Strong Limit Theorems for Dependent Random Variables

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Abstract—In This Article We establish moment inequality of dependent random variables,furthermore some theorems of strong law of large numbers and complete convergence for sequences of dependent random variables. In particular, independent and identically distributed Marcinkiewicz Law of large numbers are generalized to the case of m_0-dependent sequences.

Keywords—Lacunary System, Generalized Gaussian, NA sequences, strong law of large numbers.

I. INTRODUCTION

LET $X_1, X_2, \cdots$ denote a sequence of random variables defined on a fixed probability space (Ω, F, P) ,The partial sums of the random variables are $S_n = \sum_{i=1}^{n} X_i$,for $n \geq 1$ and $S_0 = 0$,

$$S_{b,n} = \sum_{i=b+1}^{b+n} X_i, S(i,j) = \sum_{k=i}^{j} X_k, 1 \leq i \leq j.$$

Definition 1.1 (*Fazekas* and *Klesov*, 2000, p. 447) [1] A sequence of random variables $\{X_n, n \geq 1\}$ is said to have the *r*th $(r > 0)$ moment function of superadditive structure if there exists a non-negative function $g(i,j)$ of two arguments such that for all $b \geq 0$ and $1 \leq k \leq k + l$,

$$g(b,k) + g(b+k,l) \leq g(b,k+l) \tag{1}$$

$$E|S_{b,n}|^r \leq g^{\alpha}(b,n), n \geq 1, for some \alpha > 1 \tag{2}$$

Definition1.2 Let X be a real-valued random variable,we call a Locally Generalized Gaussian, If there exists $\alpha > 0$,such that

$$E(\exp(uX)|F) \leq \exp(u^2\alpha^2/2) \quad a.s. \tag{3}$$

for any $u \in R$.

Definition 1.3 Given $p > 0$, a sequence of real-valued random variables $\{X_n, n \geq 1\}$ is called a Lacunary System or an S_p system, if there exists a positive constant K_p such that

$$E|\sum_{i=m}^{n} C_i X_i|^p \leq K_p(\sum_{i=m}^{n} C_i^2)^{p/2}$$

for any sequence of real constant $\{C_i\}$ and all $n \geq m$.

Definition 1.4 The random variables $X_1, X_2, \cdots, X_n$ are said to be negatively associated if for every n and every pair of disjoint subsets A_1, A_2 of $\{1, 2, \cdots, n\}$

$$Cov(f_1(X_i : i \in A_1), f_2(X_j : j \in A_2)) \leq 0,$$

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whenever f_1 and f_2 are coordinatewise increasing and this covariance exists.

Definition 1.5 A stationary sequence $\{X_n\}$ is called m_0-dependent for a given fixed m_0 if $(X_1, \cdots, X_i)$ and $(X_j, X_{j+1}, \cdots)$ are independent whenever $j - i > m_0$.

Lemma 1.1 (Dini Theorem, *Fikhtengolts*, 1954, p. 286) [2] Let $c_1, c_2, \cdots$ be non-negative numbers, $v_n = \sum_{k=n}^{\infty} c_k$, if $0 < v_n < \infty$ for $n \geq 1$, then

$$\sum_{n=1}^{\infty} \frac{c_n}{v_n^{\delta}} < \infty, (0 < \delta < 1). \tag{4}$$

Lemma1.2 (*Hu*, 2005,Theorem 2.1) [3] We assume that $(X_n, n \geq 1)$ has the *r*th moment function of superadditive structure, $g(0, n) = g_n$ and g_n non-decreases, $\{b_n\}$ is a non-decreasing unbounded sequence of positive numbers and

$$\sum_{n=1}^{\infty} \frac{g_n^{\alpha} - g_{n-1}^{\alpha}}{b_n^r} < \infty,$$

then

$$\lim_{n \to \infty} \frac{S_n}{b_n} = 0 a.s. \tag{5}$$

and with the growth rate

$$\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) a.s., \tag{6}$$

where

$$\beta_n = \max_{1 \leq k \leq n} b_k v_k^{\frac{\delta}{r}}, 0 < \delta < 1,$$

$$v_n = \sum_{k=n}^{\infty} \frac{\alpha_k}{b_k^r}, \lim_{n \to \infty} \frac{\beta_n}{b_n} = 0.$$

Lemma1.3 (*Hu*, 2005,Lemma 1.2) [3] Let $b_1, b_2, \cdots$ be a non-decreasing unbounded sequence of positive numbers and $a_1, a_2, \cdots$ be nonnegative numbers. Let r and C be fixed positive numbers. Assume that for each $n \geq 1$,

$$E\left(\max_{1 \leq l \leq n} |S_l|\right)^r \leq C \sum_{l=1}^{n} \alpha_l, \tag{7}$$

$$\sum_{l=1}^{n} \frac{\alpha_l}{b_l^r} < \infty, \tag{8}$$

then (5) and (6) hold .

Lemma 1.4 (*Yang*, 2000, Corollary 3, Yang, 2001; Shao, 2000)[4][6][7] Let $X_1, X_2, \cdots$ be negatively associated random variables with zero means and $E|X_j|^r < \infty$, where

$r > 1$. Then there exists a positive constant C, which does not depend on n, such that

$$E(\max_{1\leq k\leq n}|S_k|^r) \leq C\sum_{j=1}^{n} E|X_j|^r, \qquad 1 < r \leq 2,$$

$$E(\max_{1\leq k\leq n}|S_k|^r) \leq C[\sum_{j=1}^{n} E|X_j|^r + (\sum_{j=1}^{n} EX_j^2)^{r/2}], r > 2.$$

In this paper, we assume that $C, C_1, \cdots$ are some positive constants (not necessarily always the same) independent of n.

II. MAIN RESULTS

Theorem 2.1 Assume that $\{X_n, n \geq 1\}$ be a Lacunary System , exists a positive constant K_p and $p > 2$, such that $\sup_j E|X_j|^p < \infty$,then for every $\delta > 0$,

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}(\log n)^{1/p}(\log\log n)^{(1+\delta)/p}} = 0, a.s. \qquad (9)$$

and for

$$b_n = \sqrt{n}(\log n)^{1/p}(\log\log n)^{(1+\delta)/p}, n \geq n_0,$$

$$\alpha_n = K_p n^{p/2} - K_p(n-1)^{p/2},$$

$$\beta_n = \max_{1\leq k\leq n} b_k v_k^{\delta/p}, 0 < \delta < 1,$$

$$\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) a.s., \lim_{n\to\infty}\frac{\beta_n}{b_n} = 0. \qquad (10)$$

Proof From Definition1.3 ,for any sequence of real constant $\{C_i\}$,

$$E(\sum_{i=b+1}^{b+n} C_i X_i)^p \leq K_p(\sum_{i=b+1}^{b+n} C_i^2)^{p/2},$$

in particulary where $C_i \equiv 1$,we have

$$E(\sum_{i=b+1}^{b+n} C_i X_i)^p = E|S_{b,n}|^p \leq K_p n^{p/2},$$

In Definition 1.1 take

$$g_n = g(b, n) = K_p^{2/p} n, \alpha = \frac{p}{2},$$

then

$$g(b, k) = K_p^{2/p} k, g(b+k, l) = K_p^{2/p} l$$

$$g(b, k) + g(b+k, l) = K_p^{2/p}(k+l) \leq g(b, k+l),$$

$$E|S_{b,n}|^p \leq K_p n^{p/2} = g^{p/2}(b, n), n \geq 1, p > 2,$$

we know that $\{X_n, n \geq 1\}$ has the pth moment function of supersensitive structure, and

$$\sum_{n=n_0}^{\infty} \frac{g_n^{p/2} - g_{n-1}^{p/2}}{b_n^p} = \sum_{n=n_0}^{\infty} \frac{K_p(n^{p/2} - (n-1)^{p/2})}{n^{p/2}(\log n)(\log\log n)^{1+\delta}}$$

$$\leq C\sum_{n=n_0}^{\infty} \frac{1}{n(\log n)(\log\log n)^{1+\delta}} < \infty,$$

thus (9) follows from Lemma1.2.

We assume that $\alpha_n > 0$ for infinitely many n.By (8) and Lemma 1.3, we know that

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{b_n^r v_n^{\delta}} < \infty,$$

it is easy to see that $0 < \beta_k \leq \beta_{k+1}$ for $k \geq 1$,and

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{\beta_n^r} \leq \sum_{n=1}^{\infty} \frac{\alpha_n}{\beta_n^r v_n^{\delta}} < \infty,$$

$$\frac{\beta_k}{b_k} \leq \frac{\max_{1\leq l\leq k_1} b_l v_l^{\delta/r}}{b_k} + \frac{\max_{k_1\leq l\leq k} b_l v_l^{\delta/r}}{b_k}$$

$$\leq \frac{\max_{1\leq l\leq k_1} b_l v_l^{\delta/r}}{b_k} + v_{k_1}^{\delta/r},$$

from (8) and

$$\lim_{n\to\infty} b_n = \infty,$$

we get

$$\lim_{n\to\infty} \frac{\beta_n}{b_n} = 0.$$

Eq.(7) and Theorem 1.1 of Fazekas and Klesov (2000) [1] imply that

$$E\left(\max_{1\leq l\leq n}|\frac{S_l}{\beta_l}|\right)^r \leq 4C\sum_{l=1}^{n} \frac{\alpha_l}{\beta_l^r} \leq 4C\sum_{l=1}^{\infty} \frac{\alpha_l}{\beta_l^r} < \infty,$$

hence by monotone convergence theorem, we have

$$E\left(\sup_{n\geq 1}|\frac{S_n}{\beta_n}|\right)^r = \lim_{n\to\infty} E\left(\max_{1\leq l\leq n}|\frac{S_l}{\beta_l}|\right)^r \leq 4C\sum_{n=1}^{\infty} \frac{\alpha_n}{\beta_n^r} < \infty,$$

so that

$$\sup_{n\geq 1}|\frac{S_n}{\beta_n}| < \infty, a.s.,$$

and

$$0 \leq |\frac{S_n}{\beta_n}| \leq \frac{\beta_n}{b_n}\sup_{n\geq 1}|\frac{S_n}{\beta_n}| = O(\frac{\beta_n}{b_n}), a.s.,$$

this completes the proof.

Remark1: Theorem 2.1 improve result of Ryozo,Y.(Corollary 2)[8] and from strictly stationary strong mixing sequence to S_p system.

Theorem 2.2 Let (X_n, F_n) be a Locally Generalized Gaussian sequence, if $sup_n\{X_n\} = k < \infty$, then for any $r \geq 2$

$$E|\sum_{i=a+1}^{a+n} C_i X_i|^r \leq K_r(\sum_{i=a+1}^{a+n} C_i^2)^{r/2} \qquad (11)$$

furthermore,(9) and (10) hold.

Proof Let $A_n = \sum_{i=a+1}^{a+n} C_i^2, u = x/k^2 A_n$,by lemma 1 in [5], then

$$E(\exp(u\sum_{i=a+1}^{a+n} C_i X_i)) = E(\exp(uS_n)) \leq \exp(u^2 k^2 A_n/2)$$

where

$$S_n = \sum_{i=a+1}^{a+n} C_i X_i, \text{for } r \geq 2,$$

by Chebyshev's inequality, we get

$$E|\sum_{i=a+1}^{a+n} C_i X_i|^r = r \int_0^\infty x^{r-1} P(|S_n| > x)dx$$

$$\leq 2r \int_0^\infty x^{r-1} \exp(-x^2/2k^2 A_n)dx$$

$$= 2^{r/2} r k^r A_n^{r/2} \int_0^\infty x^{r/2-1} \exp(-x)dx$$

$$= K_r(\sum_{i=a+1}^{a+n} C_i^2)^{r/2}$$

where $K_r = 2^{r/2} r k^r \int_0^\infty x^{r/2-1} \exp(-x)dx$,

Therefore, Locally Generalized Gaussian sequence is a Lacunary system, by Theorem 2.1, (9) and (10) hold.

Theorem 2.3 Let $\{X_n, n \geq 1\}$ be a NA sequence, satisfying $\sup_n |X_n|^p < \infty$, then for any $0 < p < 2, \alpha p \geq 1, p/2 < \delta \leq 1, x > 0$,

$$\sum_{n=1}^\infty n^{\alpha p-(1+\delta)} P\{|S_n| \geq xn^\alpha\} < \infty, \qquad (12)$$

where $S_n = \sum_{i=1}^n X_i$.

Proof From lemma 1.3, when $C_i \equiv 1$, we have

$$E|\sum_{i=a+1}^{a+n} X_i|^p \leq K_p n^{p/2},$$

By Markov's inequality we get

$$\sum_{n=1}^\infty n^{\alpha p-(1+\delta)} P\{|S_n| \geq xn^\alpha\} \leq \sum_{n=1}^\infty \frac{n^{\alpha p-1-\delta} E|S_n|^p}{x^p n^{\alpha p}}$$

$$\leq K_p x^{-p} \sum_{n=1}^\infty \frac{n^{\alpha p-1-\delta} n^{p/2}}{n^{\alpha p}}$$

$$= K_p x^{-p} \sum_{n=1}^\infty \frac{1}{n^{1+(\delta-p/2)}} < \infty.$$

Theorem 2.4 Let $\{X_n, n \geq 1\}$ be a m_0 dependent sequence with zero mean, if $sup_n|X_n|^r < \infty$, then for any $1 \leq r < 2$,

$$\frac{S_n}{n^{1/r}} \to 0(n \to \infty), a.s., \qquad (13)$$

where $S_n = \sum_{i=1}^n X_i$.

Proof for every $t = 0, 1, \cdots, m_0 - 1, \{X_{t+m+m_0 k}, k = 1, 2, \cdots\}$ be a independent sequence, and

$$\sum_{j=m}^n C_j X_j = \sum_{t=0}^{m_0-1} \sum_{k=0}^{n_0} C_{m+t+m_0 k} X_{m+t+m_0 k},$$

by C_r inequality, we get

$$E(|\sum_{j=m}^n C_j X_j|)^2 \leq m_0 \sum_{t=0}^{m_0-1} E(\sum_{k=0}^{n_0} C_{m+t+m_0 k} X_{m+t+m_0 k})^2$$

$$= m_0 \sum_{t=0}^{m_0-1} \sum_{k=0}^{n_0} C_{m+t+m_0 k}^2 EX_{m+t+m_0 k}^2$$

$$= m_0 \sum_{j=m}^n C_j^2 EX_j^2 \leq km_0 \sum_{j=m}^n C_j^2,$$

$\{X_n, n \geq 1\}$ be a S_2 system, by Jensen's inequality, for $1 \leq r < 2$,

$$E(|\sum_{j=m}^n C_j X_j|)^r \leq C(\sum_{j=m}^n C_j^2)^{r/2},$$

in particular, when $C_j \equiv 1$, we obtained

$$E|S_n|^r \leq Cn^{r/2}.$$

In Definition 1.1 take

$$g_n = g(b, n) = K^{2/r} n, \quad \alpha = \frac{r}{2}, \quad b_n = n^{1/r},$$

then

$$g(b, k) = K^{2/r} k, g(b+k, l) = K^{2/r} l$$

$$g(b, k) + g(b+k, l) = K^{2/r}(k+l) \leq g(b, k+l),$$

$$E|S_{b,n}|^r \leq Kn^{r/2} = g^{r/2}(b, n), n \geq 1, 1 \leq r < 2,$$

we know that $\{X_n, n \geq 1\}$ has the pth moment function of superadditive structure, and

$$\sum_{n=1}^\infty \frac{g_n^{r/2} - g_{n-1}^{r/2}}{b_n^r} = \sum_{n=1}^\infty \frac{K(n^{r/2} - (n-1)^{r/2})}{n}$$

$$\leq C \sum_{n=1}^\infty \frac{n^{r/2-1}}{n} = C \sum_{n=1}^\infty \frac{1}{n^{2-r/2}} < \infty,$$

thus (13) follows from Lemma1.2.

Remark 2 This result extends independent and identically distributed Marcinkiewicz Law of large numbers for m_0-dependent sequences.

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