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Stability of Discrete Linear Systems with Periodic Coefficients under Parametric Perturbations

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***Abstract*—**This paper studies the problem of exponential stability of perturbed discrete linear systems with periodic coefficients. Assuming that the unperturbed system is exponentially stable we obtain conditions on the perturbations under which the perturbed system is exponentially stable.

***Keywords*—**Exponential stability, time-varying linear systems, periodic systems.

I. Introduction

THE theory of linear discrete-time periodic systems has received a lot of attention in the last years (see, for example [4], [11] and the references therein). In the present paper we study certain problem of robust stability of such systems.

Consider a system described by the following linear difference equation

$$x(n+1) = A(n)x(n), \tag{1}$$

where the s -by- s real matrix $A(n)$ is periodic with period T. Assume that system (1) is exponentially stable. An important problem in robustness analysis is that of determining the extent to which exponential stability is preserved under various types of parameter perturbations. To model such perturbations we consider a model of the form

$$x(n+1) = \big(A(n) + \Delta(n)\big)x(n), \tag{2}$$

where $\Delta(n)$, $n = 0,1,...$ is a sequence of s -by- s real matrices which model the parameters perturbations. The question is how large this perturbation may be without destroying stability or more precisely we are looking for the largest bound r such that stability is preserved for all perturbations $\Delta(n)$ of norm strictly less than r in a given normed perturbation set. This largest bound is called the stability radius. First time the problem was formally formulated for the continuous time invariant system in the famous paper of Hinrichsen and Pritchard [6] and since then many results have been obtained. The present state of the problem and discussion of the related literature an interested reader will find in [7]. Most of the results in the literature are about time invariant systems.

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For such systems there are formulas available for the stability radius with respect to different classes of perturbation. In particular numerical methods of calculating the stability radius for time invariant systems are good developed. Much less is known about calculating time varying stability radius (see [12]). Therefore any bounds for this quantity are very important.

Similar problem to that we consider in the present work has been investigated in [2]. However the proposed there results require full knowledge about matrices $\Delta(n)$ whereas our results are formulated in terms of upper bounds for the norm of $\Delta(n)$. In the present paper we do not assume the periodicity of perturbation sequence and present the norm bound for the perturbation which guarantees stability of the perturbed system.

II. Preliminary Results

For a sequence $A = (A(0), A(1),...)$ of matrices denote transition matrix

$$\Phi_A(m,k) = A(m-1)...A(k)$$

for $m > k$ and $\Phi(m,m) = I$, where I is the identity matrix. Denote by $\|\|$ a vector norm in R^s and the induced operator norm. By $\rho(A)$ we will denote the spectral radius of a matrix A .

Definition 1 We call the system

$$x(n+1) = A(n)x(n), \tag{3}$$

uniformly exponentially stable, if there exist constants C, $\omega > 0$, $\omega < 1$ such that $\|\Phi_A(m,k)\| \le C\omega^{m-k}$ for all $m,k = 0,1,...,$ $m \ge k$.

It is well known (see, for example, [3]) that for $T-$ periodic sequence $A(n)$, system (1) is uniformly exponentially stable if and only if the spectral radius of the monodromy matrix $A(T-1)...A(0)$ is strictly less then one i.e. $\rho(A(T-1)...A(0)) < 1$ and it is equivalent to asymptotic stability. It is also well known that in general case the uniformly exponential stability and asymptotic stability are not equivalent (see, for example, [3]). The following lemma is a straightforward consequence of Definition 1.

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Lemma 1 If sequence $A(n)$ is $T-$ periodic and $\Delta(n)$ is bounded, then system (1) is uniformly exponentially stable if and only if there exist constants C, $\omega > 0$, $\omega < 1$ such that

$$\left\|\Phi_{A+\Delta}(mT,kT)\right\| \le C\omega^{(m-k)T}$$

for all $m,k = 0,1,...,$ $m \ge k.$

In our further consideration we will use the following discrete version of Gronwall's inequality(see [3]).

Theorem 1 Suppose that for two sequences $u(n)$ and $f(n)$, $n = k_0, k_0+1,...$ of nonnegative numbers the following inequality

$$u(n) \le p + q\sum_{i=k_0}^{n-1} u(i) f(i)$$

holds for certain $p,q \in R$ and all $n = k_0, k_0+1,...$, then

$$u(n) \le p\prod_{i=k_0}^{n-1}\left(1+qf(i)\right) \tag{4}$$

for all $n = k_0, k_0+1,...$.

III. Main Results

The next theorem contains the main result of this paper. To formulate it, let introduce the following notations

$$\Delta = \sup\left\|\Delta(n)\right\|, K = \max_{i=0,...,T}\left\|\Phi_A(T,i)\right\|.$$

Theorem 2 Consider system (1) with $T-$ periodic sequence $A(n)$. If matrix $B = A(T-1)...A(0)$ is such that for certain operator norm $\left\|\cdot\right\|$ there exist $C, \omega > 0$, $\omega < 1$ such that

$$\left\|B^n\right\| \le C\omega^{Tn} \tag{5}$$

for all $n = 0,1,...$, and

$$\omega + CK\Delta\omega^{-T+1} < 1 \tag{6}$$

then system (2) is uniformly exponentially stable.

Proof We can rewrite (2) in the following form

$$x(n+1) = A(n)x(n) + \Delta(n)x(n),$$

and for $x(k) = x_0$, we have

$$x(m) = \Phi_A(m,k)x_0 + \sum_{i=k}^{m-1}\Phi_A(m,i+1)\Delta(i)x(i)$$

for all $m \ge k$. For natural number i denote by $p(i)$ and $r(i)$ the quotient and the remainder of the division i by T, that is, $i = p(i)T + r(i)$. Moreover let define

$$q(i) = \begin{cases} p(i)+1 \text{ if } r(i) \ne 0 \\ p(i) \text{ if } r(i) = 0 \end{cases}.$$

With this notation we have

$$\Phi_A\left(mT,k\right) = B^{m-q(k)}\Phi_A(q(k)T,k)$$

and consequently

$$x(mT) = B^{m-q(k)}\Phi_A(q(k)T,k)x_0 +$$

$$\sum_{i=k}^{mT-1} B^{m-q(i+1)}\Phi_A(q(i+1)T,i+1)\Delta(i)x(i). \tag{7}$$

Definition of $q(i)$ and periodicity of A imply

$$\Phi_A(q(i+1)T,i+1) = \begin{cases} \Phi_A(T,r(i+1)) \text{ if } r(i) \ne 0 \\ I \text{ if } r(i) = 0 \end{cases}$$

and consequently

$$\left\|\Phi_A(q(i+1)T,i+1)\right\| \le K. \tag{8}$$

Hence by the assumption (5) and (7)-(8) we have

$$\left\|x(mT)\right\| \le CK\omega^{(m-q(k))T}\left\|x_0\right\| +$$

$$CK\Delta\sum_{i=k}^{mT-1}\omega^{(m-q(i+1))T}\left\|x(i)\right\|.$$

Multiplying this inequality by ω^{-mT} yields

$$\omega^{-mT}\left\|x(mT)\right\| \le CK\omega^{-q(k)T}\left\|x_0\right\| +$$

$$CK\Delta\sum_{i=k}^{mT-1}\omega^{-q(i+1)T}\left\|x(i)\right\| =$$

$$CK\omega^{-q(k)T}\left\|x_0\right\| +$$

$$CK\Delta\sum_{i=k}^{mT-1}\omega^{(q(i)-q(i+1))T}\omega^{-q(i)T}\left\|x(i)\right\|.$$

Applying Gronwall's inequality (4) with $u(i) = \omega^{-q(i)}\left\|x(i)\right\|$ we obtain

$$\omega^{-mT}\left\|x(mT)\right\| \le$$

$$CK\omega^{-q(k)T}\left\|x_0\right\|\prod_{i=k}^{mT-1}\left(1+CK\Delta\omega^{(q(i)-q(i+1))T}\right)$$

Consider the last inequality with k replaced by kT, then

$$\omega^{-mT}\left\|x(mT)\right\| \le$$

$$CK\omega^{-kT}\left\|x_0\right\|\prod_{i=kT}^{mT-1}\left(1+CK\Delta\omega^{(q(i)-q(i+1))T}\right) =$$

$$CK\omega^{-kT}\left\|x_0\right\|\left(1+CK\Delta\omega^{-T}\right)^{(m-k)T},$$

because

$$\prod_{i=kT}^{mT-1}\left(1+CK\Delta\omega^{(q(i)-q(i+1))T}\right)=$$

$$\left(1+CK\Delta\omega^{-T}\right)^{(m-k)T}.$$

Finally

$$\|x(mT)\|\le CK\|x_0\|\left(\omega+CK\Delta\omega^{-T+1}\right)^{(m-k)T}$$

and

$$\|\Phi_{A+\Delta}(mT,kT)\|=$$

$$=\sup_{\|x_0\|=1}\|\Phi_{A+\Delta}(mT,kT)x_0\|\le$$

$$CK\|x_0\|\left(\omega+CK\Delta\omega^{-T+1}\right)^{(m-k)T}.$$

The last inequality implies the conclusion of the Theorem 1, because of (6) and Lemma 1.

From the Theorem 2 it follows in particular, that for each periodic exponentially stable system (1) there exists a positive constant Δ such that system (2) is exponentially stable for all perturbation sequences $\Delta(n)$ such that $\Delta>\sup\|\Delta(n)\|$. The last remark it is not trivial in the light of the next example. In this example we present a system (non periodic) which is exponentially stable however the perturbed system is unstable for exponentially decreasing perturbation.

Example 1 Consider system

$$x(n+1)=A(n)x(n), \tag{9}$$

with

$$A(n)=\begin{bmatrix} e^{a_1(n)} & 0 \\ 0 & e^{a_2(n)} \end{bmatrix},$$

where

$$a_1(n)=\begin{cases} 1 \text{ if } n\in[t_{2k},t_{2k+1}) \text{ for certain natural } k \\ -4 \text{ if } n\in[t_{2k+1},t_{2k+2}) \text{ for certain natural } k \end{cases},$$

$a_1(0)=1$, $a_2(n)=-3-a_1(n)$ and $t_k=2^k$. It is easy to find that

$$\lim_{k\to\infty}\frac{1}{2^{2k+2}}\sum_{i=0}^{2^{2k+2}-1}a_1(i)=$$

$$-\frac{7}{3}\text{ and }\lim_{k\to\infty}\frac{1}{2^{2k+1}}\sum_{i=0}^{2^{2k+1}-1}a_1(i)=-\frac{2}{3}.$$

Therefore

$$\limsup_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}a_1(i)=\limsup_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}a_2(i)=-\frac{2}{3}$$

and consequently the Lyapunov exponent of the system $\lambda(A)=\limsup_{n\to\infty}\frac{1}{n}\ln\|\Phi_A(n,0)\|$ is equal to $-\frac{2}{3}$ and the system is exponentially stable. Consider now disturbed system

$$z(n+1)=\left(A(n)+\Delta(n)\right)z(n), \tag{10}$$

with $\Delta=(\Delta(n))_{n\in N}$ given by

$$\Delta(n)=\begin{bmatrix} 0 & \delta(n) \\ \delta(n) & 0 \end{bmatrix},$$

where

$$\delta(n)=\begin{cases} e^{-\sigma n} \text{ if } n=t_k \text{ for certain natural } k \\ 0 \text{ otherwise} \end{cases}.$$

For the initial condition $z_0=[1\ 1]$ both coordinates $z_1(n)$ and $z_2(n)$ of the solution $z(n,z_0)$ of (10) are positive and therefore for each interval $[t_k,t_{k+1})$ there exists $i(k)\in\{1,2\}$ such that

$$z_{i(k)}(t_{k+1})\ge z_{i(k)}(t_k)\exp(t_{k+1}-t_k)=z_{i(k)}(t_k)\exp(t_k),$$

namely $i(k)=1$ for even k and $i(k)=2$ for odd k. Since

$$z_{i(k)}(t_k)\ge\tfrac{1}{4}\|z(t_k)\|e^{-\sigma t_k}$$

and

$$\|z(t_{k+1})\|\ge z_{i(k)}(t_{k+1})$$

we obtain

$$\|z(t_{k+1})\|\ge\frac{1}{4}\|z(t_k)\|\exp[t_k(1-\sigma)]=$$

$$\frac{1}{4}\|z(t_k)\|\exp[(t_{k+1}-t_k)(1-\sigma)].$$

From the above we get

$$\limsup_{n\to\infty}\tfrac{1}{n}\ln\|z(n,z_0)\|\ge 1-\sigma$$

and therefore the perturbed system is not exponentially stable for $0<\sigma<1$.

The main problem with application of Theorem 2 to numerical calculation is to check conditions (5) and to determine the values C and ω. In that context it is worth to mention paper [9] where three numerical algorithms are presented to compute constants C and ω for a given stable matrix B and given norm $\|\cdot\|$. Further results from the literature are collected below.

Theorem 3 [5] If for certain s-by-s matrices B,Q,H, $Q=Q^T>0$, $H=H^T>0$ the following discrete Lyapunov equation

$$H-B^THB=Q$$

is satisfied, then (5) holds with spectral norm $\|\cdot\|_2$, $C=\sqrt{\|H\|_2\|H^{-1}\|_2}$ and $\omega^T=\sqrt{1-\frac{\lambda\min(Q)}{\|H\|_2}}$, where $\lambda_{\min}(Q)$ is the smallest eigenvalues of Q.

Theorem 4 [8] If for certain s-by-s matrix B and operator norm $\|\cdot\|$ the following resolvent condition

$$\left\|(\mu I - B)^{-1}\right\| \le \frac{K}{|\mu| - \theta}$$

is satisfied for all $\mu \in \mathrm{C}$, $|\mu| > \theta > 0$, then (5) holds with $\omega^T = \theta$ and $C = esK$, where e is Euler's constant.

The previous two results do not use the fact that we consider periodic system they simply present methods of finding constant C and ω for a given stable matrix B. The next result, appeared the first time in [1] and was rediscovered in [10], is dedicated to periodic system.

Theorem 5 If for certain T -periodic sequences of s -by-s matrices $A(n), Q(n), H(n)$, $Q(n) = Q(n)^T > 0$, $H(n) = H(n)^T > 0$ the following discrete Lyapunov equation

$$H(n) - A(n)^T H(n+1) A(n) = Q(n)$$

is satisfied, then (5) holds with

$$C = \sqrt{\|H(0)\|_2 \|H^{-1}(0)\|_2}$$

and

$$\omega = \prod_{i=0}^{T-1}\left(\sqrt{1 - \frac{\lambda \min(Q(i))}{\|H(i)\|_2}}\right).$$

Using these results and Theorem 2 we are able to estimate the norm of perturbation that preserve the stability. This is demonstrated by the following numerical example.

Example 3 Consider system (1) of period $T = 2$ and

$$A(0) = \begin{bmatrix} -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} \end{bmatrix}, \; A(1) = \begin{bmatrix} -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{3} \end{bmatrix}.$$

Then $B = \begin{bmatrix} \frac{7}{36} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{7}{36} \end{bmatrix}$ and solving discrete Lyapunov equation with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we find

$$H = \frac{1}{22015}\begin{bmatrix} 24336 & -2304 \\ -2304 & 24336 \end{bmatrix}$$

and

$$C = \sqrt{\|H\|_2 \|H^{-1}\|_2} = 1.0996,$$

$$\omega = \left(1 - \frac{\lambda \min(Q)}{\|H\|_2}\right)^{\frac{1}{2T}} = 0.64551,$$

$$K = \max_{i=0,\ldots,T} \|\Phi(T,i)\| =$$

$$= \max\{\|\Phi(2,0)\|, \|\Phi(2,1)\|, \|\Phi(2,2)\|\}$$

$$= \max\{0.41667, 0.5, 1\} = 1.$$

According to Theorem 2 the perturbed system (2) is stable for all perturbation sequences $\Delta(n)$ such that $\Delta > \sup\|\Delta(n)\|$, where $\Delta = \frac{(1-\omega)\omega}{CK} = 0.35449.$

IV. Conclusion

In this note we have considered a linear discrete time periodic system with uncertain time-varying coefficients. The considered type of uncertainties is called in the literature real unstructured uncertainties. We obtained conditions on the perturbation of the system under which the perturbed system remains uniformly exponentially stable. The obtained results are illustrated on numerical examples.

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