vertex-symmetric* and *edge-symmetric*. Moreover, given any two distinct vertices (u_1, u_2) and (v_1, v_2) of Q_2^k , there is an automorphism of Q_2^k mapping (u_1, u_2) and (v_1, v_2) to $(m, 0)$ and $(0, n)$. Each vertex of Q_n^k is represented by a n -bit tuple, and we will call the d th dimension. We can partition Q_n^k over dimension d by fixing the d th element of any vertex tuple at some value a , for every $a \in \{0, 1, \dots, k - 1\}$. This results in k copies $Q_{d,n-1}^{k,0}, Q_{d,n-1}^{k,1}, \dots, Q_{d,n-1}^{k,k-1}$ of Q_{n-1}^k , with corresponding vertices in $Q_{d,n-1}^{k,0}, Q_{d,n-1}^{k,1}, \dots, Q_{d,n-1}^{k,k-1}$ joined in a cycle of length k (in dimension d) [6]. It is proven in [11], [12] that Q_n^k is hamiltonian connected for odd k and Q_n^k is hamiltonian laceable for even k .

Note that the length of a path between u and v in Q_n^k , where $k \geq 5$ is an odd integer, can not be arbitrary. For example, in Q_5^3 , for any two vertices u and v and $d(u, v) = 1$, there exists no path P between u and v with $|P| = 2$. In fact, we have the following observation. Given two vertices $u = (u_0, u_1, \dots, u_{n-1})$ and $v = (v_0, v_1, \dots, v_{n-1})$ of Q_n^k . Define the number $m_i = \min\{|u_i - v_i|, k - |u_i - v_i|\}$, where $0 \leq i \leq n - 1$. Let $s = \max\{m_i : 0 \leq i \leq n - 1\}$. Then there exists no path between u and v with length $r = d(u, v) - s + k - s - 2l = d(u, v) + k - 2s - 2l$, where l is an integer and $1 \leq l \leq \frac{k}{2} - s$. Consequently, we modify the concept of panpositionability of Q_n^k by saying that Q_n^k is *nearly-panpositionable* if for any two distinct vertices x and y of Q_n^k and for any integer l' with $d(x, y) \leq l' \leq \frac{|V(Q_n^k)|}{2}$ and $l' \notin \{r : r = d(u, v) + k - 2s - 2l \text{ for } 1 \leq l \leq \frac{k}{2} - s\}$, there exists a hamiltonian cycle C of Q_n^k with $d_C(x, y) = l'$. Therefore, in this article, we want to prove that Q_n^3 is panpositionable, Q_n^k is nearly-panpositionable if $k \geq 5$ is an odd integer, and is bipanpositionable if $k \geq 4$ is an even integer. First of all, we prove the following two lemmas.

Lemma 1. *Let k be an integer with $k \geq 3$. For any path P with length 2 in Q_2^k , there exists a hamiltonian cycle of Q_2^k that contains P .*

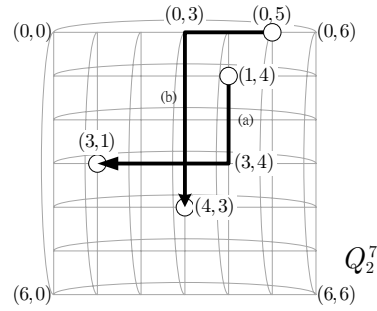


Fig. 1. (a) $f_{-3}^2(1, 4)$ and (b) $h_{-2}^4(0, 5)$.

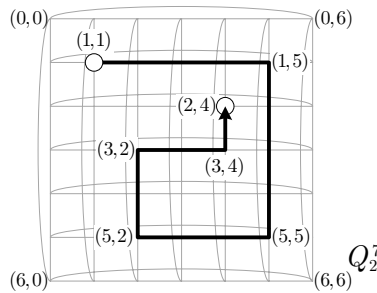


Fig. 2. $H_{b,3}^a(1, 1)$, where $\vec{a} = (4, -2, -1)$ and $\vec{b} = (4, -3, 2)$.

Proof: Let c, r, i be nonzero integers, $\frac{c}{|c|} = s$, $\frac{r}{|r|} = t$, $\vec{a} = (a_1, a_2, \dots, a_i)$ and $\vec{b} = (b_1, b_2, \dots, b_i)$. If $c = 0$, then $s = 0$. Similarly, if $r = 0$, then $t = 0$. To construct the required hamiltonian cycles, we define some path patterns in the following.

$$f_r^c(x, y) = \langle (x, y), (x + s \cdot 1, y), (x + s \cdot 2, y), \dots, (x + c, y), (x + c, y + t \cdot 1), (x + c, y + t \cdot 2), \dots, (x + c, y + r) \rangle;$$

$$h_r^c(x, y) = \langle f_r^0(x, y), f_0^c(x, y + r) \rangle;$$

$$H_{b,i}^a(x, y) = \langle h_{b_1}^{a_1}(x, y), h_{b_2}^{a_2}(x + a_1, y + b_1), h_{b_3}^{a_3}(x + a_1 + a_2, y + b_1 + b_2), \dots, h_{b_i}^{a_i}(x + \sum_{n=1}^{i-1} a_n, y + \sum_{n=1}^{i-1} b_n) \rangle.$$

Please see Fig. 1 and Fig. 2 for an illustration. Fig. 1 is examples of $f_{-3}^2(1, 4)$ and $h_{-2}^4(0, 5)$. Note that $f_{-3}^2(1, 4) = \langle (1, 4), (2, 4), (3, 4), (3, 3), (3, 2), (3, 1) \rangle$ and $h_{-2}^4(0, 5) = \langle f_{-2}^0(0, 5), f_0^4(0, 3) \rangle = \langle (0, 5), (0, 4), (0, 3), (1, 3), (2, 3), (3, 3), (4, 3) \rangle$. Fig. 2 is an example of $H_{b,3}^a(1, 1)$, where $\vec{a} = (4, -2, -1)$ and $\vec{b} = (4, -3, 2)$. Note that $H_{b,3}^a(1, 1) = \langle h_4^4(1, 1), h_{-3}^{-2}(5, 5), h_{-1}^{-1}(3, 2) \rangle = \langle f_4^0(1, 1), f_0^4(1, 5), f_{-3}^0(5, 5), f_0^{-2}(5, 2), f_2^0(3, 2), f_0^{-1}(3, 4) \rangle = \langle (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (5, 4), (5, 3), (5, 2), (4, 2), (3, 2), (3, 3), (3, 4), (2, 4) \rangle$.

Let $P = \langle u, x, v \rangle$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in Q_2^k . We have following cases.

Case 1. k is odd.

Case 1.1. Either $u_1 = v_1$ or $u_2 = v_2$. W.L.O.G., let $u = (0, 0)$, $v = (2, 0)$ and $P = \langle u, (1, 0), v \rangle$.

Let $a_i = (-1)^i(2 - k)$, for $i \leq k - 1$ and $a_k = 0$; $\vec{b} = (0, -1, -1, \dots, -1)$. There exists a hamiltonian cycle $C = \langle (0, 0), P, (2, 0), f_{k-1}^{k-3}(2, 0), H_{b,k}^a(0, k -$

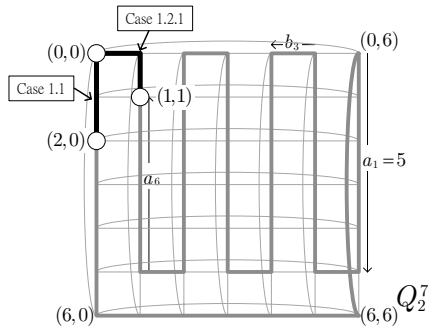


Fig. 3. Examples of Case 1.1 and Case 1.2.1 for $k = 7$.

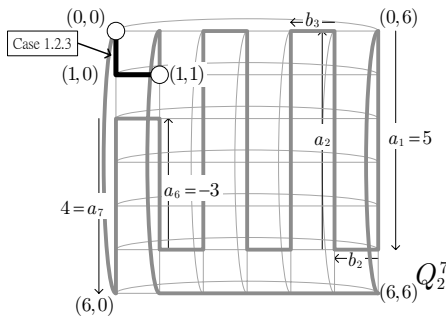


Fig. 4. An Example of Case 1.2.2 for $k = 7$.

1), (0,0)). Please see Fig. 3 for an illustration. The hamiltonian cycle in Fig. 3 is $C = \langle (0,0), P, (2,0), f_{6,7}^4(2,0), H_{b,7}^{\vec{a}}(0,6), (0,0) \rangle$ and $H_{b,7}^{\vec{a}}(0,6) = \langle h_0^5(0,6), h_{-1}^{-5}(5,6), h_{-1}^5(0,5), h_{-1}^{-5}(5,4), h_{-1}^5(0,3), h_{-1}^{-5}(5,2), h_{-1}^0(0,1) \rangle$.

Case 1.2. $u_1 \neq v_1, u_2 \neq v_2$. W.L.O.G., let $u = (0,0)$ and $v = (1,1)$.

Case 1.2.1. $P = \langle u, (0,1), v \rangle$, where $k \geq 3$.

The hamiltonian cycle is the same as in Case 1.1. Please see Fig. 3 for an illustration.

Case 1.2.2. $P = \langle u, (1,0), v \rangle$, where $k = 3$.

The hamiltonian cycle is $C = \langle (0,0), P, (1,1), f_{1,1}^{-1}(1,1), f_{2,2}^2(0,2), (2,0), (0,0) \rangle$.

Case 1.2.3. $P = \langle u, (1,0), v \rangle$, where $k \geq 5$.

Let $a_i = (-1)^i(2 - k)$, for $i \leq k - 2$, $a_{k-1} = 4 - k$ and $a_k = k - 3$; $\vec{b} = (0, -1, -1, \dots, -1)$. There exists a hamiltonian cycle $C = \langle (0,0), P, (0,1), f_{k-2}^0(k-1,1), H_{b,k}^{\vec{a}}(0, k-1), (0,0) \rangle$. Please see Fig. 4 for an illustration. The hamiltonian cycle in Fig. 4 is $C = \langle (0,0), P, (0,1), f_5^0(6,1), H_{b,7}^{\vec{a}}(0,6), (0,0) \rangle$ and $H_{b,7}^{\vec{a}}(0,6) = \langle h_0^5(0,6), h_{-1}^{-5}(5,6), h_{-1}^5(0,5), h_{-1}^{-5}(5,4), h_{-1}^5(0,3), h_{-1}^{-3}(5,2), h_{-1}^4(2,1) \rangle$.

Case 2. k is even.

Case 2.1. Either $u_1 = v_1$ or $u_2 = v_2$. W.L.O.G., let $u = (0,0)$ and $v = (2,0)$ and $P = \langle u, (1,0), v \rangle$.

Let $a_i = (-1)^i(2 - k)$, for $3 \leq i \leq k$, $a_1 = k - 3$, $a_2 = 1 - k$ and $a_{k+1} = 0$;

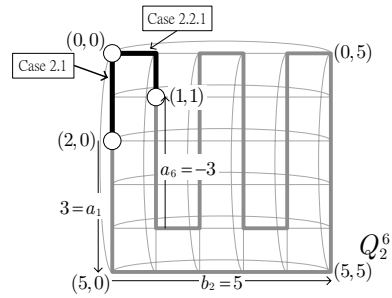


Fig. 5. Examples of Case 2.1 and Case 2.2.1 for $k = 6$.

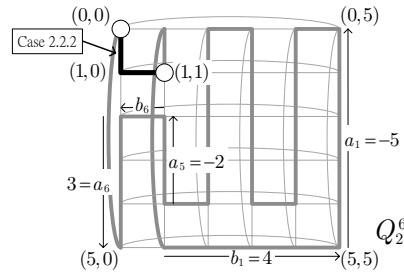


Fig. 6. An Example of Case 2.2.2 for $k = 6$.

$\vec{b} = (0, k - 1, -1, -1, \dots, -1)$. There exists a hamiltonian cycle $C = \langle (0,0), P, (2,0), H_{b,k+1}^{\vec{a}}(2,0), (0,0) \rangle$. Please see Fig. 5 for an illustration. The hamiltonian cycle in Fig. 5 is $C = \langle (0,0), P, (2,0), H_{b,7}^{\vec{a}}(2,0), (0,0) \rangle$ and $H_{b,7}^{\vec{a}}(2,0) = \langle h_0^3(2,0), h_5^{-5}(5,0), h_{-1}^4(0,5), h_{-1}^{-4}(4,4), h_{-1}^4(0,3), h_{-1}^{-4}(4,2), h_{-1}^0(0,1) \rangle$.

Case 2.2. $u_1 \neq v_1, u_2 \neq v_2$. W.L.O.G., let $u = (0,0)$ and $v = (1,1)$.

Case 2.2.1. $P = \langle u, (0,1), v \rangle$

The hamiltonian cycle is the same as in Case 2.1. Please see Fig. 5 for an illustration.

Case 2.2.2. $P = \langle u, (1,0), v \rangle$

Let $a_i = (-1)^i(k - 2)$, for $2 \leq i \leq k - 2$, $a_1 = 1 - k$, $a_{k-1} = 4 - k$ and $a_k = k - 3$; $\vec{b} = (k - 2, -1, -1, \dots, -1)$. There exists a hamiltonian cycle $C = \langle (0,0), P, (1,1), (0,1), H_{b,k}^{\vec{a}}(k-1,1), (0,0) \rangle$. Please see Fig. 6 for an illustration. The hamiltonian cycle in Fig. 6 is $C = \langle (0,0), P, (1,1), (0,1), H_{b,6}^{\vec{a}}(5,1), (0,0) \rangle$ and $H_{b,6}^{\vec{a}}(5,1) = \langle h_4^{-5}(5,1), h_{-1}^4(0,5), h_{-1}^{-4}(4,4), h_{-1}^4(0,3), h_{-1}^{-2}(4,2), h_{-1}^3(2,1) \rangle$.

The lemma is proved. ■

To facilitate our derivation in the following, we define some path patterns. We shall use $x_0^i, x_1^i, x_2^i, \dots, x_{k^n-1}^i$ to denote the k^n vertices of $Q_{d,n-1}^{k,i}$ for some d . For simplicity, denote $Q_{d,n-1}^{k,i}$ as $Q_{n-1}^{k,i}$. Let the path $p(x_a^i, x_b^i) = \langle x_a^i, x_{a_1}^i, x_{a_2}^i, \dots, x_b^i \rangle$ and $a_i = (a + i \bmod k^{n-1})$. For example, if $k^{n-1} = 64$, then $p(x_{60}^i, x_2^i) = \langle x_{60}^i, x_{61}^i, x_{62}^i, x_{63}^i, x_0^i, x_1^i, x_2^i \rangle$. It is known that there exists a hamiltonian cycle in Q_{n-1}^k [4]. Thus x_a^i and x_{a+1}^i are adjacent and so are x_a^i and x_{a+1}^i .

Lemma 2. Let k be an integer with $k \geq 3$. For any path P

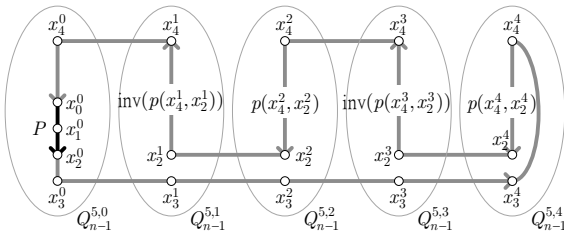


Fig. 7. An Example of Case 1 with $k = 5$.

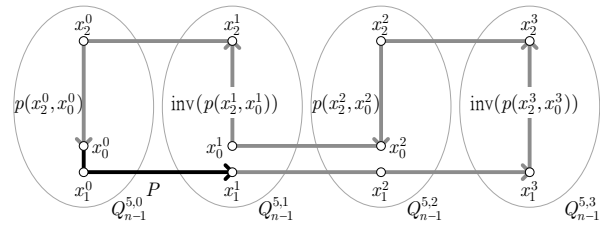


Fig. 8. An Example of Case 2 with $k = 4$.

with length 2 in Q_n^k , there exists a hamiltonian cycle of Q_n^k that contains P .

Proof: The lemma will be proved by mathematical induction. By Lemma 1, the statement holds for Q_2^k . Using the induction hypothesis, we assume that the statement holds for Q_{n-1}^k , where $n \geq 3$. Now we want to prove that the lemma is true for Q_n^k . There are three cases.

Case 1. P is in $Q_{n-1}^{k,i}$. W.L.O.G., let $i = 0$.

By the induction hypothesis, there exists a hamiltonian cycle C^0 of $Q_{n-1}^{k,0}$ that contains P . Let $P = \langle x_0^0, x_1^0, x_2^0 \rangle$ and $C^0 = \langle x_0^0, P, x_2^0, x_3^0, \dots, x_{k^{n-1}-1}^0, x_0^0 \rangle$. Since Q_{n-1}^k is hamiltonian [4], let the hamiltonian cycles in Q_{n-1}^k be $C^i = \langle x_0^i, x_1^i, x_2^i, x_3^i, \dots, x_{k^{n-1}-1}^i, x_0^i \rangle$.

- 1) k is odd. Then the hamiltonian cycle is
 $C = \langle x_0^0, P, x_2^0, x_3^0, x_4^0, \dots, x_{k-1}^0, p(x_4^{k-1}, x_2^{k-1}), \text{inv}(p(x_4^{k-2}, x_2^{k-2})), p(x_4^{k-3}, x_2^{k-3}), \text{inv}(p(x_4^{k-4}, x_2^{k-4})), \dots, p(x_4^2, x_2^2), \text{inv}(p(x_4^1, x_2^1)), p(x_4^0, x_2^0), x_0^0 \rangle$.
- 2) k is even. Then the hamiltonian cycle is
 $C = \langle x_0^0, P, x_2^0, x_3^0, x_4^0, \dots, x_{k-1}^0, \text{inv}(p(x_4^{k-1}, x_2^{k-1})), p(x_4^{k-2}, x_2^{k-2}), \text{inv}(p(x_4^{k-3}, x_2^{k-3})), p(x_4^{k-4}, x_2^{k-4}), \dots, p(x_4^2, x_2^2), \text{inv}(p(x_4^1, x_2^1)), p(x_4^0, x_2^0), x_0^0 \rangle$.

Please see Fig. 7 for an illustration, where the hamiltonian cycle in Fig. 7 is $C = \langle x_0^0, P, x_2^0, x_3^0, x_4^0, x_5^0, x_6^0, x_7^0, p(x_4^4, x_2^4), \text{inv}(p(x_4^3, x_2^3)), p(x_4^2, x_2^2), \text{inv}(p(x_4^1, x_2^1)), p(x_4^0, x_2^0), x_0^0 \rangle$.

Case 2. P passes through two $Q_{n-1}^{k,i}$. W.L.O.G., those two are $Q_{n-1}^{k,0}$ and $Q_{n-1}^{k,1}$.

Let $P = \langle x_0^0, x_1^0, x_1^1 \rangle$. In [11], [12], it has been shown that there exists a hamiltonian path $\langle x_1^i, p(x_1^i, x_0^i), x_0^i \rangle$ in $Q_{n-1}^{k,i}$.

- 1) k is odd. Then the hamiltonian cycle is
 $C = \langle x_0^0, P, x_1^1, x_2^1, x_3^1, x_4^1, \dots, x_{k-1}^1, p(x_2^{k-1}, x_0^{k-1}), \text{inv}(p(x_2^{k-2}, x_0^{k-2})), p(x_2^{k-3}, x_0^{k-3}), \text{inv}(p(x_2^{k-4}, x_0^{k-4})), \dots, p(x_2^2, x_0^2), \text{inv}(p(x_2^1, x_0^1)), p(x_2^0, x_0^0), x_0^0 \rangle$.
- 2) k is even. Then the hamiltonian cycle is
 $C = \langle x_0^0, P, x_1^1, x_2^1, x_3^1, x_4^1, \dots, x_{k-1}^1, \text{inv}(p(x_2^{k-1}, x_0^{k-1})), p(x_2^{k-2}, x_0^{k-2}), \text{inv}(p(x_2^{k-3}, x_0^{k-3})), p(x_2^{k-4}, x_0^{k-4}), \dots, p(x_2^2, x_0^2), \text{inv}(p(x_2^1, x_0^1)), p(x_2^0, x_0^0), x_0^0 \rangle$.

Please see Fig. 8 for an illustration, where the hamiltonian cycle in Fig. 8 is $C = \langle x_0^0, P, x_1^1, x_2^1, x_3^1, \text{inv}(p(x_3^3, x_0^3)), p(x_2^2, x_0^2), \text{inv}(p(x_2^1, x_0^1)), p(x_2^0, x_0^0), x_0^0 \rangle$.

Case 3. P passes through three $Q_{n-1}^{k,i}$.

It is known that we can partition Q_n^k over dimension d by fixing the d th element of any vertex tuple at some value a , for

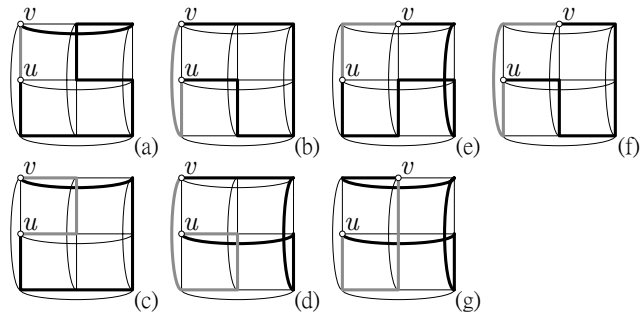


Fig. 9. Illustrations of Lemma 3.

every $a \in \{0, 1, \dots, k-1\}$. In this case, $P = \langle u, x, v \rangle$ passes through three $Q_{n-1}^{k,i}$, i.e., u, x and v have the same value in at least one element of vertex tuple. Hence this case is equivalent to Case 1.

By the mathematical induction, the lemma is proved. ■

III. THE PANPOSITIONABILITY OF Q_n^k , WHERE $k \geq 3$ IS AN ODD INTEGER AND $n \geq 2$ IS AN INTEGER.

Lemma 3. Q_2^3 is a panpositionable hamiltonian graph.

Proof: There are two cases: Case 1. $u = (0, 0)$ and $v = (1, 0)$; Case 2. $u = (1, 0)$ and $v = (0, 1)$. By brute force, we construct the required hamiltonian cycles. Please see Fig. 9. ■

Theorem 1. Q_n^3 is a panpositionable hamiltonian graph.

Proof: The theorem is proved by mathematical induction using Lemma 3 as base case. The detailed derivation is skipped. ■

Lemma 4. Let k be an odd integer with $k \geq 5$. Then Q_2^k is nearly-panpositionable.

Proof: The proof is by brute force and hence is skipped. ■

Theorem 2. Let k be an odd integer with $k \geq 5$. Q_n^k is nearly-panpositionable hamiltonian.

Proof: We will prove the theorem using the mathematical induction. By Lemma 4, Q_2^k is nearly-panpositionable hamiltonian. With the induction hypothesis, we assume that Q_{n-1}^k is nearly-panpositionable hamiltonian for some $n \geq 3$. We need to show that Q_n^k is nearly-panpositionable hamiltonian. Let $u, v \in Q_n^k$ and l be an integer with

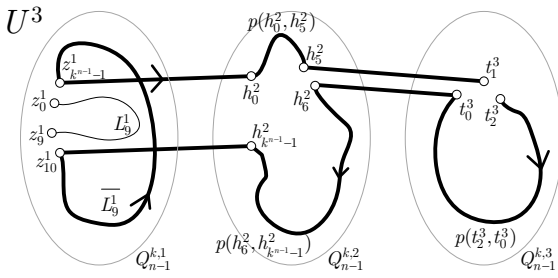


Fig. 10. U^3 for $r = 3$ and $l' = 9$.

$d \leq l \leq \frac{|Q_n^k|}{2}$, where $d = d_{Q_n^k}(u, v)$.

Case 1. $u, v \in Q_{n-1}^{k,i}$. W.L.O.G., let $i = 0$.

Obviously, $d_{Q_{n-1}^k}(u, v) = d$.

Case 1.1. $d \leq l \leq \frac{k^{n-1}-1}{2}$.

By the induction hypothesis, there exist a hamiltonian cycle $C_l^i = \langle x_0^i, x_1^i, \dots, x_l^i, \dots, x_{k^n-1}^i, x_0^i \rangle$ in $Q_{n-1}^{k,i}$ for $u = x_0^i$ and $v = x_0^i$. Then we have the hamiltonian cycle $C = \langle x_0^0, p(x_0^0, x_l^0), x_l^0, x_{l+1}^0, \dots, x_{l+1}^{k-1}, p(x_{l+1}^{k-1}, x_{l+1}^{k-2}), x_{l+1}^{k-2}, \dots, x_{l+1}^{k-3}, p(x_{l+1}^{k-3}, x_{l+1}^{k-4}), \dots, p(x_{l+1}^2, x_{l+1}^1), \text{inv}(p(x_{l+1}^1, x_{l+1}^0)), p(x_{l+1}^0, x_0^0), x_0^0 \rangle$.

Case 1.2. $\frac{k^{n-1}-1}{2} + 1 \leq l \leq \frac{|Q_n^k|}{2}$.

By the induction hypothesis, for any two vertices $x, y \in V(Q_{n-1}^k)$ and $1 \leq l' \leq k^{n-1}-1$ there exists a hamiltonian cycle C of $Q_{n-1}^{k,i}$ with $d_C(x, y) = l'$. We set $x = z_0^i$ and $y = z_{l'}^i$, then the hamiltonian cycle will be $\langle z_0^i, p(z_0^i, z_{k^n-1}^i), z_{k^n-1}^i, \dots, z_{l'}^i \rangle$. Split the hamiltonian cycle into two paths $L_{l'}^i$ and $\bar{L}_{l'}^i$ by letting $L_{l'}^i = p(z_0^i, z_{l'}^i)$ and $\bar{L}_{l'}^i = p(z_{l'+1}^i, z_{k^n-1}^i)$.

In [12], it is shown that for all $x, y \in Q_{n-1}^{k,i}$, there exists a hamiltonian path H^i of $Q_{n-1}^{k,i}$ between x and y . Define $H^i = p(h_0^i, h_{k^n-1}^i)$ with $x = h_0^i$ and $y = h_{k^n-1}^i$. By Lemma 2, for any path with length 2 denoted by $\langle t_0^i, t_1^i, t_2^i \rangle$, there exists a hamiltonian cycle $T^i = \langle t_0^i, p(t_0^i, t_{k^n-1}^i), t_{k^n-1}^i, t_0^i \rangle$ of $Q_{n-1}^{k,i}$. Let $t_0^i = h_{j+1}^i, t_1^i = h_j^i, z_{l'+1}^i = h_{k^n-1}^i, h_0^i = z_{k^n-1}^i$ and $\bar{L}_{l'}^i = p(z_{l'+1}^i, z_{k^n-1}^i)$. Then there is a unique path $U^i = \langle t_2^i, p(t_2^i, t_0^i), h_{j+1}^{i-1}, p(h_{j+1}^{i-1}, h_{k^n-1}^{i-1}), z_{l'+1}^{i-2}, \bar{L}_{l'}^{i-2}, z_{k^n-1}^{i-2}, h_0^{i-1}, p(h_0^{i-1}, h_{j+1}^{i-1}), t_1^i \rangle$. For example, let $r = 3$ and $l' = 9$, then $U^3 = \langle t_2^3, p(t_2^3, t_0^3), h_6^2, p(h_6^2, h_{k^n-1}^2), z_{10}^1, \bar{L}_9^1, z_{k^n-1}^1, h_0^2, p(h_0^2, h_5^2), t_1^3 \rangle$. Please see Fig. 10 for an illustration.

Let m and r be integers and $0 \leq r \leq \frac{k-1}{2}$ such that $\frac{k^{n-1}-1}{2} - m + r \cdot k^{n-1} + l' + 1 = l$. W.L.O.G., let $u = x_0^0$ and $v = x_{\frac{k^{n-1}-1}{2}-m}^0$. For simplicity, denote $x_{\frac{k^{n-1}-1}{2}-m}^i$ as $v^i, x_{\frac{k^{n-1}-1}{2}-m-1}^i$ as v_1^i and $x_{\frac{k^{n-1}-1}{2}-m+1}^i$ as v_1^{i+1} . If r is even, let $t_1^i = v_0^i, t_2^i = v^i, z_0^{1+r} = x_0^{1+r}$ and $z_{l'+r}^i = x_{l'+r}^i$. There is

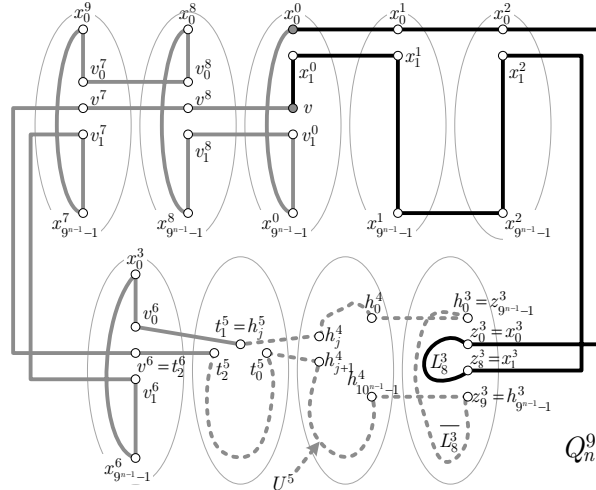


Fig. 11. An Example of Case 1.2 with $k = 9$ and $l = \frac{5 \cdot 9^{n-1} - 17}{2}$.

a hamiltonian cycle

$$C = \langle x_0^0, x_0^1, \dots, x_0^r, L_{l'}^{r+1}, p(x_1^r, x_{k^n-1}^r), \text{inv}(p(x_1^{r-1}, x_{k^n-1}^{r-1})), p(x_1^{r-2}, x_{k^n-1}^{r-2}), \text{inv}(p(x_1^{r-3}, x_{k^n-1}^{r-3})), \dots, p(x_1^2, x_{k^n-1}^2), \text{inv}(p(x_1^1, x_{k^n-1}^1)), p(x_1^0, v_0^0), v, v^{k-1}, v^{k-2}, \dots, v^{r+4}, U^{r+3}, v_0^{r+4}, \text{inv}(p(v_1^{r+4}, v_0^{r+4})), p(v_1^{r+5}, v_0^{r+5}), \text{inv}(p(v_1^{r+6}, v_0^{r+6})), p(v_1^{r+7}, v_0^{r+7}), \dots, p(v_1^{k-2}, v_0^{k-2}), \text{inv}(p(v_1^{k-1}, v_0^{k-1})), v_1^0, p(x_{\frac{k^{n-1}-1}{2}-m+2}^0, x_{k^n-1}^0), x_0^0 \rangle$$

Please see Fig. 11 for an illustration, where $m = 0, r = 2, l' = 8$ and the hamiltonian cycle is

$$C = \langle x_0^0, x_1^0, x_2^0, L_8^3, p(x_1^2, x_{9^n-1}^2), \text{inv}(p(x_1^1, x_{9^n-1}^1)), p(x_1^0, v_0^0), v, v^8, v^7, v^6, U^5, v_0^6, \text{inv}(p(v_1^6, v_0^6)), p(v_1^7, v_0^7), \text{inv}(p(v_1^8, v_0^8)), v_1^0, p(x_{\frac{9^n-1}{2}+2}^0, x_{9^n-1}^0), x_0^0 \rangle$$

If r is odd, let $t_1^i = v_1^i, t_2^i = v^i, x_0^{1+r} = z_0^{1+r}$ and $x_{k^n-1}^{1+r} = z_{l'+r}^{1+r}$. There is a hamiltonian cycle

$$C = \langle x_0^0, x_0^1, \dots, x_0^r, L_{l'}^{r+1}, \text{inv}(p(x_1^r, x_{k^n-1}^r)), p(x_1^{r-1}, x_{k^n-1}^{r-1}), \text{inv}(p(x_1^{r-2}, x_{k^n-1}^{r-2})), p(x_1^{r-3}, x_{k^n-1}^{r-3}), \dots, \text{inv}(p(x_1^3, x_{k^n-1}^3)), p(x_1^2, x_{k^n-1}^2), \text{inv}(p(x_1^1, x_{k^n-1}^1)), p(x_1^0, v_0^0), v, v^{k^{n-1}-1}, v^{k^{n-1}-2}, \dots, v^{r+4}, U^{r+3}, v_0^{r+4}, p(v_1^{r+4}, v_0^{r+4}), \text{inv}(p(v_1^{r+5}, v_0^{r+5})), p(v_1^{r+6}, v_0^{r+6}), \text{inv}(p(v_1^{r+7}, v_0^{r+7})), \dots, p(v_1^{k^{n-1}-2}, v_0^{k^{n-1}-2}), \text{inv}(p(v_1^{k^{n-1}-1}, v_0^{k^{n-1}-1})), v_1^0, p(x_{\frac{k^{n-1}-1}{2}-m+2}^0, x_{k^n-1}^0), x_0^0 \rangle$$

Case 2. $u \in Q_{n-1}^{k,i}, v \in Q_{n-1}^{k,j}$ and $i \neq j$. W.L.O.G., let $i = 0$. For any vertex x_a^j in $Q_{n-1}^{k,j}$, there exists a corresponding vertex x_a^0 . Set $u = x_0^0$ and $v = x_{d'}^j$, where d' is the length

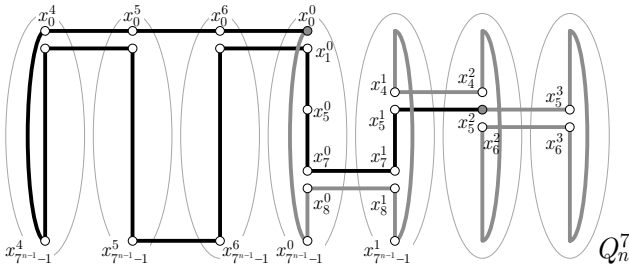


Fig. 12. An Example of Case 2.1 with $k = 7$ and $l = 3 \cdot 7^{n-1} + 11$.

of the shortest path between x_0^0 and $v = x_{d'}^0$ in $Q_{n-1}^{k,0}$. Note that for all $i \leq k - 1$, there is a hamiltonian cycle $C^i = \langle x_0^i, x_1^i, \dots, x_{d'}^i, \dots, x_{k^n-1}^i \rangle$ in $Q_{n-1}^{k,i}$.

Case 2.1. $l - d$ is even.

Let $0 \leq r \leq \frac{k-1}{2}$ be an integer, $d + 2(t - d') + r \cdot k^{n-1} = l$, $d' \leq t \leq k^{n-1} - 1$ and $e = k - 1 - r$.

Let r be an odd integer. We have the hamiltonian cycle $C = \langle x_0^0, x_0^{k-1}, x_0^{k-2}, \dots, x_0^{k-r}, \text{inv}(p(x_1^{k-r}, x_{k^n-1-1}^{k-r})), p(x_1^{k-r+1}, x_{k^n-1-1}^{k-r+1}), \text{inv}(p(x_1^{k-r+2}, x_{k^n-1-1}^{k-r+2})), p(x_1^{k-r+3}, x_{k^n-1-1}^{k-r+3}), \dots, \text{inv}(p(x_1^{k-3}, x_{k^n-1-1}^{k-3})), p(x_1^{k-2}, x_{k^n-1-1}^{k-2}), \text{inv}(p(x_1^{k-1}, x_{k^n-1-1}^{k-1})), x_1^0, p(x_2^0, x_{d'-1}^0), p(x_{d'}^0, x_t^0), \text{inv}(p(x_{d'}^1, x_t^1)), x_{d'}^2, \dots, x_{d'}^e, \text{inv}(p(x_{d'+1}^e, x_{d'-1}^e)), p(x_{d'+1}^{e-1}, x_{d'-1}^{e-1}), \text{inv}(p(x_{d'+1}^{e-2}, x_{d'-1}^{e-2})), \dots, \text{inv}(p(x_{d'+1}^3, x_{d'-1}^3)), p(x_{d'+1}^2, x_{d'-1}^2), \text{inv}(p(x_{t+1}^1, x_{d'-1}^1)), p(x_{t+1}^0, x_0^0) \rangle$. Please see Fig. 12 for an illustration, where $r = 3$, $d' = 5$, $t = 7$ and the hamiltonian cycle is $C = \langle x_0^0, x_0^6, x_0^5, x_0^4, \text{inv}(p(x_1^4, x_{7n-1-1}^4)), p(x_1^5, x_{7n-1-1}^5), \text{inv}(p(x_1^6, x_{7n-1-1}^6)), x_1^0, p(x_2^0, x_2^0), \text{inv}(p(x_5^1, x_7^1)), x_5^2, x_5^3, \text{inv}(p(x_6^3, x_4^3)), p(x_6^2, x_4^2), \text{inv}(p(x_8^1, x_4^1)), p(x_8^0, x_0^0), x_0^0 \rangle$.

Let r be an even integer. We have the hamiltonian cycle $C = \langle x_0^0, x_0^{k-1}, x_0^{k-2}, \dots, x_0^{k-r}, p(x_1^{k-r}, x_{k^n-1-1}^{k-r}), \text{inv}(p(x_1^{k-r+1}, x_{k^n-1-1}^{k-r+1})), p(x_1^{k-r+2}, x_{k^n-1-1}^{k-r+2}), \text{inv}(p(x_1^{k-r+3}, x_{k^n-1-1}^{k-r+3})), \dots, p(x_1^{k-2}, x_{k^n-1-1}^{k-2}), \text{inv}(p(x_1^{k-1}, x_{k^n-1-1}^{k-1})), x_1^0, p(x_2^0, x_{d'}^0), p(x_{d'}^0, x_t^0), \text{inv}(p(x_{d'}^1, x_t^1)), x_{d'}^2, \dots, x_{d'}^e, p(x_{d'+1}^e, x_{d'-1}^e), \text{inv}(p(x_{d'+1}^{e-1}, x_{d'-1}^{e-1})), p(x_{d'+1}^{e-2}, x_{d'-1}^{e-2}), \text{inv}(p(x_{d'+1}^{e-3}, x_{d'-1}^{e-3})), \dots, p(x_{d'+1}^2, x_{d'-1}^2), \text{inv}(p(x_{t+1}^1, x_{d'-1}^1)), p(x_{t+1}^0, x_0^0) \rangle$.

Case 2.2. $l - d$ is odd.

By the induction hypothesis, there exists a hamiltonian cycle $D^i = \langle y_0^i, y_1^i, \dots, y_{k^n-1}^i \rangle$ in $Q_{n-1}^{k,i}$ such that $x_0^i = y_0^i$, $x_{d'}^i = y_{l'}^i$ and the length l' of the path joining y_0^i to $y_{l'}^i$ in $Q_{n-1}^{k,i}$ is the smallest integer when $l' - d'$ is odd. Let $0 \leq r \leq \frac{k-1}{2}$ be an integer, $d + l' - d' + 2(t - l') + r \cdot k^{n-1} = l$, $d' \leq t \leq k^{n-1} - 1$ and $e = k - 1 - r$.

Let r be an odd integer. We have the hamiltonian cycle $C = \langle y_0^0, y_0^{k-1}, y_0^{k-2}, \dots, y_0^{k-r}, \text{inv}(p(y_1^{k-r}, y_{k^n-1-1}^{k-r})), p(y_1^{k-r+1}, y_{k^n-1-1}^{k-r+1}), \text{inv}(p(y_1^{k-r+2}, y_{k^n-1-1}^{k-r+2})), p(y_1^{k-r+3}, y_{k^n-1-1}^{k-r+3}), \dots, \text{inv}(p(y_1^{k-3}, y_{k^n-1-1}^{k-3})), p(y_1^{k-2}, y_{k^n-1-1}^{k-2}), \text{inv}(p(y_1^{k-1}, y_{k^n-1-1}^{k-1})), y_1^0, p(y_2^0, y_{d'-1}^0), p(y_{d'}^0, y_t^0), \text{inv}(p(y_{d'}^1, y_t^1)), y_{d'}^2, \dots, y_{d'}^e, p(y_{d'+1}^e, y_{d'-1}^e), \text{inv}(p(y_{d'+1}^{e-1}, y_{d'-1}^{e-1})), p(y_{d'+1}^{e-2}, y_{d'-1}^{e-2}), \text{inv}(p(y_{d'+1}^{e-3}, y_{d'-1}^{e-3})), \dots, \text{inv}(p(y_{d'+1}^3, y_{d'-1}^3)), p(y_{d'+1}^2, y_{d'-1}^2), \text{inv}(p(y_{t+1}^1, y_{d'-1}^1)), p(y_{t+1}^0, y_0^0) \rangle$.

Let r be an even integer. We have the hamiltonian cycle $C = \langle y_0^0, y_0^{k-1}, y_0^{k-2}, \dots, y_0^{k-r}, p(y_1^{k-r}, y_{k^n-1-1}^{k-r}), \text{inv}(p(y_1^{k-r+1}, y_{k^n-1-1}^{k-r+1})), p(y_1^{k-r+2}, y_{k^n-1-1}^{k-r+2}), \text{inv}(p(y_1^{k-r+3}, y_{k^n-1-1}^{k-r+3})), \dots, p(y_1^{k-2}, y_{k^n-1-1}^{k-2}), \text{inv}(p(y_1^{k-1}, y_{k^n-1-1}^{k-1})), y_1^0, p(y_2^0, y_{d'-1}^0), p(y_{d'}^0, y_t^0), \text{inv}(p(y_{d'}^1, y_t^1)), y_{d'}^2, \dots, y_{d'}^e, p(y_{d'+1}^e, y_{d'-1}^e), \text{inv}(p(y_{d'+1}^{e-1}, y_{d'-1}^{e-1})), p(y_{d'+1}^{e-2}, y_{d'-1}^{e-2}), \text{inv}(p(y_{d'+1}^{e-3}, y_{d'-1}^{e-3})), \dots, p(y_{d'+1}^2, y_{d'-1}^2), \text{inv}(p(y_{t+1}^1, y_{d'-1}^1)), p(y_{t+1}^0, y_0^0) \rangle$.

By the mathematical induction, the theorem is proved. ■

IV. Q_n^k IS BIPANPOSITIONABLE, WHERE $k \geq 4$ IS AN EVEN INTEGER AND $n \geq 2$ IS AN INTEGER.

Lemma 5. Let k be an even integer with $k \geq 4$. Then Q_2^k is bipanpositionable.

Proof: The proof is by brute force and hence is skipped. ■

Theorem 3. Let k be an even integer with $k \geq 4$. Q_n^k is bipanpositionable hamiltonian.

Proof: We will prove the theorem using the mathematical induction. By Lemma 5, Q_2^k is bipanpositionable hamiltonian. With the induction hypothesis, we assume that Q_{n-1}^k is bipanpositionable hamiltonian for some $n \geq 4$. We need to show that Q_n^k is bipanpositionable hamiltonian. Note that Q_n^2 is bipanpositionable [7]. Therefore, $Q_n^4 = Q_{2n}^2$ is bipanpositionable. It suffices to prove the cases for $k \geq 6$.

The proof is similar to Theorem 2, so we'll skip it. Readers can follow the similar techniques in Theorem 2 to construct the required hamiltonian cycles. ■

V. CONCLUSIONS

In this paper, we prove that the k -ary n -cube Q_n^k is panpositionable hamiltonian and Q_n^k is nearly-panpositionable for any odd integer $k \geq 5$. Moreover, we prove that Q_n^k is bipanpositionable hamiltonian for any even integer $k \geq 4$. It is known that the hypercube, Q_n^2 , is bipanpositionable [7]. Thus Q_n^k is bipanpositionable for all even integers $k \geq 2$.

The panpositionability of any k -ary n -cube has been completely studied and the result is optimal in the sense that given any two vertices u and v , there exists no more hamiltonian cycle on which $d_C(u, v)$ equals any of the numbers we miss in the nearly-panpositionable Q_n^k when k is odd.

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