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The Non-Uniqueness of Partial Differential Equations Options Price Valuation Formula for Heston Stochastic Volatility Model

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Abstract**—**An option is defined as a financial contract that provides the holder the right but not the obligation to buy or sell a specified quantity of an underlying asset in the future at a fixed price (called a strike price) on or before the expiration date of the option. This paper examined two approaches for derivation of Partial Differential Equation (PDE) options price valuation formula for the Heston stochastic volatility model. We obtained various PDE option price valuation formulas using the riskless portfolio method and the application of Feynman-Kac theorem respectively. From the results obtained, we see that the two derived PDEs for Heston model are distinct and non-unique. This establishes the fact of incompleteness in the model for option price valuation.

Keywords**—**Option price valuation, Partial Differential Equations, Black-Scholes PDEs, Ito process.

I. INTRODUCTION

FINANCIAL derivatives are financial contracts that are linked to an underlying asset and through which specific financial risks can be traded in a typical financial market. The value of a financial derivative is a function of the underlying asset and time from whence its price is derived. Since the future reference price of the derivative is not known with certainty, its value at maturity can only be anticipated or estimated. Options which are a type of financial derivative are used for several purposes which include risk management, hedging, etc. [1].

In the early advent of stochastic financial modeling, the Black-Scholes model [2], for option pricing, assumed that the volatility of the underlying asset was constant. The model failed to take into consideration the fact that the volatility of the underlying asset oscillates. This omission therefore necessitated the study on stochastic volatility models such as the Heston stochastic volatility model which treats price volatility as arbitrary or a random variable. This singular idea of allowing the price of the underlying asset to vary in the stochastic volatility models improved the accuracy of model calculations and predictions.

Grasping and quantifying the ingrained uncertainty in a volatility market is important for every portfolio, options and risk management. This is obvious since volatility is not directly observed but a statics of observable returns. So, estimates of it are often stochastic or probabilistic [3].

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II. THEORETICAL INSIGHT

Definition1. Self-financing trading strategy: A trading strategy is an $N-$dimensional stochastic process $a_1(t), \cdots, a_N(t)$ that represents the allocations into the assets at time, t. The time, t value of the portfolio is $\prod(t) = \sum_{i=1}^{N} a_i\,(t)S_i(t)$.

A trading strategy is self-financing if the change in the value of the portfolio is due only to changes in the value of the assets and not to inflows or outflows of funds. This implies that the strategy is self-financing if

$$d\prod(t) = d\left(\sum_{i=1}^{N} a_i(t)S_i\,(t)\right) = \sum_{i=1}^{N} a_i\,(t)dS_i(t),$$

in other words, a trading strategy is self-financing, if

$$\prod(t) = \prod(0) + \sum_{i=1}^{N} \int_0^t a_i(u)\,dS_i(u)$$

In the case of two assets the portfolio value is $\prod(t) = a_1(t)S_1(t) + a_2(t)S_2(t)$ and the strategy $(a_{1,}a_2)$ is self-financing if $d\prod(t) = a_1(t)dS_1\,(t) + a_2(t)dS_2(t)$.

Definition2. Self-financing portfolio: A portfolio allocation $(\xi_t, \eta_t)_{t\epsilon\mathbb{R}_+}$ with price (value) V_t given by

$$V_t = \xi_t S_t + \eta_t A_t, \qquad t\;\epsilon\;\mathbb{R}^+$$

is self-financing if and only if

$$dV_t = B_t dA_t + \xi_t dS_t$$

where ξ_t is the number of shares in S_t (could be any real number) and B_t is the riskless asset, which is the amount in the bank.

Theorem1. Multidimensional Version of the Feynman-Kac Theorem**:** Suppose that x_t follows the stochastic process in n dimensions

$$dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dW_t^{\mathbb{Q}}$$

where x_t and $\mu(x_t, t)$ are each vectors of dimension n, $W_t^{\mathbb{Q}}$ is a vector of dimension m of $\mathbb{Q}-$ Brownian motion, and $\sigma(x_t, t)$ is a matrix of size $n \times m$. In other words

$$d\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} \mu_1(x_t,t) \\ \vdots \\ \mu_n(x_t,t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(x_t,t) & \cdots & \sigma_{1m}(x_t,t) \\ \vdots & \ddots & \vdots \\ \sigma_{n1}(x_t,t) & \cdots & \sigma_{nm}(x_t,t) \end{pmatrix} \begin{pmatrix} dW_1^{\mathbb{Q}}(t) \\ \vdots \\ dW_m^{\mathbb{Q}}(t) \end{pmatrix}$$

The generator of the process is

$$A = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\sigma\sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \quad (1)$$

where for notational convenience $\mu_i = \mu_i(x_t,t), \sigma = \sigma(x_t,t)$, and $(\sigma\sigma^T)_{ij}$ is element (i,j) of the matrix $\sigma\sigma^T$ of size $(n \times n)$. The theorem states that the PDE in $V(x_t,t)$ given by

$$\frac{\partial V}{\partial t} + AV(x_t,t) - r(x_t,t)V(x_t,t) = 0 \quad (2)$$

and with boundary condition $V(X_T,T)|\mathcal{F}_t$ has solution

$$V(x_t,t) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(x_u,u)du} V(X_T,T) \middle| \mathcal{F}_t\right] \quad (3)$$

A. Ito Formula for Ito Processes

We now turn to the general expression of Ito's formula which applies to Ito processes of the form

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s, \qquad t \in \mathbb{R}^+ \quad (4)$$

or in differential notation

$$dX_t = \mu_t dt + \sigma_t dW_t$$

where $(\mu_t)_{t\in\mathbb{R}^+}$ and $(\sigma_t)_{t\in\mathbb{R}^+}$ are square-integrable adapted processes [4].

Lemma1. (Ito formula for Ito processes). For any Ito process $(X_t)_{t\in\mathbb{R}^+}$ of the form (4) and any $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and $Z_t = f(t,X_t)$ we have,

$$Z_t = f(0,X_0) + \int_0^t \mu_s \frac{\partial f}{\partial x}(s,X_s)ds + \int_0^t \sigma_s \frac{\partial f}{\partial x}(s,X_s)dB_s + \int_0^t \frac{\partial f}{\partial t}(s,X_s)ds + \frac{1}{2}\int_0^t |\sigma_s|^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)ds$$

or in differential form

$$dZ_t = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(dX_t)^2 = \left(\frac{\partial f}{\partial t}(t,X_t) + \frac{\partial f}{\partial x}(t,X_t)\mu_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\sigma_t^2\right)dt + \frac{\partial f}{\partial x}(t,X_t)\sigma_t dW_t (5)$$

III. Methods

The parameters for consideration in Heston model are as:

- S_t or S_u: Underlying asset
- v_t or v_u: Volatility factor
- $W_t$or W_u :Brownian motion
- σ: Measure of the standard deviation of the returns of the asset
- θ: The long-term running mean of the variance process
- κ: The speed of mean-reversion of the variance process
- ρ: The instantaneous correlation between the state process and the volatility process
- dt: Time step-size.
- r: Risk-free interest rate.
- μ: Drift factor (Measure of average rate of growth of the asset).

A. Heston Stochastic Volatility Model

Here, we go straight to use the riskless portfolio method to derive the PDEs option price valuation formula for the Heston Stochastic Differential Equation model given as [5].

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \qquad S_0 > 0$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^{(2)}, \qquad v_0 > 0$$
$$dW_t^{(1)} dW_t^{(2)} = \rho dt$$

Money market

$$dB_t = B_t r dt$$

Contingent claim

$$c(S_t, v_t, t)$$

We define a trading strategy $H_t = (\eta_t, \xi_t, \gamma_t)$, applied to the portfolio $(B_t, S_t, c(S_t, v_t, t))$. The value of the trading strategy is then

$$h_t = \eta_t B_t + \xi_t S_t + \gamma_t c(S_t, v_t, t).$$

We require the trading strategy to be self-financing, i.e.

$$dh_t = \eta_t dB_t + \xi_t dS_t + \gamma_t dc(S_t, v_t, t).$$

Hence, the value of the hedge portfolio must be equal to the value of the option

$$u(S_{t,}, v_t, t) = h_t$$

and in particular, the instantaneous changes must as well be equal. So we have

$$du(S_t, v_t, t) = dh_t.$$

Applying the Ito's formula (Lemma 1) we derive the PDE. Ito's formula directly gives the expressions for $du(S_t, v_t, t)$ and dh_tas

$$du(S_t, v_t, t) = \left(\frac{\partial u}{\partial t} + S_t\mu\frac{\partial u}{\partial s} + \kappa(\theta - v)\frac{\partial u}{\partial v} + \frac{1}{2}S_t^2 v_t \frac{\partial^2 u}{\partial s^2} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 u}{\partial v^2} + S_t\sigma v_t \rho \frac{\partial^2 u}{\partial s \partial v}\right)dt + S_t\sqrt{v_t}\frac{\partial u}{\partial s} dW_t^{(1)} + \sigma\sqrt{v_t}\frac{\partial u}{\partial v} dW_t^{(2)} \ (6)$$

$$dh_t = \gamma_t\left(\frac{\partial c}{\partial t} + S_t\mu\frac{\partial c}{\partial s} + k(\theta - v)\frac{\partial c}{\partial v} + \frac{1}{2}S_t^2 v_t\frac{\partial^2 c}{\partial s^2} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 c}{\partial v^2} + S_t\sigma v_t \frac{\partial^2 c}{\partial s \partial v}\right)dt + (\eta_t B_t r + \xi_t S_t \mu)dt +$$

$$\left(\gamma_t S_t \sqrt{v_t} \frac{\partial c}{\partial s} + \xi_t \eta_t \sqrt{v_t}\right) dW_t^{(1)} + \gamma_t \sigma \sqrt{v_t} \frac{\partial c}{\partial v} dW_t^{(2)} (7)$$

Given $|\rho| < 1$, the Ito processes $u(S_t, v_t, t)$ and h_t are identical if and only if the factors in front of $dW_t^{(1)}, dW_t^{(2)}$ and dt are equal. Equality of the first two factors implies

$$S_t \sqrt{v_t} \frac{\partial u}{\partial s} = \gamma_t S_t \sqrt{v_t} \frac{\partial c}{\partial s} + \xi_t S_t \sqrt{v_t}$$
$$\sigma \sqrt{v_t} \frac{\partial u}{\partial v} = \gamma_t \sigma \sqrt{v_t} \frac{\partial c}{\partial v}$$

Hence, the choices of

$$\gamma_t = \frac{\frac{\partial u}{\partial v}}{\frac{\partial c}{\partial v}},$$
$$\xi_t = \frac{\partial u}{\partial s} - \gamma \frac{\partial c}{\partial s} = \frac{\partial u}{\partial s} - \frac{\frac{\partial u}{\partial v} \times \frac{\partial c}{\partial s}}{\frac{\partial c}{\partial v}}$$

remove the stochastic component from dh_t which renders the portfolio riskless. Having determined ξ_t, γ_t and by replacing η_t using the relation

$$u(S_t, v_t, t) = h_t = \eta_t B_t + \xi_t S_t + \gamma_t c(S_t, v_t, t).$$

We now compare the drift terms and we have,

$$\frac{\partial u}{\partial t} + S_t \mu \frac{\partial u}{\partial s} + \kappa(\theta - v) \frac{\partial u}{\partial v} + \frac{1}{2} S_t^2 v_t \frac{\partial^2 u}{\partial s^2} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 u}{\partial v^2} + S_t \sigma v_t \rho \frac{\partial^2 u}{\partial s \partial v} = \gamma_t \left(\frac{\partial c}{\partial t} + S_t \mu \frac{\partial c}{\partial s} + k(\theta - v) \frac{\partial c}{\partial v} + \frac{1}{2} S_t^2 v_t \frac{\partial^2 c}{\partial s^2} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 c}{\partial v^2} + S_t \sigma v_t \rho \frac{\partial^2 c}{\partial s \partial v} \right) + (u - \xi_t S_t - \gamma_t c) r + \xi_t S_t \mu$$

By rearranging the terms and dividing the above equation by u'_v we see that each side of the equation is either dependent on c or on u, i.e.

$$\frac{1}{\frac{\partial u}{\partial v}} \left(\frac{\partial u}{\partial t} + S_t \mu \frac{\partial u}{\partial s} + \kappa(\theta - v) \frac{\partial u}{\partial v} + \frac{1}{2} S_t^2 v_t \frac{\partial^2 u}{\partial s^2} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 u}{\partial v^2} + S_t \sigma v_t \rho \frac{\partial^2 u}{\partial s \partial v} - ru - (\mu - r) \frac{\partial u}{\partial s} S_t \right) = \frac{1}{\frac{\partial c}{\partial v}} \left(\frac{\partial c}{\partial t} + S_t \mu \frac{\partial c}{\partial s} + k(\theta - v) \frac{\partial c}{\partial v} + \frac{1}{2} S_t^2 v_t \frac{\partial^2 c}{\partial s^2} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 c}{\partial v^2} + S_t \sigma v_t \rho \frac{\partial^2 c}{\partial s \partial v} - cr - (\mu - r) \frac{\partial c}{\partial s} S_t \right)$$

We can reproduce this result with any such option c. Given a set of these options we come to the conclusion that the left hand side of the equation does not depend on c but is a function of $S_t, v_t \; and \; t$ only. This function is denoted by $\lambda : \mathbb{R}_+^2 \times [0, T] \longrightarrow \mathbb{R}$ and we write

$$\frac{1}{\frac{\partial u}{\partial v}} \left(\frac{\partial u}{\partial t} + S_t r \frac{\partial u}{\partial s} + \kappa(\theta - v) \frac{\partial u}{\partial v} + \frac{1}{2} S_t^2 v_t \frac{\partial^2 u}{\partial s^2} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 u}{\partial v^2} + S_t \sigma v_t \rho \frac{\partial^2 u}{\partial s \partial v} - ru \right) = \lambda(S_t, v_t, t)$$

The PDE the function $u : \mathbb{R}_+^2 \times [0, T] \longrightarrow \mathbb{R}, u(s, v, t)$ has to obey is obtained by equating the drift factor to zero. Hence, we have,

$$\frac{\partial u}{\partial t} + sr \frac{\partial u}{\partial s} + \left(\kappa(\theta - v) - \lambda(s, v, t) \right) \frac{\partial u}{\partial v} + \frac{1}{2} v \left(S^2 \frac{\partial^2 u}{\partial s^2} + \sigma^2 \frac{\partial^2 u}{\partial v^2} \right) + s \sigma \rho \frac{\partial^2 u}{\partial s \partial v} - ru = 0 (8)$$

which is the PDE valuation formula for the Heston stochastic volatility model.

Now, let us examine an alternative method of using the application of Feynman-Kac theorem (Theorem 1) to obtain the PDE options price valuation formula for the Heston model as elaborated below. Given the Heston model [6],

$$dS_u = rS_u du + \sqrt{v_u} S_u dW_u, \quad (9)$$

$$dv_u = (\kappa(\vartheta - v_u) - \lambda v_u) d_u + \sigma \sqrt{v_u} dW'_u \quad (10)$$

where W, W' are now $P-$Brownian motions with instantaneous correlation ρ. Consider now the two-dimensional process X with coordinates $X^1 = S$ and $X^2 = v$. To construct W and W' as in (9) and (10), we choose independent $P-$ Brownian motions W^1, W^2 and set $W' = W^2$ and $W = \rho W^2 + \sqrt{1 - \rho^2} W^1$. Hence, by transformation equation (9) becomes

$$dX^1 = rx^1 du + \sqrt{x^2} x^1 d\left[\rho W^2 + \sqrt{1 - \rho^2} W^1\right]$$
$$dX^1 = rx^1 du + \sqrt{x^2} x^1 d(\rho w^2) + d\left[\sqrt{x^2} x^1 \sqrt{1 - \rho^2} W^1\right]$$
$$dX^1 = rx^1 du + \rho x^1 \sqrt{x^2} d(w^2) + \sqrt{1 - \rho^2} x^1 \sqrt{x^2} d(W^1)$$
$$dX^1 = rx^1 du + \sqrt{1 - \rho^2} x^1 \sqrt{x^2} d(W^1) + \rho x^1 \sqrt{x^2} d(W^2) (11)$$

and (10) becomes

$$dX^2 = \left(\kappa\left((\vartheta - x^2) - \lambda x^2 \right) du + \sigma \sqrt{x^2} d(W^2) \right) (12)$$

In matrix form, we have

$$\mu(t, x) = \begin{pmatrix} rx^1 \\ \kappa((\vartheta - x^2) - \lambda x^2) \end{pmatrix} \quad (13)$$

and

$$\sigma(t, x) = \begin{pmatrix} \sqrt{1 - \rho^2} x^1 \sqrt{x^2} & \rho x^1 \sqrt{x^2} \\ 0 & \sigma \sqrt{x^2} \end{pmatrix} \quad (14)$$

Hence, we have

$$d\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} rx^1 \\ \kappa((\vartheta - x^2) - \lambda x^2) \end{pmatrix} dt + \begin{pmatrix} \sqrt{1 - \rho^2} x^1 \sqrt{x^2} & \rho x^1 \sqrt{x^2} \\ 0 & \sigma \sqrt{x^2} \end{pmatrix} \begin{pmatrix} dW^1 \\ dW^2 \end{pmatrix}$$

To obtain the $\sigma \sigma^T$ in matrix form, we have

$$\sigma \sigma^T = \begin{pmatrix} \sqrt{1 - \rho^2} x^1 \sqrt{x^2} & \rho x^1 \sqrt{x^2} \\ 0 & \sigma \sqrt{x^2} \end{pmatrix} \begin{pmatrix} \sqrt{1 - \rho^2} x^1 \sqrt{x^2} & 0 \\ \rho x^1 \sqrt{x^2} & \sigma \sqrt{x^2} \end{pmatrix} = \begin{pmatrix} 1 - \rho^2 (x^1)^2 x^2 + \rho^2 (x^1)^2 x^2 & \sigma \rho x^1 x^2 \\ \sigma \rho x^1 x^2 & \sigma^2 x^2 \end{pmatrix}$$

Using the variables (s, v) instead of (x^1, x^2) and writing subscripts for partial derivatives, we have

$$\begin{pmatrix} 1 & \sigma\rho s v \\ \sigma\rho s v & \sigma^2 v \end{pmatrix} \quad (15)$$

Next is to apply the multi-dimensional version of the Feynman-Kac Theorem (Theorem 1), we have the generator function given as

$$A = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\sigma\sigma^T)_{ij}\frac{\partial^2}{\partial x_i \partial x_j} \quad (16)$$

Therefore substituting (15) into (16) we have

$$A = rs\frac{\partial}{\partial s} + (\kappa(\vartheta - v) - \lambda v)\frac{\partial}{\partial v} + \frac{1}{2}\frac{\partial^2}{\partial s^2} + \sigma\rho s v\frac{\partial^2}{\partial s \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2}$$

But the PDE for the multi-dimensional version of the Feynman-Kac Theorem is given by

$$\frac{\partial V}{\partial t} + AV(x_t, t) - r(x_t, t)V(x_t, t) = 0 \quad (17)$$

Therefore, the PDE in (17) for $V = v(x, v, t)$ becomes

$$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + (\kappa(\vartheta - v) - \lambda v)\frac{\partial V}{\partial v} + \frac{1}{2}\frac{\partial^2 V}{\partial s^2} + \sigma\rho s v\frac{\partial^2 V}{\partial s \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} - rV = 0 \quad (18)$$

Clearly, (18) is another form of the Heston PDEs option price valuation formula which is different from the one derived in (8). This implies that different PDEs can be derived from the Heston Model using different approaches. This in doubt makes the model incomplete.

IV. Conclusion

In this research, we examined the Heston stochastic volatility model and used two different approaches (Riskless portfolio method and application of Feynman-Kac theorem) to show that there are no unique PDE options price valuation formulas for the model. This is as a result of incompleteness in the Heston stochastic volatility model since there are two sources of uncertainty $(dW_t^{(1)}, dW_t^{(2)}$ or $dW_u, dW_u')$ in the model equation with only one risky asset S available for trade. Hence, option prices are only determined once a specific martingale measure (or, equivalently, a market price of risk) has been chosen. In particular, each ideal martingale measure also gives rise to an associated PDE and this means that many different options price valuation PDEs can be obtained from Heston stochastic volatility model.

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