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The More Organized Proof For Acyclic Coloring Of Graphs With $\Delta = 5$ with 8 Colors

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Abstract—An acyclic coloring of a graph G is a coloring of its vertices such that:(i) no two neighbors in G are assigned the same color and (ii) no bicolored cycle can exist in G. The acyclic chromatic number of G is the least number of colors necessary to acyclically color G. Recently it has been proved that any graph of maximum degree 5 has an acyclic chromatic number at most 8. In this paper we present another proof for this result.

Keywords—Acyclic Coloring,Vertex coloring.

I. Introduction

A Proper coloring of a graph G is a coloring of its vertices such that no two neighbors in G are assigned the same color. An acyclic coloring of a graph G is a proper coloring such that the graph induced by two colors α and β is a forest. The minimum number of colors necessary to acyclically color G is called the acyclic chromatic number of G and denoted by $a(G)$. For a family F of graphs, the acyclic chromatic number of F ,denoted by $a(F)$ is defined as follow : $a(F) = max\{a(G) for all G \in F\}$.

$a(F)$ has been determined for several families of graphs such as planar graphs [4], 1-planar graphs [2],planar graphs with large girth [3],outer planar graphs [11],product of trees [9], the graphs with maximum degree 3 [8], [10], the graphs with $\Delta = 4$ [5], Alon et al [1] showed that
(1):Asymptotically there exist graphs of maximum degree Δ with acyclic chromatic number in $\Omega(\frac{\Delta^{4/3}}{(\log \Delta)^{1/3}})$.
(2): Asymptotically it is possible to acyclically color any graph of maximum degree Δ with $O(\Delta^{4/3})$ colors.
(3):Trivial greedy polynomial time algorithm exists that acycliccally colors any graphs of maximum degree Δ with $\Delta^2 + 1$ colors. Fertin and Raspaud [7] proved that nine colors are enough for acyclic coloring a graph with $\Delta = 5$. Kishore Yadav etal [12] showed that any graph with $\Delta = 5$ can be acyclically colored with 8 colors. In this paper we achieve the above result by another approach which is easier than what has been presented before.

II. Preliminaries

In the following we only consider graphs of maximum degree $\Delta = 5$. Let $N(u)$ be the neighbors set of vertex u and $c(u)$ denoted the color of u. The set of colors are assigned to vertices in $N(u)$, denoted by $SCN(u)$. The color $\alpha \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ is regarded as a free color for u when:
1) If no color assigned to u , then $\alpha \notin SCN(u)$.
2) If u is colored, then $\alpha \notin SCN(u) \cup \{c(u)\}$.
The set of free colors for u is named $f(u)$. The number of different colors in the neighbors of u is denoted by $dcn(u)$ and we define the *color list* as follow:
$L_u = (n_1, n_2, ..., n_{dcn(u)})$ (where $n_1 \geq n_2 \geq ... \geq n_{dcn(u)}$) where an n_i represents for a color α in $SCN(u)$ and it is the number of times ,α is used among the colored neighbors of u.
The color α is free for u if $\alpha \in f(u)$,and α is a valid color for u if $1 \leq \alpha \leq 8$ and assigning it to u , still results in an acyclicly coloring. Let α ,β are two distinct colors. A critical cycle denoted by $C_u(\alpha, \beta)$,is a cycle such as C ,involving u in which all vertices in C are alternatively colored by α and β moreover $c(u) \notin \{\alpha, \beta\}$.We can't assign the color α and β to u (since they are not valid colors). A vertex u is called single if all its neighbors receive distinct colors.

III. Acyclic Coloring of Graphs of Maximum Degree 5

In this section we show that 8 color is enough for acyclically coloring of any graph with $\Delta = 5$. At first we prove 4 lemmas and in Theorem 1 we will achieve the goal.

Lemma *3.1:* If u is an uncolored vertex and $L_u = (1, 1, 1, 1, 1)$ then we can color u with a valid color

Proof: Since $L_u = (1, 1, 1, 1, 1)$ so $f(u) \geq 3$ and we can assign one of the color in $f(u)$ such that $1 \leq c(u) \leq 8$. ■

Lemma *3.2:* If u is an uncolored vertex and $L_u = (2, 1, 1, 1)$ and $9 \notin SCN(u)$, then we can find a valid color for u.

Proof: Let $N(u) = \{v, w, x, y, z\}$ and $c(v) = c(w) = 1, c(x) = 2, c(y) = 3$ and $c(z) = 4$,therefore $f(u) = \{5, 6, 7, 8\}$. If we can't choose a valid color from $f(u)$, then we must have$C_u(1, 5), C_u(1, 6), C_u(1, 7)$ and $C_u(1, 8)$. This means that v and w are single vertices. By assigning color 1 to u and eliminating the colors of v and w , we will have $L_v = (1, 1, 1, 1, 1), L_w = (1, 1, 1, 1, 1)$ and by **Lemma 3.1** we can color v and w with a valid color. ■

Lemma *3.3:* if u is a colored vertex with a valid color and $L_u = (2, 1, 1, 1), 9 \notin SCN(u)$. Then we can recolor u with a valid color.

Proof: Let $N(u) = \{v, w, x, y, z\}$ and $c(v) = c(w) = 1, c(x) = 2, c(y) = 3, c(z) = 4$ and $c(u) = 5$,therefore $f(u) = \{6, 7, 8\}$. If we can't find a valid color from $f(u)$ to change the color of u ,then we must have $C_u(1, 6), C_u(1, 7)$ and $C_u(1, 8)$. Now by eliminating the colors of v and w and assigning color 1 to u we have $SCN(v) = \{1, 6, 7, 8, \alpha\}$ and $SCN(w) = \{1, 6, 7, 8, \beta\}$. Let us detail the possible cases for α:

3.3.1 If $\alpha \notin \{6, 7, 8\}$ then $L_v = (1, 1, 1, 1, 1)$ and with **Lemma** 3.1 we can color v with a valid color.

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3.3.2 If $\alpha \in \{6,7,8\}$ then $L_v = (2,1,1,1)$ and by **Lemma** 3.2 we can color vertex v with a valid color.

We have similar cases for β and we can color w with a valid color. ■

Lemma *3.4:* if u is a colored with a valid color with $L_v = (2,1,1,1)$ and one of its neighbors is colored with color 9, then we can recolor vertex u with a valid color.

Proof: Let $N(u) = \{v,w,x,y,z\}$ and $c(v) = c(w) = 1, c(x) = 2, c(y) = 3, c(z) = 9$ and $c(u) = 4$ therefore $f(u) = \{5,6,7,8\}$. If we can't find a valid color from $f(u)$ to vertex u then we must have $C_u(1,5), C_u(1,6), C_u(1,7)$ and $C_u(1,8)$. In this case by eliminating the colors of v and w and assigning color 1 to vertex u , we will have $L_v = L_w = (1,1,1,1,1)$ and by **Lemma** 3.1 we can find valid colors for v and w. ■

To have acyclic coloring of a graph G with 8 colors, first we add $5 - d(u)$ new vertices (vertex) for every vertex $u \in V(G)$ and insert edges between u and these new vertices. By the above operation we get a new graph G' with the following properties:

$$d(u) = \begin{cases} 5 & \textit{if} \quad u \in V(G) \\ 1 & \textit{otherwise} \end{cases} \tag{1}$$

If we use the algorithm Fertin and Raspaud [7] for graph G', then G' can be colored acyclically with 9 colors. Then we try to recolor every vertex of G' that its color is 9 with a valid color. Finally by removing all vertices of degree 1 we achieve the goal.

Theorem *3.1:* Let G' is a graph with maximum degree 5 and acyclically colored with 9 colors and let u be a vertex such that $d(u) = 5$ and $c(u) = 9$ then we can find a valid color to u.

Proof: Let us detail possible cases:

Case 3.1.1: $L_u = (1,1,1,1,1)$
In this case by eliminating the color of u and using **Lemma** 3.1, we can find a valid color for u.
Case 3.1.2:$L_u = (2,1,1,1)$
In this case by eliminating the color of u and using **Lemma** 3.2, we can find a valid color for vertex u.
Case 3.1.3:$L_u = (3,1,1)$
Let $N(u) = \{v,w,x,y,z\}$ and $c(v) = c(w) = c(x) = 1, c(y) = 2, c(z) = 3$ and $c(u) = 9$. These assumptions imply that $f(u) = \{4,5,6,7,8\}$. If we can't choose a color from $f(u)$ to recolor u ,then we must have $C_u(1,4), C_u(1,5), C_u(1,6), C_u(1,7)$ and $C_u(1,8)$. The above discussion shows that $N(v) \cup N(w) \cup N(x) - \{u\}$ contains two vertices of color 4,two vertices of color 5,two vertices of color 6,two vertices of color 7 and two vertices of color 8. So we have at least 10 vertices in $N(v) \cup N(w) \cup N(x) - \{u\}$ such that they are of the same color pairwisly. Now by the pigeon role principle ,v, w or x has in its neighbors, 4 of that 10 vertices. This means that v, w or x is a single vertex. Without loss of generality we suppose that v is single. By eliminating the color v and using **Lemma** 3.1, we can find a valid color for vertex v such that $L_u = (2,1,1,1)$ and this case have been treated in *case 3.1.2*.

Case 3.1.4:$L_u = (4,1)$
Let $N(u) = \{v,w,x,y,z\}, c(v) = c(w) = c(x) = c(y) = 1, c(z) = 2$ and $c(u) = 9$ then $f(u) = \{3,4,5,6,7,8\}$. If we can't find a valid color from $f(u)$ for u , then we must have $C_u(1,3), C_u(1,4), C_u(1,5), C_u(1,6), C_u(1,7)$ and $C_u(1,8)$. If one of the vertices v, w, x or y is single, then by eliminating the color of this vertex and applying **Lemma** 3.1 for it we can find a valid color such that $L_u = (3,1,1)$ and this case was handled above. Now suppose that none of the vertices v, w, x and y are single. Since we have 6 critical cycles involving u therefore there exist in $N(v) \cup N(w) \cup N(x) \cup N(y) - \{u\}$ 12 vertices such that their colors are from the set $\{3,4,5,6,7,8\}$ and they are of the same color pairwisly. Whereas none of the vertices $\{v,w,x,y\}$ are single ,so the colors which assigned to the neighbors of one of them (consider v) are $\{3,4,5,\alpha,9\}$. We have two cases for α. Either $\alpha \in \{3,4,5\}$ or $\alpha = 9$ (because v isn't a single vertex). If $\alpha \in \{3,4,5\}$ (consider $\alpha = 3$) then by using **Lemma** 3.4 for v ,we have $c(v) \in \{2,6,7,8,\alpha\}$. We have two cases for $c(v)$. If $c(v) \in \{6,7,8,\alpha\}$ then $L_u = (3,1,1)$ and this case was treated. Now suppose $c(v) = 2$ and 6,7 and 8 aren't valid colors for v. suppose that two vertices s and t are neighbors of v such that $c(s) = c(t) = 3$. By this assumption $\{6,7,8\} \subset SCN(t)$ and $\{6,7,8\} \subset SCN(s)$(because we can't choose 6 and 7 and 8 for v). Now we recolor vertex v with color 1,since we have $C_u(1,3)$,then in neighbors of vertex s or t (consider t) we have two vertices such that their colors are 1(one of them is v). Since $3 \in f(t)$ and $3 \notin SCN(v)$ so we can recolor vertex t with color 3 and this yields $L_v = (1,1,1,1,1)$,therefore we can recolor v such that L_u becomes $(3,1,1)$. If $\alpha = 9$ then we can color vertex v with color 6 and we have $L_u = (3,1,1)$ and this case was handled above.
Case 3.1.5: $L_u = (5)$
Let $N(u) = \{v,w,x,y,z\}$ and $c(v) = c(w) = c(x) = c(y) = c(z) = 1$ and $c(u) = 9$ then $f(u) = \{2,3,4,5,6,7,8\}$. If we can't find a valid color from $f(u)$ for u , then we must have $C_u(1,2), C_u(1,3), C_u(1,4), C_u(1,5)$, $C_u(1,6), C_u(1,7)$ and $C_u(1,8)$. This means that we have at least 14 vertices in $N(N(u)) - \{u\}$ such that their colors are from the set $\{2,3,4,5,6,7,8\}$ and they are of the same color pairwisly. We have two cases:

Case a: One of the neighbors of u (consider v) contains 4 vertices from that 14 vertices as its neighbors.
In this case vertex v is a single vertex and we can recolor it such that $L_u = (4,1)$.
Case b: None of the vertices in $N(u)$ contains 4 vertices from that 14 vertices as its

neighbors.

In this case there exist a vertex in $N(u)$ such that it contains 3 vertices from that 14 vertices as its neighbors. Suppose that this vertex is v. Without loss of generality we can assume $SCN(v) = \{2, 3, 4, \alpha\}$. If $\alpha \in \{2, 3, 4\}$ or $\alpha = 9$ then we can recolor vertex v such that $L_u = (4, 1)$.

Case 3.1.6: $L_u = (2, 2, 1)$

Let $N(u) = \{v, w, x, y, z\}$ and $c(v) = c(w) = 1, c(x) = c(y) = 2, c(z) = 3$ and , then $f(u) = \{4, 5, 6, 7, 8\}$. If there is no valid color in $f(u)$ for u then we have some critical cycles. All critical cycles are of the two following types $C_u(1, \alpha), 4 \leq \alpha \leq 8$ or $C_u(2, \beta), 4 \leq \beta \leq 8$. We can consider two following cases. Other possibilities can be considered in the similar way.

Case a: We have $C_u(1, 4)$, $C_u(1, 5)$, $C_u(1, 6)$ and $C_u(1, 7)$

In this case the vertices v and w are single and we can find a valid color for vertex v which is neither 2 nor 3. After changing the color of v with a new color we have $L_u = (2, 1, 1, 1)$ and this case was handled above.

Case b: We have $C_u(1, 4)$, $C_u(1, 5)$, $C_u(1, 6)$ and $C_u(2, 7), C_u(2, 8)$.

By this assumptions we have $SCN(v) = \{4, 5, 6, 9, \alpha\}$. We have some cases for α. If $\alpha \notin \{4, 5, 6, 9\}$, then the vertex v is a single vertex and we can recolor it such that L_u becomes $(2, 1, 1, 1)$. If $\alpha = 9$, we can recolor v with color 7, then $L_u = (2, 1, 1, 1)$. Let $\alpha \in \{4, 5, 6\}$. Without loss of generality we can assume $\alpha = 4$, then $L_v = (2, 1, 1, 1)$, by using **Lemma** 3.4 for vertex v we have $c(v) \in \{2, 3, 7, 8, \alpha = 4\}$. We have three possible cases such as $c(v) \in \{7, 8, \alpha = 4\}$ or $c(v) = 2$ or $c(v) = 3$. If $c(v) \in \{7, 8, \alpha = 4\}$ then $L_u = (2, 1, 1, 1)$ and this case was treated above. If $c(v) = 2$ then $L_u(3, 1, 1)$ and this case was handled above.

Let $c(v) = 3$ and 2,7 and 8 aren't valid colors for v. Suppose that two vertices s and t are neighbors of v such that $c(s) = c(t) = 4$. By this assumption $\{2, 7, 8\} \subset SCN(t)$ and $\{2, 7, 8\} \subset SCN(s)$ (because we can't choose 2 and 7 and 8 for v). Now we recolor vertex v with color 1, since we have $C_u(1, 4)$, then in neighbors of vertex s or t (consider t) we have two vertices such that their colors are 1(one of them is v). Since $3 \in f(t)$ and $3 \notin SCN(v)$ so we can recolor vertex t with color 3 and this yields $L_v = (1, 1, 1, 1, 1)$ therefore we can recolor v such that L_u becomes $(2, 1, 1, 1)$.

Case 3.1.7: $L_u = (3, 2)$

Let $N(u) = \{v, w, x, y, z\}, c(v) = c(w) = 1$ and $c(x) = c(y) = c(z) = 2$, then $f(u) = \{3, 4, 5, 6, 7, 8\}$. If we can't find a valid color for u from $f(u)$ then we have 6 critical cycles containing u. Each cycle needs two vertices from $N(N(u)) - \{u\}$ of the same color. Therefore there exist at least 12 vertices in $N(N(u)) - \{u\}$ such that their colors are from $\{3, 4, 5, 6, 7, 8\}$ an they are of the same color pairwisly. We detail two possible cases:

Case a: There exist a vertex in$N(u)$ such that it contains 4 vertices from that 12 vertices as its neighbors.

This vertex is a single vertex and we can recolor it. If this vertex is v or w then L_u becomes $(3, 1, 1)$ and if this vertex is x or y or z then we have $L_u = (2, 2, 1)$.

Case b: It doesn't exist a vertex in $N(u)$ such that it contains 4 vertices from that 12 vertices as its neighbors.

In this case there is a vertex such that it contains 3 vertices from 12 vertices as its neighbors (consider v or x). First suppose that this vertex is v. We can assume $SCN(v) = \{3, 4, 5, 9, \alpha\}$ (other cases for $SCN(v)$ can be handle by similar way). If $\alpha \notin \{3, 4, 5, 9\}$ then v is single and we can recolor it such that $L_u = (3, 1, 1)$. If $\alpha \in \{3, 4, 5\}$ then $L_v = (2, 1, 1, 1)$ and by applying **Lemma** 3.4 for v, to recolor it, we will have $L_u = (3, 1, 1)$ (if $c(v) \neq 2$) or $L_u = (4, 1)$ (if $c(v) = 2$). If $\alpha = 9$ then we can assign color 6 to v and $L_u = (3, 1, 1)$.

Now suppose that x contains 3 vertices from those 12 vertices as its neighbors. By this assumption, we have $SCN(x) = \{3, 4, 5, 9, \alpha\}$, If $\alpha \notin \{3, 4, 5, 9\}$ then x is a single vertex an after recolor it, we will have $L_u = (2, 2, 1)$. If $\alpha \in \{3, 4, 5\}$ (let $\alpha = 3$) then $L_x = (2, 1, 1, 1)$ and by using **Lemma** 3.4 to recolor x, we have $c(x) \neq 1$ or $c(x) = 1$. If $c(x) \neq 1$ then $L_u = (2, 2, 1)$. Now suppose that $c(x) = 1$ and none of the colors in $f(x)$, isn't valid color for x. Let s, t are two neighbors of x such that their colors are $\alpha = 3$ (consider t). Since $c(x) = 1$ and we had $C_u(2, 3)$, therefore $SCN(t) = \{2, 6, 7, 8, 1\}$ (6, 7, 8 are in $f(x)$ but not valid, $c(x) = 1$). In this case we recolor x with color 2 and this action implies that $L_t = (2, 1, 1, 1)$. Because $1 \in f(t)$ and $1 \notin SCN(x)$, we can assign color 1 to t and obtain $L_x = (1, 1, 1, 1, 1)$. Finally we can recolor x such that $L_u = (2, 2, 1)$.

■

IV. Conclusion

In this paper, we have shown that any graph of maximum degree 5 can be acyclically colored with 8 color. As far as lower bounds are concerned. We know that $a(K_6) = 6$ then for F family of graphs with maximum degree 5 we have $a(F) \geq 6$. Closing the gap between those two bounds is a challenging open problem. In particular, we strongly suspect that the upper bound of 8 is not tight.

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