International Journal of Information, Control and Computer Sciences
ISSN: 2517-9942
Vol:2, No:7, 2008

The Game of Synchronized Triomineering and Synchronized Tridomineering

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Abstract**—**In synchronized games players make their moves simultaneously rather than alternately. Synchronized Triomineering and Synchronized Tridomineering are respectively the synchronized versions of Triomineering and Tridomineering, two variants of a classic two-player combinatorial game called Domineering. Experimental results for small $m \times n$ boards (with $m + n \leq 12$ for Synchronized Triomineering and $m + n \leq 10$ for Synchronized Tridomineering) and some theoretical results for general $k \times n$ boards (with $k = 3, 4, 5$ for Synchronized Triomineering and $k = 3$ for Synchronized Tridomineering) are presented. Future research is indicated.

Keywords**—**Combinatorial games, Synchronized games, Triomineering, Tridomineering.

I. Introduction

The game of Triomineering and Tridomineering are two-player games with perfect information, proposed in 2004 by Blanco and Fraenkel [1]. In Triomineering two players, usually denoted by Vertical and Horizontal, take turns in placing "straight" triominoes (3×1 tile) on a checkerboard. Vertical is only allowed to place its triominoes vertically and Horizontal is only allowed to place its triominoes horizontally on the board. Triominoes are not allowed to overlap and the first player that cannot find a place for one of its triominoes loses. After a time the remaining space may separate into several disconnected regions, and each player must choose into which region to place a triomino. In Tridomineering Vertical and Horizontal alternate in tiling with either a domino (2×1 tile) or a straight triomino.

Blanco and Fraenkel [1] calculated Triomineering and Tridomineering values for boards up to 6 squares and small rectangular boards.

II. Synchronized games

For the sake of self containment, we recall the previous results concerning synchronized games. Initially, the concept of synchronism was introduced in the games of Cutcake [2], Maundy Cake [3], and Domineering [4] in order to study combinatorial games where players make their moves simultaneously.

As a result, in the synchronized versions of these games there exist no zero-games (fuzzy-games), i.e., games where the winner depends exclusively on the player that makes the second (first) move. Moreover, there exists the possibility of a draw, which is impossible in a typical combinatorial game. In this work, we continue to investigate synchronized combinatorial games by focusing our attention on Triomineering and Tridomineering.

In the game of Synchronized Triomineering and Synchronized Tridomineering, a general instance and the legal moves for Vertical and Horizontal are defined exactly in the same way as defined for the game of Triomineering. There is only one difference: Vertical and Horizontal make their legal moves simultaneously, therefore, triominoes and/or dominoes are allowed to overlap if they have a 1×1 tile in common. We note that 1×1 overlap is only possible within a simultaneous move.

At the end, if both players cannot make a move, then the game ends in a draw, else if only one player can still make a move, then he/she is the winner.

In Synchronized Triomineering and Synchronized Tridomineering, for each player there exist three possible outcomes:

- The player has a *winning strategy* (ws) independently of the opponent's strategy, or
- The player has a *drawing strategy* (ds), i.e., he/she can always get a draw in the worst case, or
- The player has a *losing strategy* (ls), i.e., he/she does not have a strategy for winning or for drawing.

TABLE I
The possible outcomes in Synchronized Triomineering and Synchronized Tridomineering

	Horizontal ls	Horizontal ds	Horizontal ws
Vertical ls	$G = VHD$	$G = HD$	$G = H$
Vertical ds	$G = VD$	$G = D$	-
Vertical ws	$G = V$	-	-

Table I shows all the possible cases. It is clear that if one player has a winning strategy, then the other player has neither a winning strategy nor a drawing strategy. Therefore, the cases $ws - ws$, $ws - ds$, and $ds - ws$ never happen. As a consequence, if G is an instance of Synchronized Triomineering (Synchronized Tridomineering), then we have 6 possible legal cases:

- $G = D$ if both players have a drawing strategy, and the game will always end in a draw under perfect play, or
- $G = V$ if Vertical has a winning strategy, or
- $G = H$ if Horizontal has a winning strategy, or
- $G = VD$ if Vertical can always get a draw in the worst case, but he/she could be able to win if Horizontal makes a wrong move, or
- $G = HD$ if Horizontal can always get a draw in the worst case, but he/she could be able to win if Vertical

Manuscript received November 11, 2008.
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ISSN: 2517-9942
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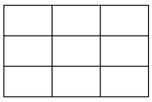
makes a wrong move, or

- $G = VHD$ if both players have a losing strategy and the outcome is totally unpredictable.

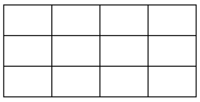
III. Examples of Synchronized Triomineering

The game



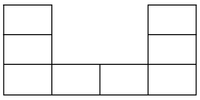
always ends in a draw, therefore $G = D$.

In the game

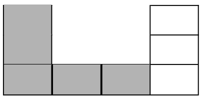
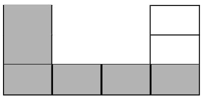


Vertical has a winning strategy moving in the second (or in the third) column, therefore $G = V$.

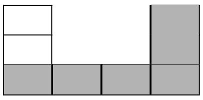
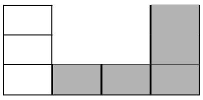
In the game



if Vertical moves in the first column we have two possibilities

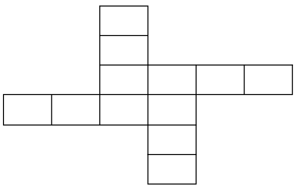
 or 

therefore, either Vertical wins or the game ends in a draw. Symmetrically, if Vertical moves in the third column we have two possibilities

 or 

therefore, either Vertical wins or the game ends in a draw. It follows $G = VD$.

In the game



each player has 4 possible moves. For every move of Vertical, Horizontal can win or draw (and sometimes lose); likewise, for every move by Horizontal, Vertical can win or draw (and sometimes lose). As a result it follows that $G = VHD$.

IV. Results for Synchronized Triomineering

Table II shows the results obtained using an exhaustive search algorithm for $n \times m$ boards with $n + m \leq 12$.

Theorem 1: Let $G = [n, 4]$ be a rectangle of Synchronized Triomineering with $n \geq 9$. Then Vertical has a winning strategy.

Proof: In the beginning, Vertical will always move into the second column of the board, i.e., $(k, b), (k+1, b), (k+2, b)$

TABLE II
OUTCOMES FOR RECTANGLES IN SYNCHRONIZED TRIOMINEERING

	1	2	3	4	5	6	7	8	9	10	11
1	D	D	H	H	H	H	H	H	H	H	H
2	D	D	H	H	H	H	H	H	H	H	
3	V	V	D	V	V	D	V	V	D		
4	V	V	H	D	V	H	H	HD			
5	V	V	H	H	D	H	H				
6	V	V	D	V	V	D					
7	V	V	H	V	V						
8	V	V	H	VD							
9	V	V	D								
10	V	V									
11	V										

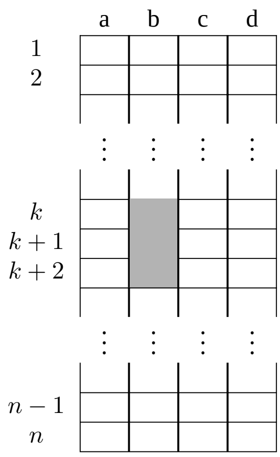
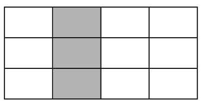


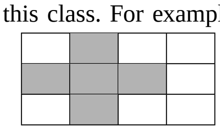
Fig. 1. Synchronized Triomineering played on $n \times 4$ rectangular board.

where $k \equiv 1 \pmod 3$, as shown in Fig. 1. When Vertical cannot move anymore in the second column, let us imagine that we divide the main rectangle into 3×4 sub-rectangles starting from the top of the board (by using horizontal cuts). Of course, if $n \not\equiv 0 \pmod 3$, then the last sub-rectangle will be of size either 1×3 or 2×3, and Horizontal will be able to make respectively either one more move or two more moves. We can classify all these sub-rectangles into 5 different classes.

- Class A. Vertical is able to make three more moves in each sub-rectangle of this class.



- Class B. Vertical is able to make one more move in each sub-rectangle of this class. For example



- Class C. Horizontal is able to make two more moves in each sub-rectangle and Vertical is able to make at least $\lceil |C|/2 \rceil$ moves where $|C|$ is the number of sub-rectangles belonging to this class. The last statement is true under the assumption that Vertical moves into the sub-rectangles of this class as long as they exist before to move into the sub-rectangles of the other classes. For example

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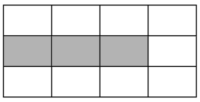
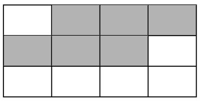
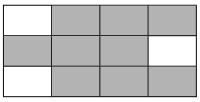


- Class D. Horizontal is able to make one more move in each sub-rectangle of this class. For example



- Class E. Neither Vertical nor Horizontal are able to make a move in the sub-rectangles of this class. For example



We show that when Vertical cannot move anymore in the central column, he/she can make a greater number of moves than Horizontal, i.e., $moves(H) < moves(V)$. We denote with $|A|$ the number of sub-rectangles in the A class, with $|B|$ the number of sub-rectangles in the B class, and so on. Both Vertical and Horizontal have placed the same number of triominoes, therefore

$$|A| = |C| + 2|D| + 3|E|$$

It follows that

$$\begin{aligned} moves(H) &\leq 2|C| + |D| + 2 \\ &= 2|A| - 3|D| - 6|E| + 2 \\ &< 3|A| + |B| + \lceil |C|/2 \rceil \\ &\leq moves(V) \end{aligned}$$

The condition $2|A| - 3|D| - 6|E| + 2 < 3|A| + |B| + \lceil |C|/2 \rceil$ is always true, as shown below.

- If $|A| = 0$ then $|C| = 0$, $|D| = 0$, $|E| = 0$, and $|B| \geq 3$ because by hypothesis $n \geq 9$,
- If $|A| = 1$ then $|C| = 1$, $|D| = 0$, $|E| = 0$, and $|B| \geq 1$ because by hypothesis $n \geq 9$,
- If $|A| = 2$ then either $|C| = 2$, $|D| = 0$, and $|E| = 0$ or $|C| = 0$, $|D| = 1$, $|E| = 0$,
- If $|A| \geq 3$ then $2|A| + 2 < 3|A|$.

■

Theorem 2: Let $G = [n, 5]$ be a rectangle of Synchronized Domineering with $n \geq 6$. Then Vertical has a winning strategy.

Proof: In the beginning, Vertical will always move into the central column of the board, i.e., (k, c), $(k+1, c)$, and $(k+2, c)$ where $k \equiv 1 \pmod 3$, as shown in Fig. 2. When Vertical cannot move anymore into the central column, let us imagine that we divide the main rectangle into 3×5 sub-rectangles starting from the top of the board (by using horizontal cuts). Of course, if $n \not\equiv 0 \pmod 3$, then the last sub-rectangle will be of size either 1×5 or 2×5, and Horizontal will be able to make respectively either one more move or two more moves.

We can classify all these sub-rectangles into 5 different classes according to:

- The number of vertical triominoes already placed in the sub-rectangle (vt),

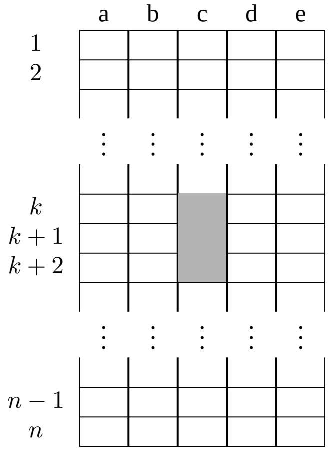
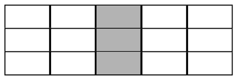
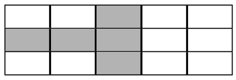
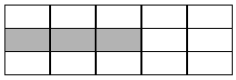
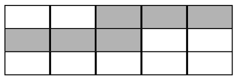
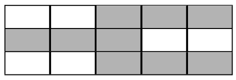


Fig. 2. Synchronized Triomineering played on $n \times 5$ rectangular board.

TABLE III
THE 5 CLASSES FOR 3×5 SUB-RECTANGLES

Class	vt	ht	vm	hm	Example
A	1	0	$4\|A\|$	0	
B	1	1	$2\|B\|$	0	
C	0	1	$\|C\|$	$2\|C\|$	
D	0	2	0	$\|D\|$	
E	0	3	0	0	

- The number of horizontal triominoes already placed in the sub-rectangle (ht),
- The number of moves that Vertical is able to make in the worst case, in all the sub-rectangles of that class (vm),
- The number of moves that Horizontal is able to make in the best case, in all the sub-rectangles of that class (hm),

as shown in Table III. We denote with $|A|$ the number of sub-rectangles in the A class, with $|B|$ the number of sub-rectangles in the B class, and so on. In the C class, Vertical is able to make $|C|$ moves under the assumption that he/she moves into the sub-rectangles of this class as long as they exist before to move into the sub-rectangles of the other classes.

When Vertical cannot move anymore into the central column, both Vertical and Horizontal have placed the same number of triominoes, therefore

$$|A| = |C| + 2|D| + 3|E| \tag{1}$$

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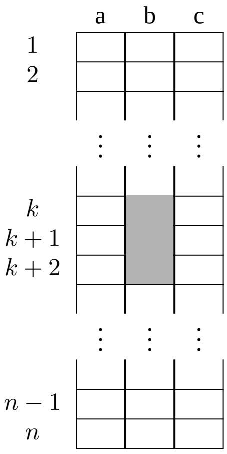


Fig. 3. Synchronized Triomineering played on $n \times 3$ rectangular board.

Let us prove by contradiction that Vertical can make a larger number of moves than Horizontal. Assume therefore $moves(V) \leq moves(H)$ using the data in Table III

$$4|A| + 2|B| + |C| \quad \leq \quad 2|C| + |D| + 2$$

and applying Equation 2

$$|C| + 2|D| + 3|E| + 3|A| + 2|B| + |C| \quad \leq \quad 2|C| + |D| + 2$$

therefore

$$3|A| + 2|B| + |D| + 3|E| \quad \leq \quad 2$$

which is false because

$$|A| + |B| + |C| + |D| + |E| = \lfloor n/3 \rfloor$$

and by hypothesis $n \geq 6$. We note that if $|A| = 0$ then $|C| = 0$, $|D| = 0$, $|E| = 0$, and $|B| \geq 2$. So $moves(V) \leq moves(H)$ does not hold and consequently $moves(H) < moves(V)$. ∎

By symmetry the following two theorems hold.

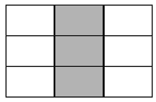
Theorem 3: Let $G = [4, n]$ be a rectangle of Synchronized Triomineering with $n \geq 9$. Then Horizontal has a winning strategy.

Theorem 4: Let $G = [5, n]$ be a rectangle of Synchronized Triomineering with $n \geq 6$. Then Horizontal has a winning strategy.

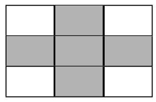
Theorem 5: Let $G = [n, 3]$ be a rectangle of Synchronized Triomineering. If $n \equiv 0 \pmod 3$, then Vertical has a drawing strategy.

Proof: In the beginning, Vertical will always move into the central column of the board, i.e., (k, b), $(k+1, b)$, and $(k+2, b)$ where $k \equiv 1 \pmod 3$, as shown in Fig. 3. When Vertical cannot move anymore into the central column, let us imagine that we divide the main rectangle into 3×3 sub-rectangles starting from the top of the board (by using horizontal cuts). We can classify all these sub-rectangles into 5 different classes.

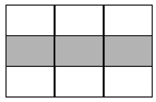
- Class A. Vertical is able to make two more moves in each sub-rectangle of this class.



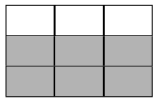
- Class B. Neither Vertical nor Horizontal are able to make another move in the sub-rectangles of this class. For example



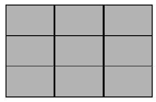
- Class C. Horizontal is able to make two more moves in each sub-rectangle of this class. For example



- Class D. Horizontal is able to make one more move in each sub-rectangle of this class. For example



- Class E. In each sub-rectangle of this class Horizontal has already made three moves.



We show that when Vertical cannot move anymore into the central column, he/she can make a number of moves greater or equal to Horizontal, i.e., $moves(H) \leq moves(V)$. We denote with $|A|$ the number of sub-rectangles in the class A, with $|B|$ the number of sub-rectangles in the class B, and so on. We observe that $|A| = |C| + 2|D| + 3|E|$ because both Vertical and Horizontal have placed the same number of triominoes.

$$\begin{aligned} moves(H) &= 2|C| + |D| \\ &= 2|A| - 3|D| - 6|E| \\ &\leq 2|A| \\ &= moves(V) \end{aligned}$$

∎

By symmetry the following theorem holds.

Theorem 6: Let $G = [3, n]$ be a rectangle of Synchronized Triomineering. If $n \equiv 0 \pmod 3$, then Horizontal has a drawing strategy.

V. Results for Synchronized Tridomineering

Table IV shows the results obtained using an exhaustive search algorithm for $n \times m$ boards with $n + m \leq 10$.

Theorem 7: Let $G = [n, 3]$ be a rectangle of Synchronized Tridomineering with $n \geq 8$. Then, Vertical has a winning strategy.

Proof: Let us imagine that we divide the main rectangle into 3×3 sub-rectangles starting from the top of the board (by using horizontal cuts). If $n \equiv 1 \pmod 3$, then the last 2 sub-rectangles on the bottom of the board will be 2×3. If $n \equiv 2 \pmod 3$, then the last sub-rectangle on the bottom of the board will be 2×3. In the beginning, Vertical will place

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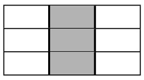
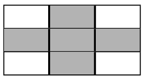
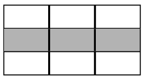
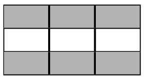
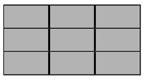
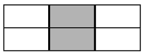
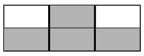
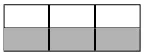
TABLE IV
OUTCOMES FOR RECTANGLES IN SYNCHRONIZED TRIDOMINEERING

	1	2	3	4	5	6	7	8	9
1	D	H	H	H	H	H	H	H	H
2	V	D	D	D	D	HD	D	H	
3	V	D	D	HD	HD	H	H		
4	V	D	VD	D	D	HD			
5	V	D	VD	D	D				
6	V	VD	V	VD					
7	V	D	V						
8	V	V							
9	V								

TABLE V
THE 9 CLASSES FOR 3×3 AND 2×3 SUB-RECTANGLES

Class	vt	ht	vm	hm	Example
A	1	0	$2\|A\|$	0	
B	1	1	0	0	
C	0	1	0	$2\|C\|$	
D	0	2	0	$\|D\|$	
E	0	3	0	0	
F	1	0	$2\|F\|$	0	
G	1	1	0	0	
H	0	1	0	$\|H\|$	
I	0	2	0	0	

triominoes into the central column of 3×3 sub-rectangles and dominoes into the central column of 2×3 sub-rectangles.

As shown in Table V, we can classify all these sub-rectangles into 9 different classes according to:

- The number of vertical triominoes (or dominoes) already placed in the sub-rectangle (vt),
- The number of horizontal triominoes (or dominoes) already placed in the sub-rectangle (ht).

For each class, vm and hm represent respectively the number of moves that Vertical is able to make in the worst case and the number of moves that Horizontal is able to make in the best case in all the sub-rectangles of that class.

We denote with $|A|$ the number of sub-rectangles in the A class, with $|B|$ the number of sub-rectangles in the B class, and so on. When Vertical cannot move anymore into the central column, both Vertical and Horizontal have placed the same number of triominoes and dominoes, therefore

$$|A| + |F| = |C| + 2|D| + 3|E| + |H| + 2|I| \qquad (2)$$

Moreover, Vertical can make a larger number of moves than Horizontal as shown below.

$$\begin{aligned} moves(V) &= 2|A| + 2|F| \\ &= 2|C| + 4|D| + 6|E| + 2|H| + 4|I| \\ &\geq 2|C| + |D| + |H| \\ &= moves(H) \end{aligned}$$

If $|D| = |E| = |H| = |I| = 0$ then $moves(V)$ should be equal to $moves(H)$. Actually, the number of effective moves that Vertical is able to make in the first and second 3×3 sub-rectangles on the top of the board considered as a single 6×3 rectangle, is greater than the previous estimation (vm) as shown in Table VI. The last statement is true under the assumption that Vertical moves into the first and second 3×3 sub-rectangles on the top of the board before to move into the other sub-rectangles. We note that the condition $n \geq 8$ is necessary to have at least a couple of 3×3 sub-rectangles on the top of the board. ■

TABLE VI
NUMBER OF EFFECTIVE MOVES FOR VERTICAL

First	Second	vm	Effective moves
A	A	4	6
A	B	2	4
A	C	2	3
B	A	2	4
B	B	0	2
B	C	0	1

By symmetry the following theorem holds.

Theorem 8: Let $G = [3, n]$ be a rectangle of Synchronized Tridomineering with $n \geq 8$. Then, Horizontal has a winning strategy.

VI. CONCLUSIONS AND FUTURE WORK

We observe that to play with triominoes (or triominoes and dominoes) increases the possibility to reach interesting game positions, i.e., games where the final outcome is not easily predictable. As a consequence, the outcomes VD and HD appear in Table II and Table IV. This seems to be the main difference between the games analyzed in this paper and the game of Synchronized Domineering [4] where the outcome VD and HD never appear in the outcome table for small boards and theoretical results seem indicate that these outcomes never appear for any $m \times n$ board.

In the field of combinatorial game theory, the concept of synchronism has been introduced recently and further efforts are necessary for a deep understanding of this topic:

- To complete the analysis of Synchronized Domineering and its variants,
- To investigate the synchronized version of other combinatorial games,
- To establish a general mathematical theory for the classification and analysis of synchronized combinatorial games.

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