International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:12, No:11, 2018

State Estimation Based on Unscented Kalman Filter for Burgers' Equation

Takashi Shimizu, Tomoaki Hashimoto

***Abstract*—** Controlling the flow of fluids is a challenging problem that arises in many fields. Burgers' equation is a fundamental equation for several flow phenomena such as traffic, shock waves, and turbulence. The optimal feedback control method, so-called model predictive control, has been proposed for Burgers' equation. However, the model predictive control method is inapplicable to systems whose all state variables are not exactly known. In practical point of view, it is unusual that all the state variables of systems are exactly known, because the state variables of systems are measured through output sensors and limited parts of them can be only available. In fact, it is usual that flow velocities of fluid systems cannot be measured for all spatial domains. Hence, any practical feedback controller for fluid systems must incorporate some type of state estimator. To apply the model predictive control to the fluid systems described by Burgers' equation, it is needed to establish a state estimation method for Burgers' equation with limited measurable state variables. To this purpose, we apply unscented Kalman filter for estimating the state variables of fluid systems described by Burgers' equation. The objective of this study is to establish a state estimation method based on unscented Kalman filter for Burgers' equation. The effectiveness of the proposed method is verified by numerical simulations.

***Keywords*—**State estimation, fluid systems, observer systems, unscented Kalman filter.

I. Introduction

CONTROLLING fluid dynamics is a challenging problem that arises in many fields such as physical, biological, and chemical systems. Burgers' equation is known as the fundamental partial differential equation that can be used to model various flow phenomena. Burgers' equation consists of the advective and diffusive terms, which can be used to represent fundamental properties of flow phenomena. Hence, using Burgers' equation can be regarded as a natural first step towards developing a method for controlling flows.

Model predictive control (MPC) is a well-established control method that optimizes control performance over a finite future horizon, and its performance index has moving initial and terminal times. In recent years, several MPC methods have been proposed for fluid systems [1]-[5], spatiotemporal dynamic systems [6]-[10], Schrödinger systems [11], [12], stochastic systems [13]-[15], and probabilistic constrained systems [16]-[18]. In particular, the MPC method for Burgers' equation has been proposed in [2]. However, the MPC method proposed in [2] is inapplicable to systems whose all state variables are not exactly known.

In general, it is usual that the state variables of systems are measured through output sensors, hence, only limited parts of them can be used for designing control inputs. In fact, it is unrealistic that the flow velocities of fluid systems are exactly known for all spatial domains. Hence, it should be supposed that the flow velocities of limited parts of spatial domain can be only used for designing control inputs.

In order to apply the MPC method proposed in [2] to the fluid systems described by Burgers' equation, we need to establish a state estimation method for Burgers' equation with limited measurable state variables. The objective of this study is to establish a state estimation method for Burgers' equation. For this purpose, we introduce an observer system for estimating the state variables of Burgers' equation.

Kalman filter is a well-known optimal estimation method that enable us to minimize the estimation errors with taking the process noise and sensor noise into consideration. The application of the Kalman filter to nonlinear systems has been studied in several decades. The simple approach is to use the Extended Kalman Filter (EKF) [19] which simply linearize all nonlinear models so that the traditional linear Kalman filter can be applied. However, it is difficult to implement the EKF to high-dimensional nonlinear systems because the computation of linearization is impracticable. Also, the EKF is only reliable for systems which are almost linear on the time scale of the update intervals. On the other hand, the different approach is to use the Unscented Kalman Filter (UKF) [20] which uses a set of appropriately chosen weighted points to parameterize the means and covariances of probability distributions. Using UKF, the estimator yields performance equivalent to the Kalman filter for linear systems yet generalizes to nonlinear systems without the linearization steps required by the EKF.

In fact, Burgers' equation is a partial differential equation that can be discretized into the high-dimensional nonlinear systems. Therefore, we apply the UKF to the state estimation method for Burgers' equation. The objective of this study is to propose a state estimation method based on the UKF for Burgers' equation.

This paper is organized as follows. In Section II, we define the system model and notations. In Section III, we consider the state estimation problem of Burgers' equation. In Section IV, we provide the results of numerical simulations that verify the effectiveness of the proposed method. Finally, some concluding remarks are given in Section V.

This work was supported in part by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C), Grant Number JP18K04216.

T. Shimizu and T. Hashimoto are with the Department of Mechanical Engineering, Osaka Institute of Technology, Asahi-ku 535-8585, Japan (e-mail: tomoaki.hashimoto@oit.ac.jp).

II. Notation and System Model

For a matrix A, the transpose of A is denoted by A^{T}. Let $s = [s_1, s_2]^{\mathrm{T}}$ and t denote a spatial vector and temporal variable, respectively. Let $v(t, s)$ denote the flow velocity. Let ν be the constant parameter that denote the kinematic viscosity. These system parameters are listed in Table I.

In this study, we restrict our attention to the range $0 \leq s_i \leq \ell$ for $i = 1, 2$. Let Ω be the set defined by

$$\Omega := \prod_{i=1}^{2} \{s_i | 0 \leq s_i \leq \ell\}.$$

Then, we consider the following system of two-dimensional Burgers' equation:

$$\frac{\partial v_1}{\partial t}(t,s) = -\left(v_1 \frac{\partial v_1}{\partial s_1}(t,s) + v_2 \frac{\partial v_1}{\partial s_2}(t,s)\right) + \nu \left(\frac{\partial^2 v_1}{\partial s_1^2}(t,s) + \frac{\partial^2 v_1}{\partial s_2^2}(t,s)\right) \tag{1a}$$

$$\frac{\partial v_2}{\partial t}(t,s) = -\left(v_1 \frac{\partial v_2}{\partial s_1}(t,s) + v_2 \frac{\partial v_2}{\partial s_2}(t,s)\right) + \nu \left(\frac{\partial^2 v_2}{\partial s_1^2}(t,s) + \frac{\partial^2 v_2}{\partial s_2^2}(t,s)\right) \tag{1b}$$

TABLE I
System Parameters

t	temporal variable
s	spatial vector
$v(t, s)$	flow velocity
ν	kinematic viscosity

The boundary conditions are considered as follows:

$$v = 0 \quad \text{for } s_1 = 0 \tag{2a}$$
$$v = 0 \quad \text{for } s_1 = \ell \tag{2b}$$
$$v = 0 \quad \text{for } s_2 = 0 \tag{2c}$$
$$v = 0 \quad \text{for } s_2 = \ell \tag{2d}$$

A schematic view of system model is shown in Fig. 1.

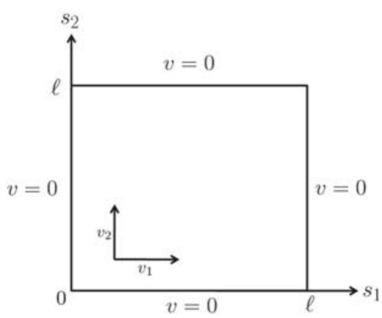

Fig. 1 A schematic view of system model

Next, the output function $y(s, t)$ are introduced as follows:

$$y(s,t) = c(s)x(s,t). \tag{3}$$

Note that $c(s)$ is a space-dependent coefficient that is introduced to account for restrictions on the allocation of sensors.

It is well known that Burgers' equation can be discretized into finite difference equation using the Crank-Nicolson finite-difference approximation method.

Let $s \in \Omega$ be divided into n grid points, where n is a positive integer. Let $\bar{s} := [\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_n]^{\mathrm{T}}$ denote the spatial vector obtained by the discretization of s. Likewise, let τ denote the discrete-time obtained by the sampling time Δt. Consequently, the discretized state can be obtained using $\bar{s}$ and τ. Let $\bar{x}$ be defined by

$$\bar{x}(\tau) = \begin{bmatrix} \bar{x}_1(\tau) \\ \bar{x}_2(\tau) \\ \vdots \\ \bar{x}_n(\tau) \end{bmatrix} := \begin{bmatrix} x(\bar{s}_1, \tau) \\ x(\bar{s}_2, \tau) \\ \vdots \\ x(\bar{s}_n, \tau) \end{bmatrix}.$$

Applying the Crank-Nicolson method and using the discretized state $\bar{x}$, we obtain the following discretized system model:

$$A(\bar{x}(\tau))\bar{x}(\tau+1) = B(\bar{x}(\tau)), \tag{4a}$$
$$\bar{y}(\tau) = C\bar{x}(\tau), \tag{4b}$$

where A, B, and C are system coefficient matrix.

The objective of this study is to propose a state estimation method for system model (4).

III. Estimation Based on Unscented Kalman Filter

In this section, we propose a state estimation method based on the UKF for system model (4). First, we introduce the following observer system:

$$A(\hat{x}(\tau))\hat{x}(\tau+1) = B(\hat{x}(\tau)) + z(\tau), \tag{5a}$$
$$\hat{y}(\tau) = C\hat{x}(\tau) + w(\tau), \tag{5b}$$

where $\hat{x}$ and $\hat{y}$ denote the estimated state and output of $\bar{x}$ and $\bar{y}$, respectively. Moreover, z and w denote the process noise and the observation noise, respectively, which can be caused by disturbances.

In the minimum mean-squared error sense, the optimal state estimate is given by the conditional mean.

Let $\hat{x}(i|j)$ be the mean of $\hat{x}(i)$ conditioned on all of the observations up to and including time j, i.e., $\hat{x}(i|j) = \mathbb{E}\left[\hat{x}(i)|\mathbf{Y}^j\right]$, where $\mathbf{Y}^j := \{\bar{y}(1), \bar{y}(2), \cdots, \bar{y}(j)\}$.

It is assumed that the means of $z(\tau)$ and $w(\tau)$ are zero for all time τ. Let $\mathbf{Q}^z(\tau)$ and $\mathbf{Q}^w(\tau)$ be the covariances of $z(\tau)$ and $w(\tau)$, respectively.

The UKF [20] first predicts the mean and covariance of a future state using the process model and weighted sigma points as follows:

$$A(\chi^i(\tau))\chi^i(\tau+1|\tau) = B(\chi^i(\tau)), \tag{6}$$

$$\hat{x}(\tau+1|\tau) = \sum_{i=0}^{2n} W^i \chi^i(\tau+1|\tau), \tag{7}$$

$$\mathbf{Q}^{\hat{x}}(\tau+1|\tau) = \mathbf{Q}^z(\tau+1) + \sum_{i=0}^{2n} W^i \left(\chi^i(\tau+1|\tau) - \hat{x}(\tau+1|\tau)\right)\left(\chi^i(\tau+1|\tau) - \hat{x}(\tau+1|\tau)\right)^{\mathrm{T}}, \tag{8}$$

where W^i and χ^i denote the weight and sigma point, respectively. The definitions of W^i and χ^i can be found in [20].

$\chi^i(\tau+1|\tau)$ can be determined from (6). Then, $\hat{x}(\tau+1|\tau)$ and $\mathbf{Q}^{\hat{x}}(\tau+1|\tau)$ are determined form (7) and (8), respectively.

After we redraw a new set of sigma points $\bar{\chi}^i$ to incorporate the effect of the additive process noise, the predicted observation is calculated by

$$\hat{y}(t+1|t) = \sum_{i=0}^{2n} W^i C(\bar{\chi}^i(\tau+1|\tau)). \tag{9}$$

Moreover, the cross covariance $\mathbf{P}$ and innovation covariance $\mathbf{R}$ are determined by

$$\begin{aligned}\mathbf{P}(\tau+1|\tau) = &\sum_{i=0}^{2n} W^i \left(\chi^i(\tau+1|\tau) - \hat{x}(\tau+1|\tau)\right)\\ &\times \left(C(\bar{\chi}^i(\tau+1|\tau)) - \hat{y}(\tau+1|\tau)\right)^{\mathrm{T}} \end{aligned} \tag{10}$$

$$\begin{aligned}\mathbf{R}(\tau+1|\tau) = &\sum_{i=0}^{2n} W^i \left(C(\bar{\chi}^i(\tau+1|\tau)) - \hat{y}(\tau+1|\tau)\right)\\ &\times \left(C(\bar{\chi}^i(\tau+1|\tau)) - \hat{y}(\tau+1|\tau)\right)^{\mathrm{T}}\\ &+ \mathbf{Q}^w(\tau+1) \end{aligned} \tag{11}$$

Consequently, the state estimate at time $\tau+1$ is obtained by updating the prediction by the linear update rule:

$$\mathbf{K}(\tau+1) = \mathbf{P}(\tau+1|\tau)\mathbf{R}^{-1}(\tau+1|\tau), \tag{12a}$$

$$\hat{x}(\tau+1|\tau+1) = \hat{x}(\tau+1|\tau) + \mathbf{K}(\tau+1)\left(\bar{y}(\tau+1) - \hat{y}(\tau+1|\tau)\right), \tag{12b}$$

$$\mathbf{Q}^{\hat{x}}(\tau+1|\tau+1) = \mathbf{Q}^{\hat{x}}(\tau+1|\tau) - \mathbf{K}(\tau+1)\mathbf{R}(\tau+1|\tau)\mathbf{K}^{\mathrm{T}}(\tau+1). \tag{12c}$$

Note that the UKF is easier to implement than an EKF, because UKF does not involve any linearization steps, eliminating the need to derive of the Jacobian matrix of $A^{-1}(\hat{x})B(\hat{x})$.

IV. Numerical Simulations

In this section, we provide numerical simulation results to verify the effectiveness of the proposed method.

The two-dimensional square domain is set as $\Omega := [0 \;\; 1] \times [0 \;\; 1]$. Here, we set the initial state and the initial estimated state as follows:

$$x(0) = \begin{bmatrix} -\cos(\pi s_1)\sin(\pi s_2) \\ \sin(\pi s_1)\cos(\pi s_2) \end{bmatrix}, \tag{13}$$

$$\hat{x}(0) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}. \tag{14}$$

Furthermore, we choose $c(s)$ so that the state variables at the points $(x_1, x_2) = (0,0), (1,0), (0,1), (1,1)$ can be measured from the outputs.

Other parameters employed in the numerical simulations are as follows: $\Delta t = 0.01$, $n = 225$, $\nu = 0.05$. z and w are set as zero-mean Gaussian noises with covariances $\mathbf{Q}^z = 0.001\mathbf{I}$ and $\mathbf{Q}^w = 0.01\mathbf{I}$, respectively, where $\mathbf{I}$ is the identity matrix. The results of numerical simulations by the proposed method are shown below.

In Figs. 2-19, the solid blue and dashed red arrows show the time histories of the real state x and the estimated state $\hat{x}$, respectively. We can see that the estimated state $\hat{x}$ converges to the real state x. Fig. 20 shows the time history of the estimate error. These figures reveal the effectiveness of the proposed method.

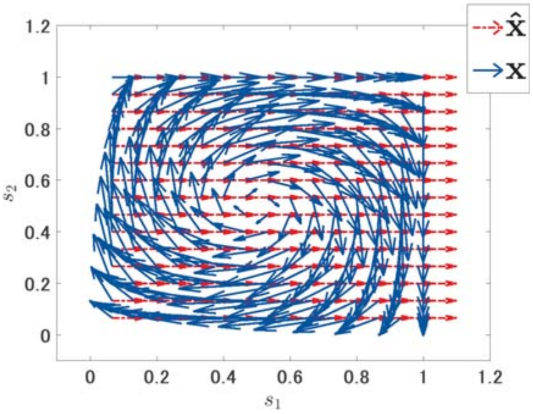

Fig. 2 Flow velocity at $t = 0$

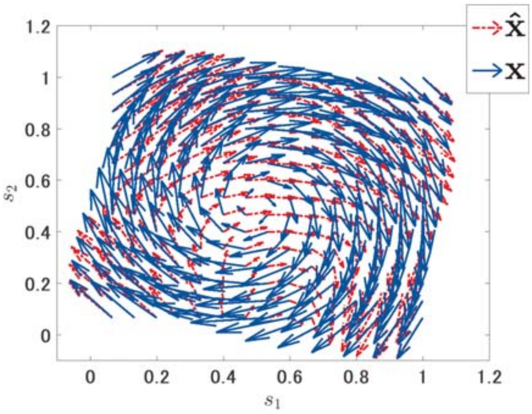

Fig. 3 Flow velocity at $t = 0.2$

V. Conclusion

The model predictive control method proposed in [2] for Burgers' equation is inapplicable when all state variables are not exactly known. In general, the state variables of systems are measured through output sensors, hence, only limited parts of them can be directly known. Thus, it is unrealistic that the flow velocities of fluid systems are exactly known for all spatial domains. Hence, it should be assumed that the flow velocities of limited parts of spatial domain can be only known.

To apply the MPC method proposed in [2] to the fluid systems described by Burgers' equation, we need to establish a state estimation method for Burgers' equation with limited measurable state variables. In this study, we established a state estimation method for Burgers' equation. We proposed a state observer system using the unscented Kalman filter for estimating the state of Burgers' equation. The effectiveness of the proposed method was verified by numerical simulations.

It is known that time delays may cause instabilities of the state observer system and lead to more complex analysis [21]-[26]. The state estimation problem of the systems with time delays is a possible future work.

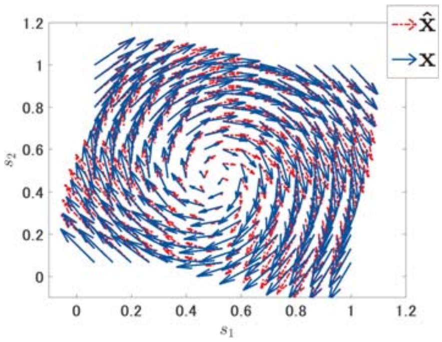

Fig. 4 Flow velocity at $t = 0.4$

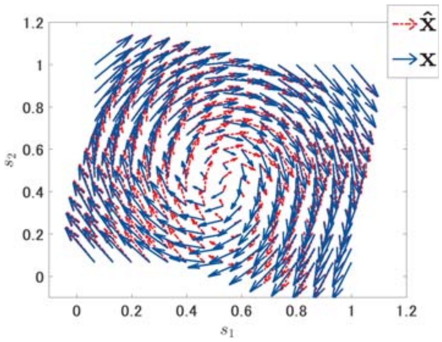

Fig. 5 Flow velocity at $t = 0.6$

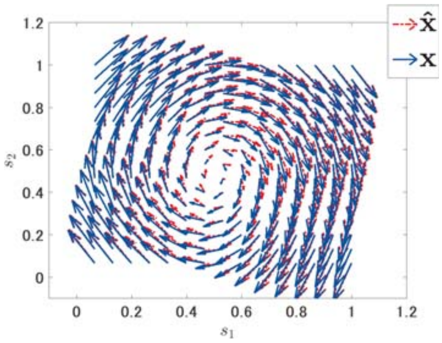

Fig. 6 Flow velocity at $t = 0.8$

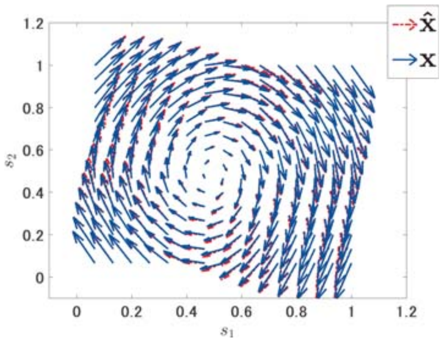

Fig. 7 Flow velocity at $t = 1.0$

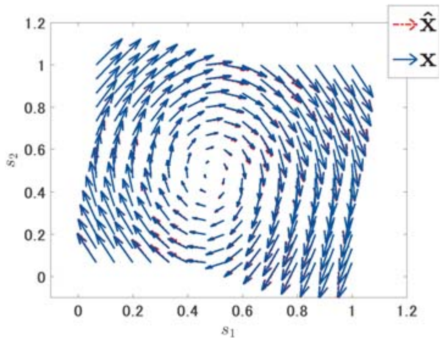

Fig. 8 Flow velocity at $t = 1.2$

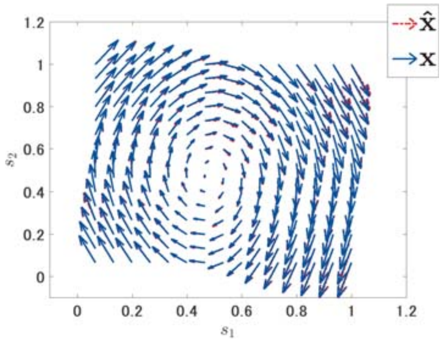

Fig. 9 Flow velocity at $t = 1.4$

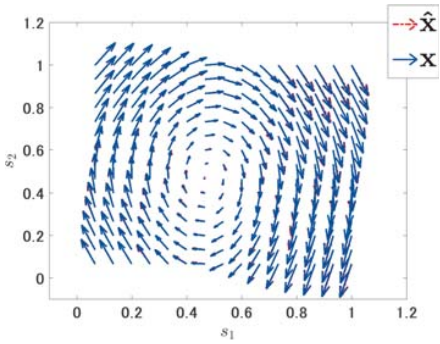

Fig. 10 Flow velocity at $t = 1.6$

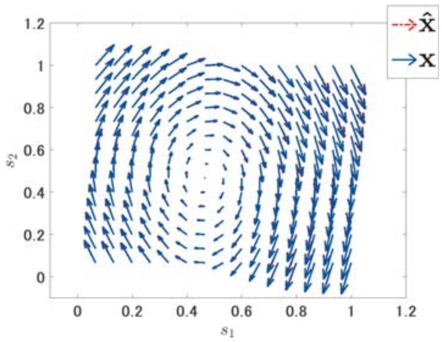

Fig. 11 Flow velocity at $t = 1.8$

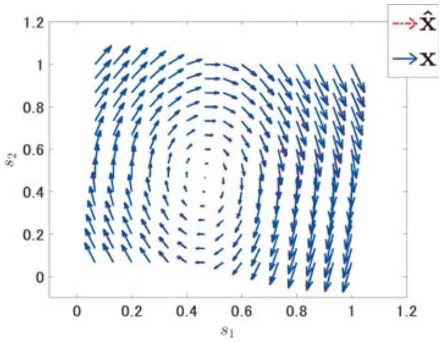

Fig. 12 Flow velocity at $t = 2.0$

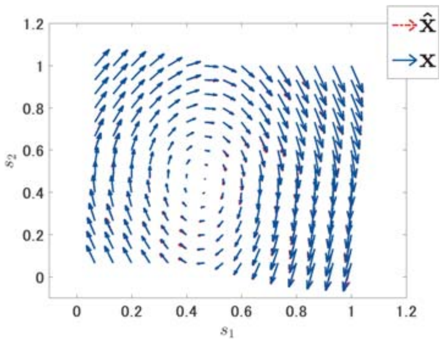

Fig. 13 Flow velocity at $t = 2.2$

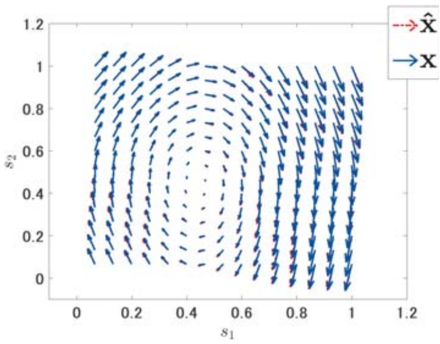

Fig. 14 Flow velocity at $t = 2.4$

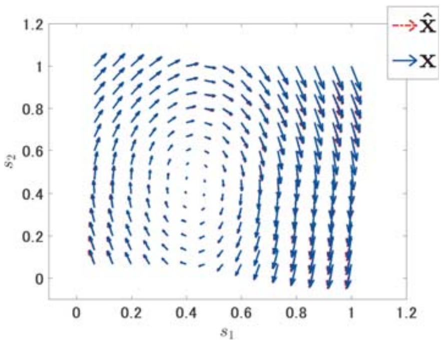

Fig. 15 Flow velocity at $t = 2.6$

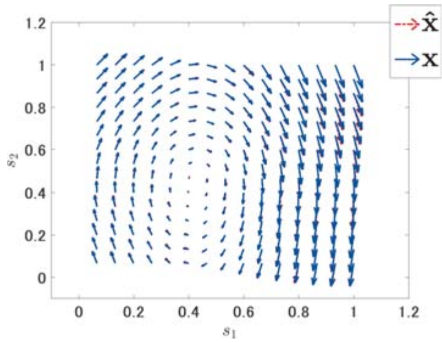

Fig. 16 Flow velocity at $t = 2.8$

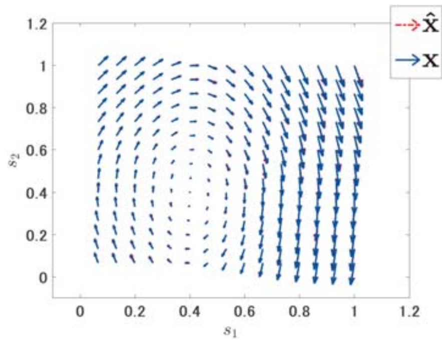

Fig. 17 Flow velocity at $t = 3.0$

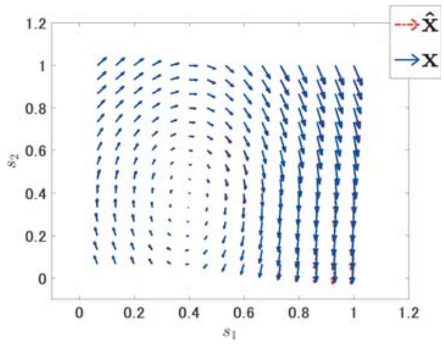

Fig. 18 Flow velocity at $t = 3.2$

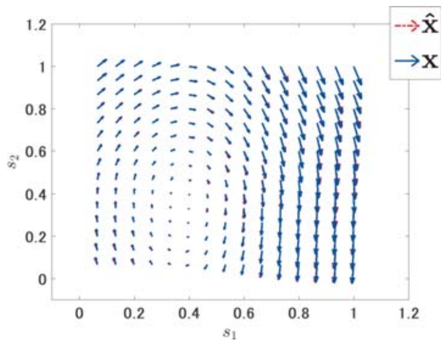

Fig. 19 Flow velocity at $t = 3.4$

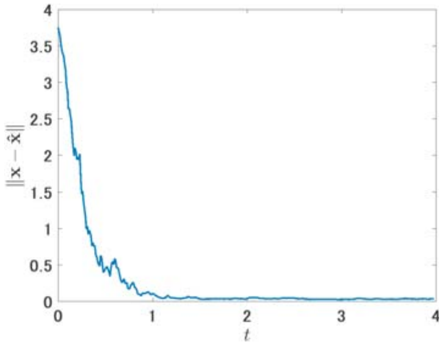

Fig. 20 Time history of state estimation error

REFERENCES

[1] T. Hashimoto, Y. Yoshioka, T. Ohtsuka, Receding Horizon Control with Numerical Solution for Thermal Fluid Systems, *Proceedings of SICE Annual Conference*, pp. 1298-1303, 2012.

[2] T. Hashimoto, Y. Takiguchi and T. Ohtsuka, Receding Horizon Control for High-Dimensional Burgersf Equations with Boundary Control Inputs, *Transactions of the Japan Society for Aeronautical and Space Sciences*, Vol. 56, No.3, pp. 137-144, 2013.

[3] R. Satoh, T. Hashimoto and T. Ohtsuka, Receding Horizon Control for Mass Transport Phenomena in Thermal Fluid Systems, *Proceedings of Australian Control Conference*, pp. 273-278, 2014.

[4] T. Hashimoto, Receding Horizon Control for a Class of Discrete-time Nonlinear Implicit Systems, *Proceedings of IEEE Conference on Decision and Control*, pp. 5089-5094, 2014.

[5] T. Hashimoto, Optimal Feedback Control Method Using Magnetic Force for Crystal Growth Dynamics, *International Journal of Science and Engineering Investigations*, Vol. 4, Issue 45, pp. 1-6, 2015.

[6] T. Hashimoto, Y. Yoshioka, T. Ohtsuka, Receding Horizon Control with Numerical Solution for Spatiotemporal Dynamic Systems, *Proceedings of IEEE Conference on Decision and Control*, pp. 2920-2925, 2012.

[7] T. Hashimoto, Y. Takiguchi and T. Ohtsuka, Output Feedback Receding Horizon Control for Spatiotemporal Dynamic Systems, *Proceedings of Asian Control Conference*, 2013.

[8] T. Hashimoto, Y. Yoshioka and T. Ohtsuka, Receding Horizon Control for Hot Strip Mill Cooling Systems, *IEEE/ASME Transactions on Mechatronics*, Vol. 18, No. 3, pp. 998-1005, 2013.

[9] T. Hashimoto, Y. Yoshioka and T. Ohtsuka, Receding Horizon Control With Numerical Solution for Nonlinear Parabolic Partial Differential Equations, *IEEE Transactions on Automatic Control*, Vol. 58, No. 3, pp. 725-730, 2013.

[10] T. Hashimoto, R. Satoh and T. Ohtsuka, Receding Horizon Control for Spatiotemporal Dynamic Systems, *Mechanical Engineering Journal*, Vol. 3, No. 2, 15-00345, 2016.

[11] T. Hashimoto, I. Yoshimoto, T. Ohtsuka, Probabilistic Constrained Model Predictive Control for Schrödinger Equation with Finite Approximation, *Proceedings of SICE Annual Conference*, pp. 1613-1618, 2012.

[12] T. Hashimoto, Stability of Stochastic Model Predictive Control for Schrödinger Equation with Finite Approximation, *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering*, Vol. 11, No. 1, pp. 12-17, 2017.

[13] T. Hashimoto, Probabilistic Constrained Model Predictive Control for Linear Discrete-time Systems with Additive Stochastic Disturbances, *Proceedings of IEEE Conference on Decision and Control*, pp. 6434-6439, 2013.

[14] T. Hashimoto, Computational Simulations on Stability of Model Predictive Control for Linear Discrete-time Stochastic Systems, *International Journal of Computer, Electrical, Automation, Control and Information Engineering*, Vol. 9, No. 8, pp. 1385-1390, 2015.

[15] T. Hashimoto, Stochastic Model Predictive Control for Linear Discrete-time Systems with Random Dither Quantization, *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering*, Vol. 11, No. 2, pp. 130-134, 2017.

[16] T. Hashimoto, Conservativeness of Probabilistic Constrained Optimal Control Method for Unknown Probability Distribution, *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering*, Vol. 9, No. 9, pp. 11-15, 2015.

[17] T. Hashimoto, A Method for Solving Optimal Control Problems subject to Probabilistic Affine State Constraints for Linear Discrete-time Uncertain Systems, *International Journal of Mechanical and Production Engineering*, Vol. 3, Issue 12, pp. 6-10, 2015.

[18] T. Hashimoto, Solutions to Probabilistic Constrained Optimal Control Problems Using Concentration Inequalities, *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering*, Vol. 10, No. 10, pp. 441-446, 2016.

[19] H. W. Sorenson, Ed., Kalman Filtering: Theory and Application, Piscataway, NJ: IEEE, 1985.

[20] S. Julier, J. Uhlmann and H. F. Durrant-Whyte, A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators, *IEEE Transactions on Automatic Control*, Vol. 45, 2000, pp. 477-482.

[21] T. Hashimoto, T. Amemiya and H. A. Fujii, Stabilization of Linear Uncertain Delay Systems with Antisymmetric Stepwise Configurations, *Journal of Dynamical and Control Systems*, Vol. 14, No. 1, pp. 1-31, 2008.

[22] T. Hashimoto, T. Amemiya and H. A. Fujii, Output Feedback Stabilization of Linear Time-varying Uncertain Delay Systems, *Mathematical Problems in Engineering*, Vol. 2009, Article ID. 457468, 2009.

[23] T. Hashimoto and T. Amemiya, Stabilization of Linear Time-varying Uncertain Delay Systems with Double Triangular Configuration, *WSEAS Transactions on Systems and Control*, Vol. 4, No.9, pp.465-475, 2009.

[24] T. Hashimoto, Stabilization of Abstract Delay Systems on Banach Lattices using Nonnegative Semigroups, *Proceedings of the 50th IEEE Conference on Decision and Control*, pp. 1872-1877, 2011.

[25] T. Hashimoto, A Variable Transformation Method for Stabilizing Abstract Delay Systems on Banach Lattices, *Journal of Mathematics Research*, Vol. 4, No. 2, pp.2-9, 2012.

[26] T. Hashimoto, An Optimization Algorithm for Designing a Stabilizing Controller for Linear Time-varying Uncertain Systems with State Delays, *Computational Mathematics and Modeling*, Vol.24, No.1, pp.90-102, 2013.