

# The direct Ansätze method for finding exact multi-wave solutions to the (2+1)-dimensional extension of the Korteweg de-Vries equation

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**Abstract**—In this paper, the direct Ansätze method is used for constructing the multi-wave solutions to the (2+1)-dimensional extension of the Korteweg de-Vries (shortly EKdV) equation. A new breather type of three-wave solutions including periodic breather type soliton solution, breather type of two-solitary solution are obtained. Some cases with specific values of the involved parameters are plotted for each of the three-wave solutions. Mechanical features of resonance interaction among the multi-wave are discussed. These results enrich the variety of the dynamics of higher-dimensional nonlinear wave field.

**Keywords**—EKdV equation; Breather; Soliton; Bilinear form; The direct Ansätze method.

## I. INTRODUCTION

**N**ONLINEAR evolution equations (NLEEs) have an important role in many nonlinear science fields such as fluid dynamics, nonlinear optics, elasticity theory, plasma physics, the propagation of long internal waves and many other fields. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations [1]. Exact soliton solutions may help us to find and explain physical phenomena and experimental results. KdV equation is the earliest soliton equation which was firstly derived by Korteweg and de Vries to model the evolution of shallow water wave in 1895 ([1],[2],[3]). In order to search for soliton solutions and study interaction of solitons for nonlinear partial differential equations, many effective methods have been developed, such as Inverse scattering transformation[1], Bäcklund transformation [2], Darboux transformation ([3],[4]), Painlevé expansion method [5], Hirota bilinear method [6], Painlevé analysis method [7], similarity reductions method ([8],[9]), homogeneous balance method [10], homotopy perturbation method [11] variational method [12], Adomian method [13], F-expansion method [14], Exp-function method [15], Extended Homoclinic test function ([16],[17]),The mapping approach [18],The improved mapping approach [19],Conditional Similarity Reduction Method [20], Projective equation method ([21],[22]) and so on.

In this paper, we use the direct Ansätze method to obtain the exact multi-wave solutions of the (2+1)-dimensional EKdV

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equation. The (2+1)-dimensional EKdV[23] can be shown in the form of

$$u_t + 3uu_y + u_{xxy} + 3u_x \int_{-\infty}^x u_y dx = 0. \quad (1)$$

In Ref.[23], a type of bell-shape soliton and exact two-soliton solution had been obtained for EKdV equation. Equation (1) now becomes an alternative form with  $u = w_x$ :

$$w_{xt} + 3w_x w_{xy} + w_{xxx} + 3w_{xx} w_y = 0. \quad (2)$$

Integrating Eq. (2) ones with respect to  $x$ , we obtain

$$w_t + 3w_x w_y + w_{xxy} = \lambda(y, t). \quad (3)$$

where  $\lambda(y, t)$  is an arbitrary function. For convenience, Eq. (3) is called (2+1)-dimensional potential EKdV (shortly PEKdV) equation with  $\lambda(y, t) = 0$ . In Ref.[24], Shen discussed Lie symmetries, Lie algebra of symmetry vector fields and similarity reductions, and found the PEKdV equation is not Painlevé integrable by means of the WTC-Painlevé analysis method. In Ref.[17], kinky periodic solitary-wave solutions, periodic soliton solutions, and cross kink-wave solutions of the (2+1)-dimensional potential EKdV equation were obtained.

## II. THE DIRECT ANSÄZ METHOD

We consider general form of higher dimensional nonlinear evolution equation

$$F(u, u_t, u_x, u_y, u_{xx}, u_{yy}, \dots) = 0, \quad (4)$$

where  $u = u(x, y, t)$  and  $F$  is a polynomial about  $u$  and its derivatives.

By Painlevé analysis, a transformation is made

$$u = T(f), \quad (5)$$

where  $f$  is a new unknown function. Then, the NLEE (4) is reduced to Hirota's bilinear equation

$$G(D_x, D_t, D_y; f, f) = 0, \quad (6)$$

where the  $D$ -operator [6] is defined by

$$D_x^m D_t^n f(x, t) \cdot g(x, t) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n [f(x, t)g(x', t')] \Big|_{x'=x, t'=t},$$

In this section, we shall seek the multi-wave solution for a given partial differential equation in the following form:

$$f(x, y, t) = e^{-\xi_1} + \delta_1 \cos \xi_2 + \delta_2 \cosh \xi_3 + \delta_3 e^{\xi_1}, \quad (7)$$

where  $\xi_i = a_i x + b_i y + c_i t$ ,  $a_i, b_i, c_i, \delta_i$  ( $i = 1, 2, 3$ ) are some constants to be determined later from the solution of (6).

Substituting Eq.(7) into Eq.(6) and setting the coefficients of the same power of  $e^{j(\xi_1)}$ ,  $\cos \xi_2$ ,  $\sin \xi_2$ ,  $\cosh \xi_3$ ,  $\sinh \xi_3$  ( $j = -1, 1$ ) equal to zero, we obtain algebraic equations. Solving the set of algebraic equations, we can find solutions  $a_i, b_i, c_i$  and  $\delta_i$  ( $i = 1, 2, 3$ ). Substituting solutions  $a_i, b_i, c_i$  and  $\delta_i$  ( $i = 1, 2, 3$ ) into Eq.(7) and Eq.(5) we can obtain exact multi-wave solutions of Eq.(4).

### III. APPLICATION TO THE THE (2+1)-DIMENSIONAL EXTENSION OF THE KORTEWEG DE-VRIES EQUATION

By the dependent variable transformation

$$u = 2(\ln f)_{xx}, \tag{8}$$

where the function  $f(x, y, t)$  is an unknown real function.

Eq. (1) is transformed into the bilinear differential equation

$$G(D_x, D_y, D_t) f \cdot f = D_x(D_t + D_x^2 D_y) f \cdot f = 0, \tag{9}$$

By using the simplest direct Ansatz method, we may choose the solution of (9) in the form

$$f(x, y, t) = e^{\xi_1} + \delta_1 \cos \xi_2 + \delta_2 \cosh \xi_3 + \delta_3 e^{-\xi_1}, \tag{10}$$

where  $\xi_1 = a_1 x + b_1 y + c_1 t$ ,  $\xi_2 = a_2 x + b_2 y + c_2 t$ ,  $\xi_3 = a_3 x + b_3 y + c_3 t$  and  $a_i, b_i, c_i, \delta_i$  ( $i = 1, 2, 3$ ) are some constants to be determined later from the solution of (9).

Substituting Eq.(10) into Eq.(9), and setting the coefficients of the same power of  $e^{\pm \xi_1}$ ,  $\cos \xi_2$ ,  $\sin \xi_2$ ,  $\cosh \xi_3$ ,  $\sinh \xi_3$  equal to zero, we obtain algebraic equations about  $a_i, b_i, c_i, \delta_i$  ( $i = 1, 2, 3$ ). Solving the set of algebraic equations, we can find solution

Case 1.

$$a_1 = 0, a_2 = ia_3, b_2 = -ib_3, c_1 = -b_1 a_3^2, \\ c_2 = ib_3 a_3^2, c_3 = -b_3 a_3^2, \delta_1 = \delta_2$$

under a transformation  $a_3 \rightarrow a_3 i, b_3 \rightarrow b_3 i$  in the above relations, (10) can be represented by the following form

$$f(x, y, t) = e^{b_1 \xi} + 2\delta_2 \cos(a_3 x) \cos(b_3 \xi) + \delta_3 e^{-b_1 \xi} \tag{11}$$

Where  $\xi = y + a_3^2 t$ .

Substituting Equation (11) into (8), then we obtain the  $x$ -periodic breather type soliton solution for EKdV equation as follows

$$u(x, y, t) = \frac{2\delta_2 a_3^2 \cos(b_3 \xi) (\sqrt{\delta_3} \cos(a_3 x) \cosh(b_1 \xi + \theta_0) + \delta_2 \cos(b_3 \xi))}{(\sqrt{\delta_3} \cosh(b_1 \xi + \theta_0) + \delta_2 \cos(a_3 x) \cos(b_3 \xi))^2}, \tag{12}$$

Where  $\xi = y + a_3^2 t, \theta_0 = \ln \sqrt{\delta_3}$  and  $\delta_2 \neq 0, \delta_3 > 0$ .

Obviously, the solution (12) is a type of periodic breather solitons which is a periodic standing wave in the propagating direction  $x$  with period  $\frac{2\pi}{a_3}$ , and at the same time is also a breather solitary wave with  $\xi = y + a_3^2 t$  and exponentially decay in  $y$ , so it is called the periodic breather-type soliton. Note that the denominator of expression (12) is greater than zero when  $\xi$  and  $x$  take arbitrarily values with  $\sqrt{\delta_3} > \delta_2$ . Therefore we see that (12) has no poles and should be well

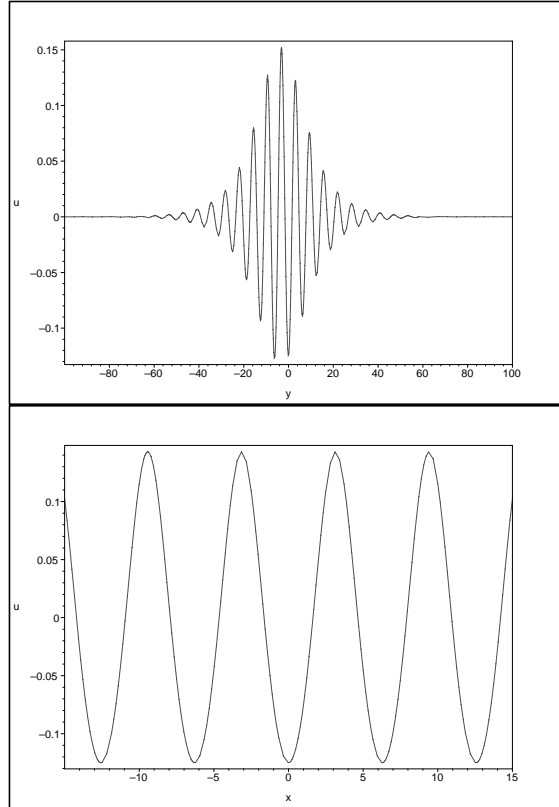


Fig. 1. (a)The breather behavior of solution (12) in  $y$ -direction. (b) The periodic behavior of solution (12) in  $x$ -direction.

behaved everywhere. So the solution is a nonsingular periodic breather soliton solution. The periodic and breather behavior of the solution (12) are shown in Figs. 1 (a)-(b).

Case 2.

$$a_1 = a_2 = c_3 = b_3 = 0, c_1 = -a_3^2 b_1, b_2 = -\frac{c_2}{a_3^2}. \tag{13}$$

Substituting Equation (10) into (8) with (13), we get the breather-type two-soliton solution of Equation (1) as follows:

$$u(x, y, t) = \frac{2\delta_2 a_3^2 (\delta_1 \cosh(a_3 x) \cos(\frac{c_2}{a_3^2} \xi) + 2\sqrt{\delta_3} \cosh(a_3 x) \cosh(b_1 \xi - \theta_0) + \delta_2)}{(\delta_1 \cos(\frac{c_2}{a_3^2} \xi) + \delta_2 \cosh(a_3 x) + 2\sqrt{\delta_3} \cosh(b_1 \xi - \theta_0))^2}, \tag{14}$$

Where  $\xi = y - a_3^2 t, \theta_0 = \ln \sqrt{\delta_3}, \delta_3 > 0$  and  $\delta_2 \neq 0$ .

The expression (14) is the breather-type two-soliton solution of EKdV equation which is a breather solitary wave on the  $y$ -axis, and meanwhile is a soliton on the  $x$ -axis. In fact, the solution (14) reflects the interaction between two solitons. There are three types of resonance interactions in two soliton solutions, namely full resonance, partial resonance and non-resonance interactions. The value of  $\delta_3$  plays an important role in determining the type of resonance interactions occurrence. Resonance only occurs when the value of  $\delta_3$  approaches 0. Because when the value of  $\delta_3$  approaches 0, then  $\ln \sqrt{\delta_3}$  approaches  $\infty$ . If  $\delta_3 = 0$  or  $\delta_3 \rightarrow 0$ , then the partial resonance

and full resonance interactions will occur. For the partial resonance interaction, the length of the resonant breather soliton increases with  $\delta_3 \rightarrow 0$ . If the value of  $\delta_3$  is not equal to zero or approaches 0, then the resonance interaction will not exist. Fig. 2 shows that the partial resonance interaction between two soliton solutions. It is not completely elastic. That is, when two initial solitons come into interaction it will produce some particularly high and steep wave humps in the vicinity of the crossing point and later break up again into two solitons which are actually the original soliton respectively (see Figs. 2 (a)). These particularly high and steep wave humps represent the localized oscillation, namely, they express a breather soliton solution. It is called the resonance breather-soliton solution. The resonance breather-soliton solution is converted into the line-soliton solution accordingly as the value of  $|\delta_1|$  becomes small.

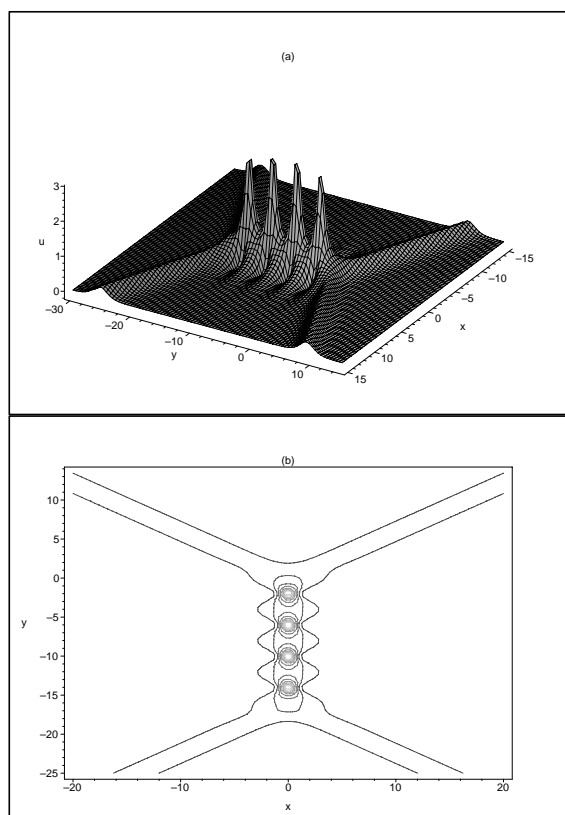


Fig. 2. The partial resonance interaction of two-soliton solution (14) and contourplot map of  $u$  in  $(x,y)$ -plane.

#### IV. CONCLUSION

In this letter, we have applied the direct Ans z method to obtain the exact multi-wave solutions of the (2+1)-dimensional EKdV equation. The obtained solutions have very concise and explicit forms. It is also shown that the simplest direct Ans z method is a direct, concise and effective method. The properties of the obtained solutions are discussed and shown in Figures 1 and 2. These obtained results enrich the variety of the dynamics of higher-dimensional nonlinear wave field. The

direct Ans z method can also be applied to solve other types of higher dimensional integrable and non-integrable systems.

#### ACKNOWLEDGMENT

The authors would like to thank the National Natural Sciences Foundation of Peoples Republic of China under Grant No.11261049 .

#### REFERENCES

- [1] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution and Inverse Scattering, Cambridge Univ. Press, 1991.
- [2] M. R. Miurs, Backlund Transformation, 1978, Springer, Berlin.
- [3] Gu CH, Soliton Theory and Its Application, Springer, Berlin (1995).
- [4] Chen AH, Multi-kink solutions and soliton fission and fusion of Sharma-Tasso-Olver equation. Phys.Lett.A. 2010; **374**:2340-2345.
- [5] Wang S, Tang XY, Lou SY, Soliton fission and fusion: Burgers equation and Sharma-Tasso-Olver equation. Chaos Solitons Fractals. 2004; **21**:231-239.
- [6] Hirota R, Exact solution of the Korteweg-de-Vries equation for multiple collisions of solitons, Phys. Lett. A. 1971; **27**:1192-1194.
- [7] The Painlev  property for partial differential equations. J. Math.Phys. 1983;**24**:522-526.
- [8] Clarkson PA, Kruskal, New similarity solutions of the Boussinesq equation. J. Math. Phys.1989; **30**:2202-2213.
- [9] Lou SY, Ruan HY, Chen DF and Chen WZ, Similarity reductions of the KP equation by a direct method, J. Phys. A Math. Gen. 1991; **24**:1455-1467.
- [10] Wang ML, Exact solutions for a compound KdV-Burger equation, Phys. Lett. A 1996; **213**:279-287.
- [11] He JH, Wu XH. Exp-function method for nonlinear wave equations. Chaos, Solitons and Fractals 2006; **30**(3):700-708.
- [12] He JH. Variational iteration method-a kind of non-linear analytical technique: some examples. International Journal of Non-linear Mechanics 1999; **34**(4):699-708.
- [13] Abassy TA, El-Tawil MA and Saleh HK, The solution of KdV and mKdV equations using adomian Pade approximation. Int.J.Nonlinear Sci.Numer.Simul. 2004; **5**:327-340.
- [14] Liu JB, Yang KQ, The extended F-expansion method and exact solutions of nonlinear PDEs. Chaos Solitons Frac. 2004; **22**:111-121.
- [15] He JH, Wu XH. Exp-function method for nonlinear wave equations. Chaos, Solitons and Fractals 2006; **30**(3):700-708.
- [16] Dai ZD, Liu ZJ, Li DL. Exact periodic solitary-wave solution for KdV equation. Chin. Phys. Lett. 2008; **25**:1531-1532.
- [17] Wang CJ, Dai ZD, Mu G, Lin SQ, New exact periodic solitary-wave solutions for new (2+1)-dimensional KdV equation. Commun. Theor. Phys.2009; **52**:862-864.
- [18] Ma SH and Fang JP, Hong BH and Zheng CL, New exact solutions for the (3+1)-dimensional Jimbo-Miwa system. Chaos, Solitons and Fractals 2009; **40**:1352-1355.
- [19] Ma SH and Fang JP, Hong BH, Zheng CL, Complex wave excitations and chaotic patterns for a general (2+1)-dimensional Korteweg-de Vries system. Chin. Phys. B 2008; **17**(8):2767-2773.
- [20] Ma SH and Fang JP, New Exact Solutions and Localized Excitations in a (2+1)-Dimensional Soliton System . Z. Naturforsch. 2009; **64**:37-43.
- [21] Ma SH and Fang JP, Multi Dromion-Solitoff and Fractal Excitations for (2+1)-DimensionalBoiti-Leon-Manna-Pempinelli System. Commun. Theor. Phys. 2009; **52**:641-647.
- [22] Ma SH and Fang JP, Ren QB, Instantaneous embed soliton and instantaneous taper-like soliton for the (3+1)-dimensional Burgers system. Acta. Phys. Sin. 2010; **59**:4420-4425.
- [23] Zhang YF,Tam Honwah, Zhao J, Higher-Dimensional KdV Equations and Their Soliton Solutions, Commun. Theor. Phys. 2006; **45**:411-413.
- [24] Shen SF, Lie symmetry analysis and Painlev  analysis of the new (2+1)-dimensional KdV equation, Appl. Math. J. Chinese Univ. Ser. B. 2007; **22**(2):207-212.