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The Fluid Limit of the Critical Processor Sharing Tandem Queue

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Abstract—A sequence of finite tandem queue is considered for this study. Each one has a single server, which operates under the egalitarian processor sharing discipline. External customers arrive at each queue according to a renewal input process and having a general service times distribution. Upon completing service, customers leave the current queue and enter to the next. Under mild assumptions, including critical data, we prove the existence and the uniqueness of the fluid solution. For asymptotic behavior, we provide necessary and sufficient conditions for the invariant state and the convergence to this invariant state. In the end, we establish the convergence of a correctly normalized state process to a fluid limit characterized by a system of algebraic and integral equations.

Keywords—Fluid Limit, fluid model, measure valued process, processor sharing, tandem queue.

I. Introduction

CONSIDER a finite sequence of queues indexed from 1 to J. Each queue j is assumed to have a single server and an infinite storage capacity. All customers present in the system are served simultaneously according to the egalitarian processor sharing rule: at any time, each of them is served at a rate that is the inverse of the total number of customers in the system. Customers arrive at queue j from the outside and receive some service. Upon service completion, customers may leave the queue, or become to some next queue i i.e. $i = j+1$. We call this system a PS Tandem Queue. This paper is devoted to study the dynamics of this system which described by a deterministic system of algebric and integral equations. Our goal in this paper is to prove the existence, uniqueness, and asymptotic behavior of the solutions to the fluid model based on the limit theorems (law of large numbers).

In the literature, several authors were studied the system of $GI/GI/1$ queue serving customers according to Processor Sharing (PS) policy. Ben Tahar and Jean-Marie [1] generalized the PS queue to multiclass case, they established the convergence of a properly normalized state process to a fluid limit. They showed the existence of a unique solution, both for a stable and an overloaded queue. Gromoll [2] established a diffusion approximation for a measure-valued descriptor of a single server processor sharing queue. Gromoll et al. [3] described the fluid limit results for the PS queue with the so-called state descriptor measure. Puha and Williams [4] studied the critical fluid of model of PS queue, they provided sufficient conditions for a fluid model solution to converge to an invariant state and gave a rate of this convergence. Jean-Marie and Robert [5] established that the queue length of an overloaded, single class PS queue grows asymptotically linearly with time, and gave the value of the growth rate.

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In this paper we generalized the PS queue to a PS Tandem Queue, for the most part we use the same notation and terminologies as [1]. We model the dynamics system by means of three J-dimensional random processes: A, D and μ taking values respectively in $\mathcal{N}^J$, $\mathcal{N}^J$ and $\mathcal{M}^J$ the space of finite, nonnegative Borel measures on $\mathcal{R}^+$ endowed with the topology of weak convergence. This tandem network has a set J of GI/GI/1 PS queue. Based on this link between queues, the fluid scaled processes $(\bar{A}^r(t), \bar{D}^r(t), \bar{\mu}^r(t))$ converges in distribution to a solution of some fluid model.

The components of these vectors correspond, respectively, to the number of arrivals $A_j(t)$ and departures $D_j(t)$ from queue j up to time t, and $\mu_j(t)$ is a measure valued process that keeps track of all residual service times of jobs in queue j.

A fluid model is a system of dynamic equations associated with data (α, ν, P) where α is the vector of exogenous arrival rates, ν is a vector of probability laws in which each component ν_j corresponds to the distribution of i.i.d. service times within queue j and $P = (p_{kl})$ corresponds to the routing matrix associated with an open network. A solution to the fluid model characterized by (α, ν, P) is a family of two real-valued, and one measure-valued vectors of continuous functions A, D, and μ respectively, that satisfy the flow conservation equations:

$A(t) = \alpha t + {}^tPD(t)$, $\langle 1, \mu_j(t)\rangle = \langle 1, \mu_j(0)\rangle + A_j(t) - D_j(t)$ and,

$$\begin{aligned} \mu_j(t)([x,\infty)) &= \mu_j(0)([x+S_j(t),\infty)) \\ &+ \int_0^t \nu_j([x+S_j(s,t),\infty))dA_j(s) \quad (1) \end{aligned}$$

for all $j \in \mathcal{J}$ where $\mathcal{J} = \{1, \ldots, J\}$, and $t \geq s \geq 0$, $S_j(s,t)$ is the accumulated service quantity devoted to any customer present in queue j over $[s,t]$, in the fluid model and $S_j(t) = S_j(0,t)$.

In a tandem queue, $P = (p_{ji} \; j, i \in \mathcal{J})$ is a superior triangular matrix defined by

$$p_{ji} = \begin{cases} 1 & \text{if } i = j+1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

That resumes the routing matrix from a queue j to a queue $j+1$. The transition of customers between queues is uniquely possible from the queue j to queue $j+1$, the analyse of first queue is equivalent to that obtained for $GI/GI/1$ PS queue with data (α_1, ν_1) where its fluid solution is the restrictive fluid solution to queue 1 of our system of PS Tandem Queue. For the next queue, our approach is based on the backward link between queues. Therefore, we reduced the fluid model that evolves as (1) to some equation of the form

$$\begin{aligned} Y(t) &= g(S(t)) + \int_0^t {}^tG(S(t) - S(s))\; d\; \mathcal{D}(s) \\ &+ \int_0^t K(S(t) - S(s))\; ds \qquad (3) \\ S(t) &= \int_0^t \frac{1}{Y(u)} du. \qquad (4) \end{aligned}$$

where $g,\ K\ : [0,\infty) \mapsto \mathcal{R}^+$, $G : [0,\infty) \mapsto \mathcal{R}^{+J}$ are continuous and nonincreasing functions and $\mathcal{D} : [0,\infty) \mapsto \mathcal{R}^{+J}$ is a continuous and nondecreasing function. In analysing each queue j, the functions g, K, G are given in functions of data (α_j, ν_j, P), and the function $\mathcal{D}(t)$ is defined in function of $\mathcal{D}_i(t)$ for all $i \leq j-1$ and $\mathcal{D}(t) = 0$ for $j = 1$. When $\mathcal{D} = 0$ the equation (3) can be reduce to a renewal equation for which existence and uniqueness is known to hold. This reduction is not possible for a general $\mathcal{D}$. A different approach to prove existence and uniqueness is used. We prove that (3) admits a solution and under Lipschitz condition on initial state, this solution is unique. Thereafter, using induction reasoning, for each j, it is shown that the fluid solution exists and unique.

The paper is organized as follows: Section I-A contains the preliminaries and definition of fluid solution model which are necessary for stating the results. This includes the dynamics description of the processor sharing tandem queue, the basic equations for the system and a discussion of the fluid model (Sections I-A1, I-A2 and I-A3). Section II presents the main results theorems (existence, uniqueness and the asymptotic behavior of fluid solution). Finally, in Section III we summarize the main ideas of this work.

Notations. Let $\mathcal{J}$ be the set $\{1, \ldots, J\}$ of all queues and $\mathcal{R}_+$ denote the non-negative real numbers, $\mathcal{R}_+^*$ denote the positive real numbers and $\mathcal{R}_+^J$ denote the J-dimensional Euclidean space. Vectors will be normally arranged as a column. As an exception, the vector e stands for a row vector of ones. The transpose of a vector or matrix A is denoted by ${}^tA$. the $J \times J$ diagonal matrix whose entries are given by the components of x will be denoted by $diag\{x\}$. For two matrices of measurable functions $F(\cdot)$ and $G(\cdot)$ defined on $\mathcal{R}_+$, we denote by the matrix-valued functions $(F * G)(x)$ for $x \in \mathcal{R}_+$, the matrix convolution formed of the elements: $(F * G)_{ij}(x) = \sum_k (F_{ik} * G_{kj})(x)$. This operation is associative and distributive over matrix addition. The multiplication by a constant matrix C can be seen as a convolution, where each element C_{ij} is interpreted as the function $C_{ij} 1_{x \geq 0}$. Associativity therefore holds for mixed scalar products and convolutions. The n^{th} convolution power of a matrix $F(x)$ is denoted with $F^{*n}(x)$. For a continuously differentiable function g, we write $g'(x) = \frac{d}{dx} g(x)$.

The set of finite, nonnegative Borel measures on $\mathcal{R}_+$ is denoted by $\mathcal{M}$. The measure δ_x^+ denotes the element of $\mathcal{M}$ with mass one at $x > 0$. We write $\langle g, \mu \rangle = \int g\, d\mu$ for $\mu \in \mathcal{M}$ and a Borel measurable function g which is integrable with respect to μ. When $g = 1_A$ for a measurable set A, we simply write $\mu(A)$. The space $\mathcal{M}$ is endowed with the weak topology, for which it is a Polish space. For a sequence $(\mu_n, n \geq 1)$ and μ of $\mathcal{M}$, the weak convergence of $(\mu_n, n \geq 1)$ to μ is denoted as $\mu_n \xrightarrow{w} \mu$. The symbol $\mathbf{0}$ denotes the zero measure of $\mathcal{M}^J$, the dimension J being always clear from the context. Let $\mathcal{M}^{c,J} = \{\xi \in \mathcal{M}^J : \xi_j(\{x\}) = 0$ for all $x \in \mathcal{R}_+$ and $j \in \mathcal{J}\}$ be the set of vectors of finite, non-negative Borel measures on $\mathcal{R}_+$ that have no atoms, and let $\mathcal{M}^{c,p,J} = \{\xi \in \mathcal{M}^{c,J} : \xi_j \neq \mathbf{0}$ for all $j \in \mathcal{J}\}$ be the set of positive measures of $\mathcal{M}^{c,J}$. We will use $\Rightarrow$ to denote convergence in distribution of a sequence of random elements of a metric space.

A. The Model

We construct in this section the evolution equations for the system, which will be the basis for the analysis.

1) Queueing Model, Primitive Processes and Initial Conditions: We consider a Processor Sharing Tandem Queue Networks composed of J queues. For each $j \in \mathcal{J}$, as already presented in [1], we assume that there are two i.i.d. sequences of random variables, $u_j = \{u_j(i), i \geq 1\}$ and $v_j = \{v_j(i), i \geq 1\}$. Each element of u_j and v_j takes values respectively in $\mathcal{R}_+$ and $\mathcal{R}_+^*$. For each $i \geq 1$ and $j \in \mathcal{J}$, $u_j(i)$ is the interarrival time between the $(i-1)th$ and the ith arriving job in queue j, and $v_j(i)$ is the service time for the ith job of queue j.Let then, $U_j(i) = \sum_{k=1}^{i} u_j(k)$ is the time at which the ith arrival enters the queue j, which has a value as zero if $i = 0$, and $V_j(i) = \sum_{k=1}^{i} v_j(k)$ is the total amount of time required from the server to process the first i arrivals at queue j. The sequences

$$u_1, \ldots, u_J, v_1, \ldots, v_J$$

are assumed mutually independent. They constitute the *primitive data* of the system.

It is assumed that for each $j \in \mathcal{J}$, the distribution ν_j does not charge the origin, $\nu_j(\{0\}) = 0$, and satisfies: $\langle \chi, \nu_j \rangle < \infty$ (finite expectation). For each j, let $\mathbf{E}_j(t) = \sup\{n : \sum_{i=1}^n u_j(i) \leq t\}$ be the number of exogenous arrivals in queue j by time t. Denote by $\mathbf{E}(t) = (\mathbf{E}_1(t), ..., \mathbf{E}_J(t))$. Any costumer that is present in the system at time zero is called an *initial job*. For each $j \in \mathcal{J}$, we assume that there exists an integer random variable with finite mean $Z_j(0)$ and an i.i.d. sequence of strictly positive random variables $v_j^0 = \{v_j^0(i), i \geq 1\}$ with a common Borel probability measure ν_j^0, such that

$$v_1^0, \ldots, v_J^0, Z_1(0), \ldots, Z_J(0)$$

are mutually independent. Then, $Z_j(0)$ be the number of initial job at queue j and $v_j^0(i)$ be the service time requirement of the ith initial job in queue j.

The routing matrix P is assumed to be open, that is, the matrix

$$Q = I + {}^tP + ({}^tP)^2 + ... \qquad (5)$$

is finite, which is equivalent to requiring that $(I - {}^tP)$ be inversible. We denote $Q = (I - {}^tP)^{-1}$.

2) Queuing Equations: Let

$$\mathbf{A}(t) = (\mathbf{A}_1(t), ..., \mathbf{A}_J(t)) \, , \mathbf{D}(t) = (\mathbf{D}_1(t), ..., \mathbf{D}_J(t)),$$
$$\mathbf{Z}(t) = (\mathbf{Z}_1(t), ..., \mathbf{Z}_J(t))$$

be a random processes such that $\mathbf{A}_j(t), \mathbf{D}_j(t)$ and $\mathbf{Z}_j(t)$ are respectively, the total number of arrivals by time t at, the number of departures by time t from, and the number of customers present at time t in, queue j. For each $j \in \mathcal{J}$, those processes satisfy the following queueing equations:

$$\mathbf{A}_1(t) = \alpha_1 t \quad (6)$$
$$\mathbf{A}_j(t) = \alpha_j t - \sum_{i=1}^{j-1} \mathbf{Z}_i(t) + \sum_{i=1}^{j-1} \mathbf{Z}_i(0) \quad (7)$$
$$\mathbf{D}_j(t) = \sum_{i=1}^{\mathbf{Z}_j(0)} 1_{\{ v_j^0(i) \leq \mathbf{S}_j(t) \}} + \sum_{i=1}^{\mathbf{A}_j(t)} 1_{\{ v_j(i) \leq \mathbf{S}_j(t) - \mathbf{S}_j(U_j(i)) \}} \quad (8)$$
$$\mathbf{Z}(t) = \mathbf{Z}(0) + \mathbf{A}(t) - \mathbf{D}(t) \quad (9)$$
$$\mathbf{S}_j(t) = \int_0^t \frac{1}{\mathbf{Z}_j(s)} ds \, . \quad (10)$$

Equation (6) is the arrival for the first queue i.e., $j = 1$. The functions in equations (7) and (8) are, respectively, the arrivals and departures by time t for queue j where $j = 1, \ldots, J$. Equation (9) is the queue lengths per station with input/outputs. And the function in (10) means the accumulative amount of service time allocated per job up to time t at station j. Hence, $\mathbf{S}_j(t) - \mathbf{S}_j(s)$ is the amount of service received in the interval $[s, t]$ at queue j. For each $j \in \mathcal{J}$, we define the measure-valued function of time $\mu_j : [0, +\infty) \to \mathcal{M}$ by

$$\mu_j(t) = \sum_{i=1}^{\mathbf{Z}_j(0)} \delta^+{}_{(v_j^0(i) - \mathbf{S}_j(t))} + \sum_{i=1}^{\mathbf{A}_j(t)} \delta^+{}_{(v_j(i) - \mathbf{S}_j(t) + \mathbf{S}_j(U_j(i)))}. \quad (11)$$

Recall that δ^+ is the Borel measure on $\mathcal{R}^+$ with mass one at $x > 0$. At each time t, $(v_j^0(i) - \mathbf{S}_j(t))^+$ and $(v_j(i) - \mathbf{S}_j(t) + \mathbf{S}_j(U_j(i)))^+$ are the residual service times for queue j of, respectively, the ith initial job, and the ith job. This defines $\mu_j(.)$ as a measure-valued stochastic process with sample path in the Polish space $\mathcal{D}([0, +\infty), \mathcal{M})$ of r.c.l.l. function from $[0, +\infty)$ to $\mathcal{M}$. In [3], this process is referred to as the state descriptor. The number of customers in queue j at time t is given by

$$\mathbf{Z}_j(t) = \langle 1, \boldsymbol{\mu}_j(t) \rangle \, . \quad (12)$$

3) Fluid Model: The fluid model shares the following parameters with the discrete model: the nonnegative vector $\alpha = (\alpha_1, ..., \alpha_J)$, the vector of Borel probability measures $\nu = (\nu_1, ..., \nu_J)$ and the nonnegative routing matrix P satisfies (2). The required assumptions on this parameters are the same as above: for $j \in \mathcal{J}$, the measure ν_j does not charge the origin, $\langle \chi, \nu_j \rangle < \infty$, and the routing matrix P has spectral radius strictly less than one. Hence the matrix $Q = (I - {}^tP)^{-1}$ is well defined. Define the vector $\lambda = Q\alpha$. The global arrival rate in the queue j is then λ_j, and the load factor of the queue j is $\rho_j = \lambda_j \langle \chi, \nu_j \rangle$.

Definition 1 (Fluid Solution Model): Let (α, ν, P) be some data and $\xi \in \mathcal{M}^{c,p,J}$ be an initial state. A fluid solution is a triple $(A(t), D(t), \bar{\mu}(t))$ of two real-, and one measure-valued vectors of functions: $A, \, D : \mathcal{R}_+ \to \mathcal{R}_+^J$, and $\bar{\mu} = (\bar{\mu}_1, ..., \bar{\mu}_J) : \mathcal{R}_+ \to \mathcal{M}^J$ such that $\bar{\mu}(0) = \xi$, and

i) A and D are continuous and increasing componentwise,
ii) The triple satisfies the relations

$$A_1(t) = \alpha_1 t \quad (13)$$
$$A_j(t) = \alpha_j t - \sum_{i=1}^{j-1} Z_i(t) + \sum_{i=1}^{j-1} Z_i(0) \qquad j = 2, \ldots, N. \quad (14)$$
$$\langle 1, \bar{\mu}_j(t) \rangle = \langle 1, \xi_j \rangle + A_j(t) - D_j(t) \quad (15)$$
$$\bar{\mu}_j(t)(I_x) = \xi_j(I_{x+S_j(t)}) + \int_0^t \nu_j(I_{x+(S_j(t)-S_j(s))}) \, dA_j(s) \quad (16)$$

for every $j \in \mathcal{J}$ and for all $t, \; x \in \mathcal{R}_+$, where

$$S_j(t) = \int_0^t \frac{1}{\langle 1, \mu_j(s) \rangle} \, ds \quad (17)$$

for all $t < t_j = \inf\{t : \mu_j(t) = \mathbf{0}\}$.
Denote the total mass of $\bar{\mu}_j(t)$ by

$$Z_j(t) = \langle 1, \bar{\mu}_j(t) \rangle \, . \quad (18)$$

Since ξ_j is a finite measure for each $j \in \mathcal{J}$, let ν_j^0 be a probability measure such that $\xi_j = Z_j(0)\nu_j^0$, where $Z_j(0) = \langle 1, \xi_j \rangle$. For each $j \in \mathcal{J}$, let v_j and v_j^0 be a random variables with respectively a distribution ν_j and ν_j^0. Denote by $B^0 = diag\{B_j^0; j \in \mathcal{J}\}$ and $\beta^0 = diag\{\beta_j^0; j \in \mathcal{J}\}$, where B_j^0 the distribution function of ν_j^0 and $\beta_j^0 = \langle \chi, \nu_j^0 \rangle$. As a particular case of (16), we have the law of evolution for Z_j :

$$Z_j(t) = Z_j(0)\mathcal{P}(v_j^0 > S_j(t)) + \int_0^t \mathcal{P}(v_j > S_j(t) - S_j(s)) \, dA_j(s) \quad (19)$$

with $\xi_j(I_x) = Z_j(0)\mathcal{P}(v_j^0 > x)$ for all $j \in \mathcal{J}$ and $x \geq 0$.
Observe that the state of each queue j of the such system is defined in functions of $(A_j(t), D_j(t), Z_j(t))$. We represent in vectoriel forms for each $j \in \mathcal{J}$ the system of equations (14), (15) and (19) by using the backward link between queues,

$$A(t) = \alpha t + {}^tPD(t) \quad (20)$$
$$Z(t) = Z(0) + A(t) - D(t) \quad (21)$$
$$Z(t) = (I - B^0)(S_j(t))Z(0) + \int_0^t (I - B)(S_j(t) - S_j(s)) dA(s) \quad (22)$$

Where $j \in \mathcal{J}$.

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II. Summary of Results

The main results of the paper are presented in this section. on the one hand, we discuss the existence and uniqueness result (Theorem 1), and the asymptotic result as $t \to \infty$ (Theorem 2). Finally, we introduce the framework for the convergence of the discrete process to a fluid limit, and state the result (Theorem 3).

A. Existence and Uniqueness

The next result provides existence and uniqueness of the fluid solution. In the case of a critical system which in initially not empty, we have a reduction of the evolution equation for fluid model corresponding to queue j where the functional equations (3)-(4) play an important role to proved the theorem below.

Theorem 1: Given a critical data (α, ν, P) such that the matrix P satisfies (2) and $\xi \in \mathcal{M}^{c,p,J}$ be an initial state. There exists a fluid solution $(A(t), D(t), \mu(t))$ of the model such that $\mu(0) = \xi$. This solution is such that

$$\bar{W}_j(t) = \left(\bar{W}_j(0) + (\rho_j - 1)t\right)^+, \tag{23}$$

for $j = 1, \ldots, J$. Where $\bar{W}_j(\cdot)$ is the total workload at queue j. Moreover, if $x \to \xi([0, x])$ is a lipschtzian function with a Lipschitz constant b. Then the fluid solution is unique.

B. Asymptotic Results

In this section, we give the convergence results of the fluid model to an invariant state. For that we start with the following definition of the invariant state.

Definition 2: A measure $\xi \in \mathcal{M}^{\mathcal{J}}$ is an invariant state for the fluid model associated with data (α, ν, P), if the component μ of the fluid solution satisfies

$$\mu(t) = \xi \qquad \text{for all } t \geq 0 \tag{24}$$

Let $\Delta(x)$ be measure valued $J \times J$-matrix with components $\Delta_{ij}(x)$ for $i, j \in \mathcal{J}$ defined by

$$\Delta(x) = A^{-1} B \, diag(\nu^e \, [x, +\infty)) \tag{25}$$

where A and B are two $J \times J$ matrices defined by $A = I + \beta\beta^e Q$ and $B = \beta^0 \beta^e + \beta\beta^e Q$.

The following asymptotic result concern the trajectories of the fluid limit when $t \to \infty$.

Theorem 2: Given a critical data (α, ν, P) and $\xi \in \mathcal{M}^{c,J}$ and assuming that $\beta^0_j = \langle \chi, \xi_j \rangle < \infty$, $\beta_j = \langle \chi, \nu_j \rangle < \infty$ and $\beta^e_j = \langle \chi, \nu^e_j \rangle < \infty$ for all $j \in \mathcal{J}$, then, as $t \to \infty$, $\mu(t) \, [x, +\infty)$ converge to the limit denoted $\mu(\infty) \, [x, +\infty)$, we have:

$$\mu(\infty) \, [x, +\infty) = \Delta(x) \, Z(0). \tag{26}$$

We have in the following the consequence of the last theorem which is the limit of the total mass when $t \to \infty$.

Corollary 1: Given a critical data (α, ν, P) and $\xi \in \mathcal{M}^{c,J}$ and assuming that the convergence (26) holds. Then the vector $(Z_1(\infty), \ldots, Z_J(\infty))$ limit $(Z_1(t), \ldots, Z_J(t))$, where $\langle 1, \mu_j(t) \rangle = Z_j(t)$,

$$Z_i(\infty) = \frac{2\beta^0_i}{\beta^{(2)}_i} Z_i(0) - \frac{2\beta_i}{\beta^{(2)}_i} \left(\sum_{k=1}^{i-1} (Z_k(\infty) - Z_k(0)) \right). \tag{27}$$

Therefore,

$$Z(\infty) = \Delta(0) \; Z(0). \tag{28}$$

C. Convergence to the Fluid Model

Consider a sequence of multiclass processor sharing queues indexed by integer numbers r. Assume that this model is defined on probability space $(\Omega, \mathcal{P}^r)$, and that it has the same basic structure as described in [1]. The primitive increments are denoted by $u^r_j = \{u^r_j(i), i \geq 1\}$, $v^r_j = \{v^r_j(i), i \geq 1\}$ and $\varphi^{r,j} = \{\varphi^{r,j}(i), i \geq 1\}$, for all $j \in \mathcal{J}$. The data of the r-th queue is (α^r, ν^r, P^r).

Assumptions for primitive data. It is assumed that

$$u^r_1, \ldots, u^r_J, v^r_1, \ldots, v^r_J \tag{29}$$

are mutually independent and $\langle \chi, \nu^r_j \rangle < \infty$, for all $j \in \mathcal{J}$. When $r \to \infty$,

$$\alpha^r_j \quad \to \quad \alpha_j, \text{ for all } j \in \mathcal{J} \tag{30}$$

Each matrix P^r has a spectral radius < 1.

$$\nu^r_j \quad \xrightarrow{w} \quad \nu_j \tag{31}$$

$$\langle \chi, \nu^r_j \rangle \quad \to \quad \langle \chi, \nu_j \rangle \tag{32}$$

$$\mathcal{E}(u^r_j(1); u^r_j(1) > r) \quad \to \quad 0 \; . \tag{33}$$

Recall that $\lambda^r = Q^r \alpha^r$ and $\rho^r_j = \lambda^r_j \langle \chi, \nu^r_j \rangle$. Assumptions (30) and (32) guarantee the following convergence $\lambda^r_j \to \lambda_j$ for all $j \in \mathcal{J}$, and $\rho^r_j \to \rho_j = \lambda_j \langle \chi, \nu_j \rangle$.

The fluid-scaled processes. The fluid-scaled processes that will give rise to a fluid limit are defined as:

$$\begin{aligned}
\bar{A}^r_j(t) &= \frac{\mathbf{A}^r_j(rt)}{r}, \bar{D}^r_j(t) = \frac{\mathbf{D}^r_j(rt)}{r}, \bar{E}^r_j(t) = \frac{\mathbf{E}^r_j(rt)}{r}, \\
\bar{Z}^r_j(t) &= \frac{\mathbf{Z}^r_j(rt)}{r}, \bar{W}^r_j(t) = \frac{\mathbf{W}^r_j(rt)}{r}, \bar{\mu}^r_j(t) = \frac{\boldsymbol{\mu}^r_j(rt)}{r} \; .
\end{aligned}$$

In particular, if we define $\bar{S}^r_j(t, u) = \mathbf{S}^r_j(rt, ru)$, then:

$$\bar{S}^r_j(t, u) = \int_{rt}^{ru} \frac{1}{Z^r_j(s)} ds.$$

Assumption for initial conditions. We assume that the sequence of initial service times for each $j \in \mathcal{J}$ is denoted by $v^{0,r}_j = \{v^{0,r}_j(i), i \geq 1\}$ with $\nu^{0,r}_j$ its distribution, and the initial number of customers is $\bar{Z}^r_j(0)$.

$$v^{0,r}_1, \ldots, v^{0,r}_J, v^r_1, \ldots, v^r_J, \bar{Z}^r_1(0), \ldots, \bar{Z}^r_J(0) \tag{34}$$

are mutually independent and $\langle \chi, \nu^{0,r}_j \rangle < \infty$, for all $j \in \mathcal{J}$. Moreover, assume that there exist a vector $\bar{Z}(0) =$

$(\bar{Z}_1(0), \ldots, \bar{Z}_J(0)) \in \mathcal{R}_+^J$ and a measure-valued vector $\nu^0 = (\nu_1^0, \ldots, \nu_J^0) \in \mathcal{M}^J$ such that for each $j \in \mathcal{J}$ we have

$$\bar{Z}^r(0) \Rightarrow \bar{Z}(0) \quad (35)$$

$$\nu_j^{0,r} \xrightarrow{w} \nu_j^0 \quad (36)$$

$$\langle \chi, \nu_j^{0,r} \rangle \to \langle \chi, \nu_j^0 \rangle \quad (37)$$

$$\langle 1_{\{x\}}, \nu_j^0 \rangle = 0 \qquad \text{for all } x \in \mathcal{R}_+. \quad (38)$$

Fluid limit result.

Theorem 3: Consider a sequence of PS Tandem Queue as defined above, satisfying assumptions (29)-(38). Then the sequence of fluid-scaled processes converges as:

$$(\bar{A}^r, \bar{D}^r, \bar{\mu}^r) \Rightarrow (A, D, \mu),$$

where (A, D, μ) is a fluid solution such that $\mu_j(0) = Z_j(0)\nu_j^0$ for all $j \in \mathcal{J}$.

III. Conclusion

We have provided results available for the fluid approximation to the critical PS Tandem Queue. This includes the existence, uniqueness and asymptotic of fluid solution. Furthermore, the convergence to the invariant state has been presented.

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