

# The Effects of Various Boundary Conditions on Thermal Buckling of Functionally Graded Beam with Piezoelectric Layers Based on Third order Shear Deformation Theory

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**Abstract**—This article attempts to analyze functionally graded beam thermal buckling along with piezoelectric layers applying based on the third order shearing deformation theory considering various boundary conditions. The beam properties are assumed to vary continuously from the lower surface to the upper surface of the beam. The equilibrium equations are derived using the total potential energy equations, Euler equations, piezoelectric material constitutive equations and third order shear deformation theory assumptions. In order to fulfill such an aim, at first functionally graded beam with piezoelectric layers applying the third order shearing deformation theory along with clamped-clamped boundary conditions are thoroughly analyzed, and then following making sure of the correctness of all the equations, the very same beam is analyzed with piezoelectric layers through simply-simply and simply-clamped boundary conditions. In this article buckling critical temperature for functionally graded beam is derived in two different ways, without piezoelectric layer and with piezoelectric layer and the results are compared together. Finally, all the conclusions obtained will be compared and contrasted with the same samples in the same and distinguished conditions through tables and charts. It would be noteworthy that in this article, the software MAPLE has been applied in order to do the numeral calculations.

**Keywords**—Thermal Buckling, Functionally Graded Beam, Piezoelectric Layer, Various Boundary Conditions

## I. INTRODUCTION

UTILIZING piezoelectric materials, which would deform mechanically and vice-versa when being impacted by electric circuits, is supposed as one efficient way in order to achieve the aims mentioned above. Bonding or embedding of many pieces of such materials would make it easier in order to sense their strain and then monitor them. Considering their small dimensions, it could be possible to apply them in different pieces without altering a remarkable amount of their germs [1].

Piezoelectric materials are abundantly applied in various transducers such as, pressure transducers, strain gauges and accelerometers. In addition, it could be mentioned that such materials could be utilized in turbo-machines mechanical pieces [2] and also vibration control of simple machines in an experimental way [3].

Tiersten [4] was the first one who reached to the equations monitoring vibration of piezoelectric sheets; he also managed to survey electro-mechanic basics. Furthermore, Tzou [5] also succeeded in obtaining equations monitoring a consistent piezoelectric condition through energy ways in curvilinear coordinates and solving several of them by engineering theories.

Reddy [6] is another one who could obtain the equations monitoring piezoelectric sheets applying different theories like classic theory, first and third order shearing theory. Saravanas [7] and his partners managed to obtain the equations monitoring compound sheets with piezoelectric layers through layer wise theory and solve them as well and in addition, they attempted to compare their founding with analytic solutions. Brooks and Heyliger [8] made an effort to solve monitoring equations on piezoelectric sheets analytically in cylindrical bending condition; all the results claim that in short distance with thin layers the existence of a consistent electric circuit or linear change within it would be utterly logical and match the experimental conclusions as well.

In order to test such solutions, it would be claimed that the best way could be obtaining and then solving equations in their certain conditions with analytical solutions. As it was mentioned earlier, in this article the buckling critical temperature of functionally graded beams would be obtained through various boundary conditions in two different ways, without piezoelectric layer and also with layers, and according to the third order shearing theory in various boundary conditions.

## II. PIEZOELECTRIC CONSTITUTIVE MATERIALS EQUATIONS

Since piezoelectric materials treat as capacitors do when being polarized in order to obtain their constitutive equations, one should define the equations concerning dielectric materials and the capacitor. Capacity in dielectric materials would be defined as the following:

$$C = \xi_0 \xi_r \left( \frac{A}{t} \right) = \xi \left( \frac{A}{t} \right) \quad (1)$$

In such a relation,  $\xi_0$  stands for dielectric capacitors constancy,  $\xi_r$  stands for dielectric constancy,  $\xi$  stands for total dielectric constancy,  $A$  stands for area of capacitor surfaces and  $C$  stands for capacity. On the other hand, electric charge for a flat capacitor would be obtained as the following:

$$Q = CV \quad (2)$$

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Then  $C$  replacement would be done:

$$Q = \frac{\xi A}{t} V \quad (3)$$

In the relationships above,  $Q$  stands for electric charge on sheets of the capacitor,  $V$  stands for the voltage on two sides of the capacitor,  $D$  stands for the amount of electric translation which would be obtained as the following:

$$D = \frac{Q}{A} \quad (4)$$

Finally, the total amount of  $D$  would be obtained as the following:

$$D = \frac{\xi V}{t} \quad (5)$$

Thus, the electric field as the following:

$$E = \frac{V}{t} \quad (6)$$

Finally, obtained equations as the following:

$$D = \xi E \quad (7)$$

The equation above is true for all isotropic dielectrics, but piezoelectric materials would remain as isotropic unless they are polarized; afterward, they turn into orthotropic. In such materials, electric field and electric translation would be defined through three elements. In other words the general equation for electric translation would be as the following:

$$D_i = \xi_{ij} E_j \quad (8)$$

Except  $\xi_{11}$ ,  $\xi_{22}$ ,  $\xi_{33}$  all  $\xi_{ij}$  for piezoelectric materials would be zero. Now, direct and reverse feature of piezoelectric materials mentioned earlier let us define piezoelectric quotient as the following in which explains tension and strain relationship with the imposed field and also electric replacement.

*Direct impact position:* in this way, applying force produces electric field. As observed in Hook's law is applied in order to relate mechanical displacement and mechanical stress, shown as  $\sigma = E\varepsilon$  in which  $E$  stands for elasticity modulus for a certain substance. It could be possible even to think of a relation for piezoelectric substances which defines the relation between mechanical stress and electric displacement as the following.

$$D_i = d_{ij} \sigma_j \quad (9)$$

In which  $D_i$  stands for electric displacement,  $\sigma_j$  mechanical stress and  $d_{ij}$  piezoelectric modulus.

*Reverse impact position:* in this way, applying voltage produces strain or mechanical displacement. In such a position, we'd better apply a kind of relation which defines the relation between strain and the applied electric field on piezoelectric substances as the following,

$$\xi_i = d_{ij} E_{ej} \quad (10)$$

In which  $\xi_i$  stands for strain,  $E_{ej}$  electric field and  $d_{ij}$  piezoelectric modulus in reverse position.

$$\xi_{ij} E_{ej} = d_{ij} \sigma_j \quad (11)$$

Now, in order to obtain structural relations defining relations between stress and strain and also between electric field and electric displacement, all  $D_i$  values are to be added.

Furthermore, the equations would be summed up,

$$D^T = d \sigma + \xi E_e \quad (12)$$

Also, mechanical and electric deformation would be added:

$$\xi^T = \frac{1}{E} \sigma + d E_e \quad (13)$$

As it seems relationships concerning piezoelectric materials consists of two parts which one of them is electric charge equations defining the relationship among electric translation, mechanic stress and electric field and the other one defines the relationship among mechanic deformation, mechanic stress and electric field.

### III. FUNCTIONALLY GRADED BEAM

The material properties are assumed to be graded through the thickness direction. The constituent materials are assumed to be ceramic and metal. The volume fractions of the ceramic  $V_c$  and metal  $V_m$  corresponding to the power law are expressed as:

$$V_c = \left( \frac{2z+h}{2h} \right)^k, \quad V_m = 1 - V_c \quad (14)$$

Here, subscripts  $m$  and  $c$  are the metal and ceramic constituents, respectively,  $z$  is the thickness coordinate ( $-h/2 \leq z \leq h/2$ ), and  $k$  is the power law index that takes values greater than or equal to zero. The variation of the composition of ceramic and metal is linear for  $k=1$ . The value of  $k$  equal to zero represents a fully ceramic beam. The properties of functionally graded beam are determined from the volume fraction of the material constituents. The Young's modulus,  $E$ , and coefficient of thermal expansion,  $\alpha$ , are assumed to change in the thickness direction.

$$E(z) = E_c V_c + E_m V_m \quad (15)$$

$$\rho(z) = \rho_c V_c + \rho_m V_m$$

The Poisson' ratio,  $\nu$ , is assumed to be constant across the plate thickness. Substituting Eq. (1) into (2), the material properties of the FG plate are determined as:

$$E(z) = E_m + (E_c - E_m) \left( \frac{2z+h}{2h} \right)^k \quad (16)$$

$$\rho(z) = \rho_m + (\rho_c - \rho_m) \left( \frac{2z+h}{2h} \right)^k$$

### IV. OBTAINING FUNCTIONALLY GRADED BEAM EQUILIBRIUM EQUATIONS BASED ON THE THIRD ORDER SHEAR DEFORMATION THEORY

Supposing the third order shear deformation theory, displacement field would be defined as the following:

$$\begin{cases} u(x, z, t) = u_0(x, t) + z\phi(x, t) - cz^3(\phi + w_{0,x}) \\ w(x, z, t) = w_0(x, t) \end{cases} \quad (17)$$

In equation (17),  $u$  stands for translation on  $x$  axis,  $w$  stands for translation on  $z$  axis. In such a relationship  $u$  is considered as the third order to  $z$  while  $w$  is considered as constant to  $z$ .

Considering such elements  $\epsilon_z$  would be zero. Kinematic relationships, relations between translations and deformation would be as the following:

$$\epsilon_x = u_{,x} + \frac{1}{2} w_{,x}^2 \quad \epsilon_y = 0 \quad (18)$$

$$\epsilon_{xz} = u_{,z} + w_{,x} \quad \epsilon_{yz} = 0$$

When one replaces (17) in (18), non-linear translations would be as the following:

$$\begin{aligned} \epsilon_x &= u_{0,x} + Z u_{1,x} - c z^3 u_{1,x} - c z^3 w_{0,xx} + \frac{1}{2} w_{0,x}^2 \\ \epsilon_{xz} &= u_1 - 3 c z^2 u_1 - 3 c z^2 w_{,x} + w_{0,x} \end{aligned} \quad (19)$$

Strain-deformation relationships considering thermal effects would be as the following:

$$\begin{aligned} \sigma_x &= Q_{11}(\epsilon_x - \alpha T) \\ \sigma_{xz} &= C_7 \epsilon_{xz} \end{aligned} \quad (20)$$

Thus:

$$Q_{11} = \frac{E}{1-\nu^2}, \quad C_7 = \frac{E}{2(1+\nu)} \quad (21)$$

$E$  Stands for elasticity,  $\nu$  stands for Poisson ratio,  $\alpha$  stands for linear expansion ratio of the substance the beam is made of and  $T$  stands for difference in temperature.

In general, when the beam is under the impact of the mechanic and thermal elements the total potential energy relationship would be as the following:

$$V = U + \Omega \quad (22)$$

$U$  Stands for deformation energy of the beam and it would be defined as the following according to the first shearing theory:

$$U = \frac{1}{2} \iiint [\sigma_x (\epsilon_x - \alpha T) + \sigma_{xz} \epsilon_{xz}] dx dy dz \quad (23)$$

Also,  $\Omega$  stands for the potential energy of the impacted mechanic forces on the beam which is zero here.

Therefore:

$$V = U \quad (24)$$

And then:

$$\delta V = \delta U = 0$$

$$\delta U = \frac{b}{2} \iint (\sigma_x \delta(\epsilon_x - \alpha T) + \sigma_{xz} \delta \epsilon_{xz}) dx dz = 0$$

$$\delta U = \frac{b}{2} \iint \left( \sigma_x \delta(u_{0,x} + Z u_{1,x} - c z^3 u_{1,x} - c z^3 w_{0,xx} + \frac{1}{2} w_{0,x}^2) + \sigma_{xz} \delta(u_1 - 3 c z^2 u_1 - 3 c z^2 w_{0,x} + w_{0,x}) \right) dx dz = 0 \quad (25)$$

$$\delta U = \frac{b}{2} \iint (\sigma_x \delta \epsilon_x - \sigma_x \delta(\alpha T) + \sigma_{xz} \delta \epsilon_{xz}) dx dz = 0$$

Stresses values would obtain as the following:

$$\begin{aligned} Q_x &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} dz, \quad M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_x dz, \quad N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z dz \\ R_x &= \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 \sigma_{xz} dz, \quad P_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^3 \sigma_x dz \end{aligned} \quad (26)$$

Thus, replacements would occur as the following:

$$\delta U = \frac{1}{2} \left[ - \int \frac{\partial N_x}{\partial x} \delta u_0 dx + \int \left( - \frac{\partial M_x}{\partial x} + c \frac{\partial P_x}{\partial x} + Q_x - 3 c R_x \right) \delta u_1 dx + \int \left( 3 c \frac{\partial R_x}{\partial x} - \frac{\partial Q_x}{\partial x} \right) \delta w_0 - \frac{\partial (N_x w_{0,x})}{\partial x} - c \frac{\partial P_x^2}{\partial x^2} \right) dx = 0 \quad (27)$$

Equilibrium equations would be obtained one considers integrand as zero:

$$\begin{cases} N_{x,x} = 0 \\ Q_x - 3 c R_x - M_{x,x} + c P_{x,x} = 0 \\ 3 c R_{x,x} - Q_{x,x} - c P_{x,xx} - N_x w_{0,xx} = 0 \end{cases} \quad (28)$$

#### V. BUCKLING OF FUNCTIONALLY GRADED BEAM ANALYSIS WITH UNIVERSAL PIEZOELECTRIC LAYERS UNDER THE EFFECT OF CONSTANT INCREASING TEMPERATURE ACCORDING TO THE THIRD ORDER SHEARING DEFORMATION THEORY

Normal stress values  $\sigma_x$ , shearing stress values  $\sigma_{xz}$  and the relationships concerning deformation based on translation for the beam with piezoelectric layers would be:

$$\sigma_x = Q_{11}(\epsilon_x - \alpha T) - e_{31} E_z \quad (29)$$

$$\sigma_{xz} = b \int Q_{11}^p dz = c_7 dz$$

$$\epsilon_x = u_{0,x} + z u_{1,x} - c z^3 u_{1,x} - c z^3 w_{0,xx} + \frac{1}{2} w_{0,x}^2$$

$$\epsilon_{xz} = u_1 - 3 c z^2 u_1 - 3 c z^2 w_{0,x} + w_{0,x}$$

Now, let us calculate the stress values regarding translation elements.

Thus:

$$N_x = b \int_{-\frac{h}{2}-h_B}^{\frac{h}{2}} \sigma_x^{(1)} dz + b \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x^{(2)} dz + b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} \sigma_x^{(3)} dz \quad (30)$$

Stress values would be:

$$\begin{aligned} \sigma_x^{(1)} &= Q_{11}^P (\epsilon_x - \alpha_P T) - e_{31} E_z^B \\ \sigma_x^{(2)} &= Q_{11} (\epsilon_x - \alpha T) \\ \sigma_x^{(3)} &= Q_{11}^P (\epsilon_x - \alpha_P T) - e_{31} E_z^T \end{aligned} \quad (31)$$

In above equations, index P is related to properties of piezoelectric material.  $E_z^B, E_z^T$  Are electrical field of lower and upper layer respectively.  $\alpha, \alpha_P$  Are also linear expansion factor of middle layer and piezoelectric material respectively.

Suppose:

$$c_1 = b \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} Q_{11}^p dz, \quad c_2 = b \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} Q_{11}^p z dz, \quad c_3 = b \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} Q_{11}^p z^3 dz$$

$$c_4 = b \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11} dz, \quad c_5 = b \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11} z dz, \quad c_6 = b \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11} z^3 dz,$$

$$c_8 = b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} Q_{11}^p z dz, \quad c_9 = b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} Q_{11} dz \quad (32)$$

$$c_{20} = b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} Q_{11}^p dz, \quad d_{11} = \frac{b}{1-\nu^2} \int_{\frac{-h}{2}}^{\frac{h}{2}} E_{(z)} \alpha_{(z)} dz,$$

$$u = zQ - \frac{4z^3}{3h^2} (Q + \frac{\partial w}{\partial x})$$

$$G_{11} = c_1 + c_4 + c_{20}, \quad G_{22} = c_2 + c_5 + c_8, \quad G_{33} = c_3 + c_6 + c_9$$

$N_x$  Would be:

$$N_x = G_{11}(u_{0,x} + \frac{w_{0,x}^2}{2}) + G_{22}u_{1,x} - cG_{33}(u_{1,x} + w_{0,xx}) - be_{31}(E_z^B h_B + E_z^T h_T) - (c_1 + c_{20})\alpha_p + d_{11} \quad (33)$$

$Q_x$  Is another stress deviator force which would be given through the following equation:

$$Q_x = k_{11}b \int_{\frac{-h}{2}-h_B}^{\frac{h}{2}} \sigma_{xz}^{(1)} dz + k_{11}b \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{xz}^{(2)} dz + bk_{11} \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} \sigma_{xz}^{(3)} dz \quad (34)$$

Stress values would be:

$$\sigma_{xz}^{(1)} = c_7(u_1 - 3cz^2u_1 - 3cz^2w_{0,x} + w_{0,x})$$

$$\sigma_{xz}^{(2)} = c_7(u_1 - 3cz^2u_1 - 3cz^2w_{0,x} + w_{0,x}) \quad (35)$$

$$\sigma_{xz}^{(3)} = c_7(u_1 - 3cz^2u_1 - 3cz^2w_{0,x} + w_{0,x})$$

$k_{11}$  Is correction quotient

Supposing:

$$E_1 = bk_{11} \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} c_7^p dz, \quad E_2 = bk_{11} \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} z^2 c_7^p dz, \quad E_3 = bk_{11} \int_{\frac{-h}{2}}^{\frac{h}{2}} c_7 dz$$

$$E_4 = bk_{11} \int_{\frac{-h}{2}}^{\frac{h}{2}} c_7 z^2 dz \quad (36)$$

$$E_5 = bk_{11} \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} c_7^p dz, \quad E_6 = bk_{11} \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} c_7^p z^2 dz$$

$$E_7 = E_1 - 3cE_2 + E_3 - 3cE_4 + E_5 - 3cE_6$$

$Q_x$  Would be:

$$Q_x = E_7(u_1 + w_{0,x}) \quad (37)$$

Then torque deviator force out of stress would be calculated as the following:

$$M_x = b \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} \sigma_x^{(1)} z dz + b \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_x^{(2)} z dz + b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} \sigma_x^{(3)} z dz \quad (38)$$

Stress values would be:

$$\sigma_x^{(1)} = Q_{11}^p(\epsilon_x - \alpha_p T) - e_{31}E_z^B$$

$$\sigma_x^{(2)} = Q_{11}(\epsilon_x - \alpha T) \quad (39)$$

$$\sigma_x^{(3)} = Q_{11}^p(\epsilon_x - \alpha_p T) - e_{31}E_z^T$$

Supposing:

$$c_2 = b \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} Q_{11}^p z dz, \quad c_5 = b \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11} z dz, \quad c_8 = b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} Q_{11}^p z dz,$$

$$c_{10} = b \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} Q_{11}^p z^2 dz \quad (40)$$

$$c_{11} = b \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} Q_{11}^p z^4 dz, \quad c_{12} = b \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11} z^2 dz, \quad c_{13} = b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} Q_{11} z^4 dz,$$

$$c_{14} = b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} Q_{11}^p z^2 dz, \quad c_{15} = b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} Q_{11}^p z^4 dz,$$

$$d_{22} = \frac{b}{1-\nu^2} \int_{\frac{-h}{2}}^{\frac{h}{2}} zE(z)\alpha(z) dz$$

$$G_{22} = c_2 + c_5 + c_8, \quad G_{44} = c_{10} + c_{12} + c_{14},$$

$$G_{55} = c_{11} + c_{13} + c_{15}$$

$M_x$  Would be:

$$M_x = G_{22}(u_{0,x} + \frac{w_{0,x}^2}{2}) + G_{44}c_{1,xx} - cG_{55}(u_{1,x} + w_{0,xx})$$

$$-[(c_2 + c_8)\alpha_p + d_{22}]T + \frac{e_{31}bE_z^B}{2}(hh_B + h_B^2) - \frac{e_{31}bE_z^T}{2}(hh_T + h_T^2) \quad (41)$$

$R_x$  Is another stress deviator force which is given in the following equation:

$$R_x = k_{11}b \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} z^2 \sigma_{xz} dz + b \int_{\frac{-h}{2}}^{\frac{h}{2}} z^2 \sigma_{xz}^{(1)} dz + b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} z^2 \sigma_{xz}^{(2)} dz + b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} z^2 \sigma_{xz}^{(3)} dz \quad (42)$$

Stress values would be:

$$\sigma_{xz}^{(1)} = c_7(u_1 - 3cz^2u_1 - 3cz^2w_{0,x} + w_{0,x})$$

$$\sigma_{xz}^{(2)} = c_7(u_1 - 3cz^2u_1 - 3cz^2w_{0,x} + w_{0,x}) \quad (43)$$

$$\sigma_{xz}^{(3)} = c_7(u_1 - 3cz^2u_1 - 3cz^2w_{0,x} + w_{0,x})$$

Supposing:

$$\begin{aligned} E_2 &= k_{11}b \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} c_7^{(p)} z^2 dz, E_4 = k_{11}b \int_{\frac{-h}{2}}^{\frac{h}{2}} c_7^{(2)} z^2 dz \\ E_6 &= k_{11}b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} c_7^{(p)} z^2 dz, E_8 = k_{11}b \int_{\frac{-h}{2}}^{\frac{-h}{2}-h_B} c_7^{(p)} z^4 dz \\ E_9 &= k_{11}b \int_{\frac{-h}{2}}^{\frac{h}{2}} c_7^{(2)} z^4 dz, E_{10} = k_{11}b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} c_7^{(p)} z^4 dz \end{aligned} \quad (44)$$

$$E_{11} = E_2 - 3cE_8 + E_4 - 3cE_9 + E_6 - 3cE_{10}$$

$R_x$  Would be:

$$R_x = E_{11}(u_1 + w_{0,x}) \quad (45)$$

Finally,  $P_x$  another stress deviator force would be given as the following:

$$P_x = b \int_{x_0}^{x_1} z^3 \sigma_x dz = b \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} \sigma_x^{(1)} z^3 dz + b \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_x^{(2)} z^3 dz + b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} \sigma_x^{(3)} z^3 dz \quad (46)$$

Stress values would be:

$$\begin{aligned} \sigma_x^{(1)} &= Q_{11}^p(\varepsilon_x - \alpha_p T) - e_{31}E_z^B \\ \sigma_x^{(2)} &= Q_{11}(\varepsilon_x - \alpha T) \\ \sigma_x^{(3)} &= Q_{11}^p(\varepsilon_x - \alpha_p T) - e_{31}E_z^T \end{aligned} \quad (47)$$

Supposing:

$$\begin{aligned} c_3 &= b \int_{\frac{-h}{2}-h_B}^{\frac{-h}{2}} z^3 Q_{11}^p dz, c_{11} = b \int_{\frac{-h}{2}}^{\frac{h}{2}} z^4 Q_{11}^p dz, c_{16} = b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} z^6 Q_{11}^p dz, \\ c_6 &= b \int_{\frac{-h}{2}}^{\frac{h}{2}} z^3 Q_{11} dz \end{aligned} \quad (48)$$

$$c_{13} = b \int_{\frac{-h}{2}}^{\frac{h}{2}} z^4 Q_{11}^p dz, c_{17} = b \int_{\frac{-h}{2}}^{\frac{h}{2}} z^6 Q_{11} dz, c_9 = b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} z^3 Q_{11}^p dz,$$

$$c_{15} = b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} z^4 Q_{11}^p dz, b \int_{\frac{h}{2}}^{\frac{h}{2}+h_T} z^6 Q_{11}^p dz = c_{18},$$

$$d_{33} = \frac{b}{1-\nu^2} \int_{\frac{-h}{2}}^{\frac{h}{2}} z^3 E(z) \alpha(z) dz$$

$$G_{33} = c_3 + c_6 + c_9, G_{55} = c_{11} + c_{13} + c_{15}, G_{66} = c_{16} + c_{17} + c_{18}$$

$P_x$  Would be:

$$\begin{aligned} P_x &= G_{33}(u_{0,x} + \frac{w_{0,x}^2}{2}) + G_{55}u_{1,x} - cG_{66}(u_{1,x} + w_{0,xx}) \\ &- [(c_3 + c_9)\alpha_p + d_{33}]T + be_{31}E_z^B(h_B^4 + \frac{h^3 h_B}{2}) \\ &+ 2h_B^3 h + \frac{3}{2}h^2 h_B^2 - be_{31}E_z^T(h_T^4 + \frac{h^3 h_T}{2} + 2h_T^3 h + \frac{3}{2}h^2 h_B^2) \end{aligned} \quad (49)$$

Now, let us suppose the thickness of the practical piezoelectric layers up and down the beam be equal; that is,  $h_T = h_B = h_P$

On the other hand, as it was mentioned in the previous equations; the amount of electric field  $E = \frac{V}{h}$  would be

constant in which  $h$  stands for piezoelectric layers thickness,  $V$  stands for the imposed voltage. It is supposed here that voltage for upper and lower layer is equal,  $V^T = V^B = V$  supposing such data, the values of  $N_x$ ,  $Q_x$  and  $M_x$  would be finally obtained upon translation elements,

if  $h = h_T = h_B = 0$

$$\begin{aligned} E &= \frac{V}{h} = E_z^B = E_z^T = \frac{V^T}{h} = \frac{V^B}{h} \Rightarrow G_{22} = G_{33} = c_2 + c_8 = c_3 + c_9 \\ \Rightarrow G_{11} &= c_1 + c_4 + c_7, G_{22} = c_2 + c_5 + c_8 = c_2 + c_8 = 0, \\ G_{33} &= c_3 + c_6 + c_9 = c_3 + c_9 = 0, G_{44} = c_{10} + c_{12} + c_{14}, \\ G_{55} &= c_{11} + c_{13} + c_{15}, G_{66} = c_{16} + c_{17} + c_{18}, c_1 = c_4 = c_7, \\ c_2 &= -c_8, c_3 = -c_9 \\ \Rightarrow G_{22} &= G_{33} = c_2 + c_8 = c_3 + c_9 = c_5 = c_6 = 0 \end{aligned} \quad (50)$$

Thus,

$$N_x = G_{11}(u_{0,x} + \frac{w_{0,x}^2}{2}) - F - ((c_1 + c_{20})\alpha_p + d_{11})T \quad (51)$$

$$Q_x = E_7(u_1 + w_{0,x})$$

$$M_x = G_{44}u_{1,x} - cG_{55}(u_{1,x} + w_{0,xx}) - d_{22}T$$

$$R_x = E_{11}(u_1 + w_{0,x})$$

$$P_x = G_{55}u_{1,x} - cG_{66}(u_{1,x} + w_{0,xx}) - d_{33}T$$

In the relationships above  $F = 2e_{31}bV$  stands for piezoelectric force.

Now, having the elements of  $N_x, Q_x, M_x, R_x$  and  $P_x$  their derivations upon translation elements and their translation in balance equations and solving them afterward, it would be possible to obtain the critical temperature difference.

#### VI. BEAM WITH CLAMPED-CLAMPED BOUNDARY CONDITIONS

Let us begin the process from the first equation of relationship (25). Thus,

$$N_{x,x} = 0 \Rightarrow N_x = cte = \text{constant} = k \quad (52)$$

$$\Rightarrow k = -(F + ((c_1 + c_{20})\alpha_p + d_{11})T) = -(F + XT)$$

Therefore:

$$N_x = G_{11}(u_{0,x} + \frac{w_{0,x}^2}{2}) - 2e_{31}V - ((c_1 + c_{20})\alpha_p + d_{11})T = k \quad (53)$$

With relation (53) displacement in relation (28) the remained equilibrium equation would be:

$$\begin{cases} E_7(u_1 + w_{0,x}) - 3cE_{11}(u_1 + w_{0,x}) - G_{44}u_{1,xx} + cG_{55}(u_{1,xx} + \frac{\partial^3 w_0}{\partial x^3}) \\ + cG_{55}u_{1,xx} - c^2G_{66}(u_{1,xx} + \frac{\partial^3 w_0}{\partial x^3}) = 0 \\ 3cE_{11}(u_{1,x} + w_{0,xx}) - E_7(u_{1,x} + w_{0,xx}) - cG_{55}\frac{\partial u_1^3}{\partial x^3} \\ + c^2G_{66}(\frac{\partial u_1^3}{\partial x^3} + \frac{\partial^4 w_0}{\partial x^4}) + (F + XT)w_{0,xx} = 0 \end{cases} \quad (54)$$

Thus,

$$\begin{cases} A_1u_1 + A_1w_{0,x} + A_2u_{1,xx} + A_3u_{1,xx} + A_3\frac{\partial^3 w_0}{\partial x^3} = 0 \\ -A_1u_{1,x} - A_1w_{0,xx} - A_3\frac{\partial^3 u_1}{\partial x^3} + A_4\frac{\partial^4 w_0}{\partial x^4} + (F + XT)w_{0,xx} = 0 \end{cases} \quad (55)$$

Both equations above are equilibrium equations upon translation elements in which parameter  $x$  and other variables would be obtained as the following:

$$A_1 = E_7 - 3cE_{11}$$

$$A_2 = cG_{55} - G_{44}$$

$$A_3 = cG_{55} - c^2G_{66} \quad (56)$$

$$A_4 = c^2G_{66}$$

$$A_5 = A_2 + A_3$$

Now, the possible response in such boundary conditions would be reckoned:

$$w_0 = \sum_{m=1}^n w_{0m} \left( 1 - \cos\left(\frac{m\pi x}{l}\right) \right) \quad m = 2, 4, \dots \quad (57)$$

$$u_1 = \sum_{m=1}^n u_{1m} \left( \sin\left(\frac{m\pi x}{l}\right) \right)$$

$w_{0m}$  And  $u_{1m}$  would be constant,  $l$  stands for beam length and  $m$  stands for half-wave in  $x$  axis direction.

Thus:

$$\begin{cases} (A_1 - A_5\beta^2)u_{1m} + (A_1\beta - A_3\beta^3)w_{0m} = 0 \\ (A_1\beta - A_3\beta^3)u_{1m} - (A_1\beta^2 - A_4\beta^4 + (F + XT)\beta^2)w_{0m} = 0 \end{cases} \quad (58)$$

By solving above equation we have:

$$\Delta T_{cr} = \frac{(A_3^2\beta^4 - 2A_1A_3\beta^2 + A_4A_5\beta^4 + A_1A_5\beta^2 - A_1A_4\beta^2) - F(A_1 - A_5\beta^2)}{-X(A_1 - A_5\beta^2)} \quad (59)$$

Buckling critical temperature difference would be Slight

minimum would make us obtain relationship (51) upon  $\beta = \frac{2\pi}{l}$

#### VII. BEAM WITH SIMPLY-SIMPLY BOUNDARY CONDITIONS

Similar to clamped-clamped boundary conditions, let us solve the equations and reach an equations system in such a step; let us reckon  $u_1$  and  $w_0$  in a different way.

Thus:

$$w_0 = \sum_{m=1}^n w_{0m} \left( \sin\left(\frac{m\pi x}{l}\right) \right), \quad m = 1, 2, 3, \dots \quad (60)$$

$$u_1 = \sum_{m=1}^n u_{1m} \left( \cos\left(\frac{m\pi x}{l}\right) \right), \quad m = 1, 2, 3, \dots$$

$w_{0m}$  And  $u_{1m}$  would be constant,  $l$  stands for beam length and  $m$  stands for half-wave in  $x$  axis direction.

Therefore:

$$\begin{cases} (A_1 - A_5\beta^2)u_{1m} + (A_1\beta - A_3\beta^3)w_{0m} = 0 \\ (A_1\beta - A_3\beta^3)u_{1m} - (A_1\beta^2 - A_4\beta^4 + (F + XT)\beta^2)w_{0m} = 0 \end{cases} \quad (61)$$

Finally, solving the above temperature difference equation would define  $\Delta T_{cr}$ .

$$\Delta T_{cr} = \frac{(A_3^2\beta^4 - 2A_1A_3\beta^2 + A_4A_5\beta^4 + A_1A_5\beta^2 - A_1A_4\beta^2) - F(A_1 - A_5\beta^2)}{-X(A_1 - A_5\beta^2)} \quad (62)$$

Buckling critical temperature difference Slight minimum would make us obtain relationship (54) upon  $\beta = \frac{\pi}{l}$ .

#### VIII. BEAM WITH CLAMPED-SIMPLY BOUNDARY CONDITIONS

Similar to previous boundary conditions, at first solve the equations and reach an equations system in such a step; let us reckon  $u_1$  and  $w_0$  in a different way.

Thus:

$$u_1(x) = u_{1m}(P \cos Px + P^2 l \sin Px - P), \quad m = 1, 2, 3, \dots \quad (63)$$

$$w_0(x) = w_{0m}(\sin Px - Pl \cos Px + P(l - x)), \quad m = 1, 2, 3, \dots$$

$w_{0m}$  And  $u_{1m}$  would be constant,  $l$  stands for beam length and  $m$  stands for half-wave in  $x$  axis direction.

Therefore:

$$\begin{cases} (A_1 - A_5\beta^2)u_{1m} + (A_1\beta - A_3\beta^3)w_{0m} = 0 \\ (A_1\beta - A_3\beta^3)u_{1m} - (A_1\beta^2 - A_4\beta^4 + (F + XT)\beta^2)w_{0m} = 0 \end{cases} \quad (64)$$

Finally, solving the above temperature difference equation would define  $\Delta T_{cr}$ .

$$\Delta T_{cr} = \frac{(A_3^2\beta^4 - 2A_1A_3\beta^2 + A_4A_5\beta^4 + A_1A_5\beta^2 - A_1A_4\beta^2) - F(A_1 - A_5\beta^2)}{-X(A_1 - A_5\beta^2)} \quad (65)$$

Buckling critical temperature difference Slight minimum would make us obtain relationship (54) upon  $\beta = \frac{\sqrt{2.04\pi}}{l}$ .

#### FIGURES AND TABLETS

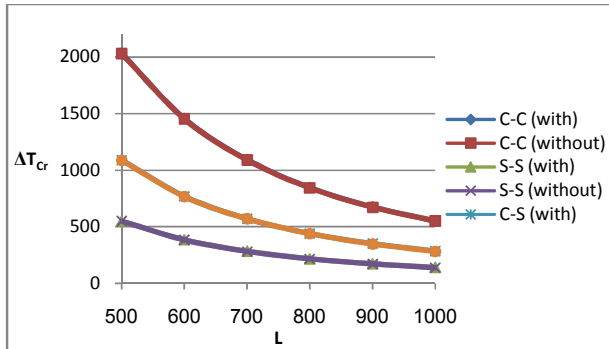


Fig. 1 Comparison of critical temperature of FGM beam With layer and without layer versus the length of Beam for various boundary conditions. (h=50 mm, hp=0.1, n=5)

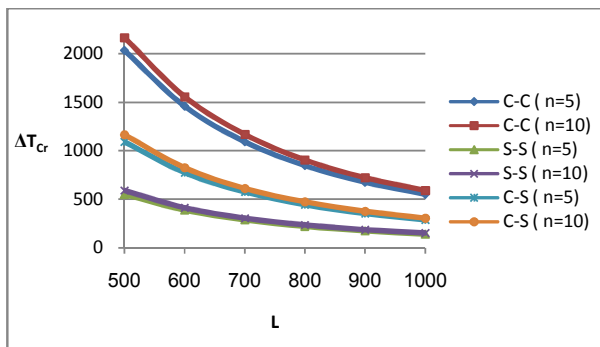


Fig. 2 Comparison of critical temperature of FGM beam With layer versus the length of beam for various Boundary conditions for various n. (h=50 mm, hp=0.1)

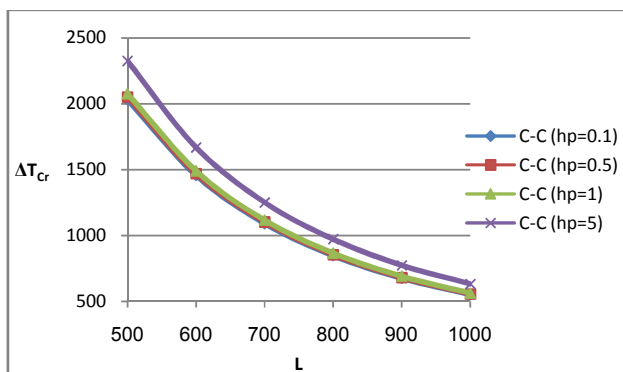


Fig. 3 Comparison of critical temperature of FGM beam with Layer with clamped-clamped boundary conditions Versus the length of beam for various hp. (h=50 mm, n=5)

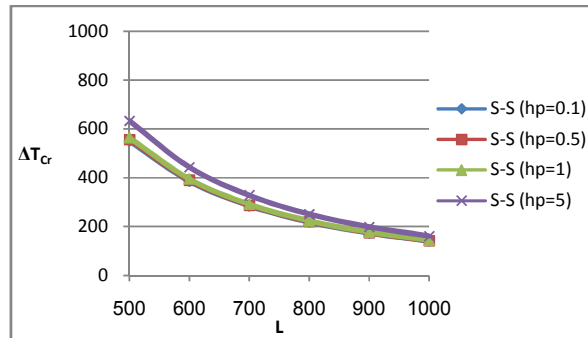


Fig. 4 Comparison of critical temperature of FGM beam with Layer with simply- simply boundary conditions Versus the length of beam for various hp. (h=50 mm, hp=0.1, n=5)

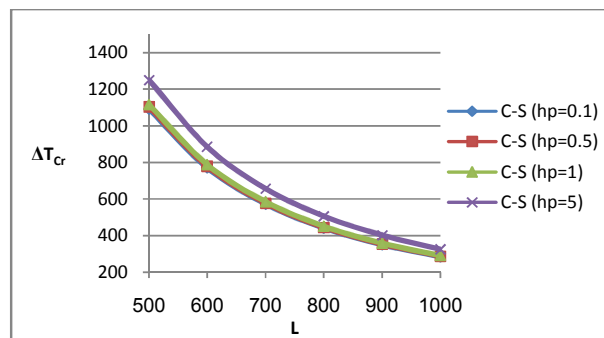


Fig. 5 Comparison of critical temperature of FGM beam with Layer with clamped- simply boundary conditions Versus the length of beam for various hp. (h=50 mm, hp=0.1, n=5)

#### IX. CONCLUSION

Studying charts I-V the following results are obtained:

Charts 1-5 are related to functionally graded beam critical temperature along with piezoelectric layers with thickness of  $h_p=0.1$  based on the third order shear deformation theory in clamped-clamped, clamped –simply and simply-simply boundary conditions. The highest difference in temperature in boundary conditions belongs to clamped-clamped one while the lowest one belongs to simply-simply one. On the other hand, difference in critical temperature an all 3 conditions upon all the charts falls down when the length stretches and it rises when the thickness increases. It is noteworthy that such changes in temperature occur with a sharp bent in clamped-clamped condition compared to the other two conditions. Also, according to the chart 3-5 which are related to functionally graded beam critical temperature difference along with piezoelectric layers with variable thickness it would be concluded that increasing piezoelectric layer thickness increase would increase the critical temperature difference and also in all 3 conditions length stretch would decrease it while thickness increase would increase it.

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