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# Stability Bound of Ruin Probability in a Reduced Two-Dimensional Risk Model

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***Abstract*****—**In this work, we introduce the qualitative and quantitative concept of the strong stability method in the risk process modeling two lines of business of the same insurance company or an insurance and re-insurance companies that divide between them both claims and premiums with a certain proportion. The approach proposed is based on the identification of the ruin probability associate to the model considered, with a stationary distribution of a Markov random process called a reversed process.

Our objective, after clarifying the condition and the perturbation domain of parameters, is to obtain the stability inequality of the ruin probability which is applied to estimate the approximation error of a model with disturbance parameters by the considered model. In the stability bound obtained, all constants are explicitly written.

***Keywords*****—**Markov chain, risk models, ruin probabilities, strong stability analysis.

## I. Introduction

IN the actuarial literature, the evolution in time of the capital of insurance company is often modeled by the stochastic process of reserve resulting from the difference between the premium-income and the pay-out process.

The ruin probability is one of the basic characteristic of risk models. Various authors investigate the problem of its evaluation, but it cannot, however, be found in an explicit form for many risk models. Furthermore, parameters governing these models are often unknown and one can only give some bounds for their values. In such a situation the question of stability becomes crucial (see [2], [10]).

In the stability theory, we establish the domain within which a model may be used as a good approximation or idealization of the real system under consideration. Such results give the possibility of approximating some complicated systems by other systems more exploitable or much simpler.

There exist numerous results on perturbation bounds of Markov chains. The strong stability method has been developed in the early 1980s by V. Kartashov (see [1], [8]). It allows both to make qualitative and quantitative analysis of some complex systems. This approach assumes that the perturbation of the transition kernel is small with respect to a certain norm. Such a strict condition allows us to obtain better estimations on the stationary characteristics of the perturbation chain. In addition, using this method, it is possible to obtain inequalities of stability with an exact computation of the constants.

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Using the strong stability method, the academician V. Kalashnikov realized the first application of this method in risk model and investigated the estimation of ruin probabilities in the univariate risk models. Then, many authors extend the application of this approach for different type of risk process.

In a general form, this approach (Strong stability method) is based on the following three steps. The first step consists of identification of the ruin probability associate to the considered risk model with a stationary distribution for a specific random process which is called a reversed process. The second step consists of the embedding into a Markov process by equipping it with supplementary coordinates. The third step consists of the application of the quantitative aspect of this method, giving estimation for the deviations of stationary distributions of the two Markov process under comparison (see [7], [4], [6], [9]).

In this paper, we present the principle and some details about the strong stability approach and its application in a specific two-dimensional risk model where we divide, with a certain proportion, claims and premiums of two lignes of business in the same insurance company or between an insurance and re-insurance company (see [3]).

In order to obtain a strong stability bound of the ruin probability which is an approximation error of the disturbance risk model by the ideal model where the calcul of its ruin probability is explicit, we will delimit the perturbation domain of parameters under the conditions of the strong stability method.

## II. A Two-Dimensional Risk Model

### A. Presentation

We consider a particular two dimensional risk model where two compagnies divide the claim amounts in positive proportions $\delta_1$ and $\delta_2$ with $\delta_1+\delta_2 = 1$ and the according premiums rates $c_1$ and $c_2$.

Then, the evolution in time of the i'th company is modeled by the process of reserve $\{X_i(t), t \geq 0\}$ described by: (see [3])

$$X^i(t) = u_i + c_i t - \delta_i S(t), \qquad t \geq 0, i = 1, 2, \tag{1}$$

where $u_i$ are the initial reserves and

$$S(t) = \sum_{k=1}^{N(t)} Z_k,$$

where $\{N(t), t \geq 0\}$ is a Poisson process with intensity $\lambda$ and $\{Z_k\}_k$ is a sequence of i.i.d. positive random variables independent of $\{N(t), t \geq 0\}$ with common distribution function $F$ such that mean $E(Z_k) = \frac{1}{\mu}$.

We shall assume that the second company, to be called the reinsurer, gets smaller profits per amount paid:

$$p_1 = \frac{c_1}{\delta_1} > \frac{c_2}{\delta_2} = p_2.$$

In addition, we assume that $p_i > \frac{\lambda}{\mu}, i = 1, 2$.

The concept of ruin in multi-dimensional cases could have different meanings and interpretations.

In this paper, we consider the following type of ruin time:

$$\tau(u_1, u_2) = \inf\{t \geq 0 : X^1(t) < 0 \text{ } or \text{ } X^2(t) < 0\}. \quad (2)$$

For the ruin time considered, the corresponding ruin probability is denoted by:

$$\psi(u_1, u_2) = P(\tau(u_1, u_2) < \infty). \quad (3)$$

The ruin probabilities under multi-dimensional models rarely admet analytical solutions. It is possible to obtain a closed solution for $\psi(u_1, u_2)$ if $Z_k$ are exponentially distributed with intensity $\mu$.

The solution of the two dimensional ruin problem strongly depends on the relative sizes of the proportion $(\delta_1, \delta_2)$ and premium rates $(c_1, c_2)$.

### *B. One Dimensional Reduction*

The reduction of the considered risk model to one dimensional model is based on the following important observation:(see [3])

$$\tau(u_1, u_2) = \inf\{t \geq 0 : S(t) > b(t)\},$$

where

$$b(t) = \min\{(u_1 + c_1 t)/\delta_1, (u_2 + c_2 t)/\delta_2\}$$
$$\Rightarrow b(t) = \min\{\frac{u_1}{\delta_1} + \frac{c_1}{\delta_1} t, \frac{u_2}{\delta_2} + \frac{c_2}{\delta_2} t\}.$$

We suppose that the initial reserves $u_1$ and $u_2$ are such that

$$(u_1, u_2) \in \mathcal{C} = \{(u_1, u_2) : u_2 \leq (\frac{\delta_2}{\delta_1}) u_1\}. \quad (4)$$

In this case, the barrier $b(t) = (u_2 + c_2 t)/\delta_2$ is linear and the ruin happens always for the second company.

Thus, as we already observed, the problem considered reduces in fact to the classical one-dimensional ultimate ruin probability with premium $c_2$ and claims $\delta_2 Z$:

$$\psi(u_1, u_2) = \psi_2(u_2) = P(\tau_2(u_2) < \infty), \quad (5)$$

where $\tau_2(u_2) = \inf\{t \geq 0, X^2(t) < 0\}$ and $\psi_2(u_2)$ is the ruin probability of the second process $\{X^2(t), t \geq 0\}$.

It is well known that in the case of the exponential claims sizes with intensity $\mu$, it reduces to:

$$\psi_2(u_2) = C_2 e^{-(\gamma_2/\delta_2) u_2},$$

where $\gamma_2 = \mu - \lambda\delta_2/c_2$ and $C_2 = \frac{\lambda\delta_2}{\mu c_2} = \frac{\lambda}{\mu p_2}, \quad p_2 = \frac{c_2}{\delta_2}$.

An analyze of the opposite case $\frac{u_2}{\delta_2} > \frac{u_1}{\delta_1}$ have been realized bye F. Avram and al. in [3].

For general distribution of the claim amounts, where the evaluation of the ruin probabilities is not explicite (see [5]), we propose, in the case of one dimensional reduction where $\frac{u_2}{\delta_2} \leq \frac{u_1}{\delta_1}$, the application of the strong stability method to obtain un estimation of the ruin probability deviation which will be given as a stability inequality with respect to a certain norm.

## III. The Strong Stability Concept

Let $m\mathcal{E}$ be the space of finite measures on the probabilisable space $(E, \mathcal{E})$, and $f\mathcal{E}$ the space of bounded measurable function on $E$. We associate with each transition kernel $P$ the linear mapping

$$\mu P(A) = \int_E \mu(dx) \ P(x, A), \quad \forall \quad A \in \mathcal{E}, \quad (6)$$

$$Pf(x) = \int_E \ P(x, dy) f(y), \quad \forall \quad x \in E. \quad (7)$$

Introduce on $m\mathcal{E}$ the class of norms of the form

$$\|\mu\|_v = \int_E v(x)|\mu|(dx), \quad (8)$$

where $v$ is an arbitrary measurable function (not necessarily finite) bounded below away from a positive constant, and $|\mu|$ is the variation of the measure $\mu$.

This norm induces in the space $f\mathcal{E}$ the norm

$$\|f\|_v = \sup\{|\mu f|, \ \|\mu\|_v \leq 1\} = \sup_{x \in E} [v(x)]^{-1} |f(x)|, \forall f \in f\mathcal{E}. \quad (9)$$

Let us consider $\mathcal{B}$, the space of linear operators, with the norm

$$\|P\|_v = \sup_{x \in E} \left( \ [v(x)]^{-1} \int_E v(y) |P(x, dy)| \ \right). \quad (10)$$

*Definition 1:*
A Markov chain $X$ with a transition kernel $P$ and invariant measure $\pi$ is said to be strongly $v-$stable with respect to the norm $\| \ . \ \|_v$, if $\|P\|_v < \infty$, and each stochastic kernel $Q$ on the space in some neighborhood $\{Q : \|Q - P\| \ < \epsilon\}$ has a unique invariant measure $\nu = \nu(P)$ and $\|\nu - \pi\|_v \to 0$ as $\|Q - P\|_v \to 0$.

In the sequel, we use the following results:

*Theorem 1 (see [1]):*
The Markov chain $X$ with the transition kernel $P$ and invariant measure $\pi$ is strongly $v$-stable with respect to the norm $\| \ . \ \|_v$ if and only if there exist a measure $\alpha$ and a nonnegative measurable function $h$ on $E$ such that $\pi h > 0$, $\alpha \mathbf{1} = 1, \alpha h > 0$, and

- The operator $T = P - h \circ \alpha$ is nonnegative.
- There exist $\rho < 1$ such that $Tv(x) \leq \rho v(x)$ for $x \in E$
- $\|P\|_v < \infty$.

Here $\mathbf{1}$ is the function identically equal to 1 and $\circ$ denotes the convolution between a measure and a function.

The following result was proved in [8].

*Theorem 2 (see [7]):*
Let $v$ be the fixed weight function and assume that a Markov

chain with the transition probability $P$, satisfying $\|P\|_v < \infty$, possess a unique stationary distribution $\pi$. Assume also that there exist a non-negative function $h$ and a probability measure $\alpha$ such that $P$ can be splitted as follows:

$$P(x,.) = T(x,.) + h(x).\alpha(.), \tag{11}$$

where

$$\|\pi\|_h > 0, \quad \|\alpha\|_h > 0 \tag{12}$$

and

$$\|T\|_v \leq \rho < 1. \tag{13}$$

Then each Markov chain with the transition probability $P'$ satisfying the inequality

$$\triangle = \|P - P'\|_v < \triangle_0 \equiv \frac{(1-\rho)^2}{1-\rho+\rho\|\alpha\|_v} \tag{14}$$

has a unique stationary $\pi'$ and, furthermore

$$\|\pi - \pi'\|_v \leq \frac{\triangle\|\alpha\|_v}{(1-\rho)(\triangle_0 - \triangle)}. \tag{15}$$

## IV. Strong Stability of the Reduced Two-Dimensional Risk Model

In this section, we are interested to apply the qualitative and quantitative aspects of the strong stability approach which serve for delimiting domain where the tow-dimensional classical risk model considered can be a good approximation of another disturbance two-dimensional risk model and to estimate the error of approximation.

In order to simplify this study, we suppose that $(u_1, u_2) \in \mathcal{C} = \{(u_1, u_2) : u_2 \leq (\frac{\delta_2}{\delta_1})u_1\}$ which is the condition for the reduction to one dimensional model.

In first, we present the reversed process associate to the reduced considered model.

### A. Reversed Process

Since the ruin can only happen at the claim occurrence times $\{T_n\}$, the probability of ruin $\Psi_2(u_2)$, defined by the relation (5), can be expressed in terms of the process $\{X^2_{T_n}\}$ as

$$\Psi_2(u_2) = P\left( \inf_{n\geq 1}(X^2_{T_n}) < 0/ \quad X^2_0 = u_2 \right), \tag{16}$$

where $\{T_n,\ n \geq 1\}$ be a successive i.i.d. occurrence times.

The reversed process $\{V_n\}_n$ associate to our risk model can be defined by the equation

$$\forall\ n \geq 0, V_{n+1} = \left( V_n - c_2\theta_{n+1} + \delta_2 Z_{n+1} \right)_+,\ V_0 = 0, \tag{17}$$

with $T_n = \theta_1 + \theta_2 + \cdots + \theta_n$ and $\theta_n$ be successive i.i.d inter-occurrence times.

According to the recursive form of the reversed process $\{V_n\}_{n\geq 0}$, we have that $V_{n+1}$ depend only on $V_n$, $\theta_{n+1}$ and $\delta_2 Z_{n+1}$ where the random variables $\theta_{n+1}$ and $Z_{n+1}$ are independent on $n$ and on the state of the system before $n$.

Then $\{V_n\}_{n\geq 0}$ is a homogenous Markov chain and its state space is $E = \mathcal{R}^+$. Denote by

$$P(x, A) = P(V_{n+1} \in A/V_n = x) \tag{18}$$

its transition probability.

Using this Markov chain $\{V_n\}_{n\geq 0}$, it is well-known that

$$\Psi_2(u_2) = \lim_{n\to\infty} P(V_n > u_2). \tag{19}$$

### B. Transition Kernel

The transition kernel associate to the chain $\{V_n\}_{n\geq 0}$ defined on the probabilisable space $(E, \mathcal{E})$ can be splitted as follows: $\forall\ x\ \in R^+$ and $\forall\ \ A\ \in \mathcal{E}$, we have:

$$\begin{aligned} P(x,A) &= P(V_1 \in A/V_0 = x) \\ &= P\left( (V_0 - c_2\theta_1 + \delta_2 Z_1)_+ \in A/V_0 = x \right) \\ &= P\left( 0 < (x - c_2\theta_1 + \delta_2 Z_1) \in A \right) \\ &+ P\left( 0 \in A \right) \ P\left( x - c_2\theta_1 + \delta_2 Z_1 \leq 0 \right) \\ &= T(x,A) + \alpha(A).h(x), \end{aligned} \tag{20}$$

with

$$T(x,A) = P\left( 0 < (x - c_2\theta_1 + \delta_2 Z_1) \in A \right),$$

$$\alpha(A) = \delta_0(A),$$

where $\delta_0$ is a probability measure concentrated at 0 ( Dirac measure), and

$$h(x) = P\left( c_2\theta_1 - \delta_2 Z_1 \geq x \right), \quad x \in R^+.$$

To apply the Theorem 1 to the Markov chain $\{V_n\}_{n\geq 0}$, we choose the function $v(x) = e^{\epsilon x}, \quad x \in R^+$.

All conditions of this theorem are satisfied for

- $T(x,A) = P\left( 0 < (x - c_2\theta_1 + \delta_2 Z_1) \in A \right)$,
- $\alpha(A) = \delta_0(A)$ (Dirac measure),
- $h(x) = P\left( c_2\theta_1 - \delta_2 Z_1 \geq x \right), \quad x \in R^+$,

obtained by the precedent decomposition of the transition kernel $P$, with

$$\rho = E\left( \exp\{\epsilon(\delta_2 Z_1 - c_2\theta_1)\} \right). \tag{21}$$

Finally, the Markov chain $\{V_n\}_{n\geq 0}$ is strongly stable for the weight function $v(x) = e^{\epsilon x}, \quad x \in R^+$ which means that a small deviation of parameters led to a small deviation of the characteristics.

Let us now illustrate how Theorem 2 can be applied to obtain stability bounds.

### C. Stability Inequalities

Under le condition given in the relation (4), the two-dimensional classical risk model is completely determined by the vector of parameters $a = (c_2, \lambda, F)$. As we have seen, the probability of ruin $\Psi_2(u_2)$ coincides with the stationary distribution of the reversed process $\{V_n\}_{n\geq 0}$ (see (19)) to exceed the level $u_2$.

Let $a' = (c'_2, \lambda', F')$ be the vector parameter governing another reduced bivariate risk model, its ruin probability being $\Psi'_2(u_2)$ and $\{V'_n\}_{n\geq 0}$ its reversed process associate.

To be able to estimate numerically the margin between the stationary distributions of the Markov chains $\{V_n\}_{n\geq 0}$ and $\{V'_n\}_{n\geq 0}$, we estimate the the deviation of transition kernel with respect to the norm $\|\ .\ \|_v$.

According to [7], the deviation $\|P - P'\|_v$ can be estimated as follows:

$$\|P - P'\|_v \leq 2 \ E \quad e^{\epsilon Z} \quad \ln|\frac{\lambda c_2'}{\lambda' c_2}| + \|B - B'\|_v, \qquad (22)$$

where $B$ and $B'$ are, respectively, the distribution functions of the random variables $\delta_2 Z$ and $\delta_2' Z'$ with

$\forall x \in R,\ B(x) = P(\delta_2 Z \leq x) = F(\frac{x}{\delta_2})$
and $B'(x) = F'(\frac{x}{\delta_2})$.

Denote

$$\mu(a, a') = 2 \ E \quad e^{\epsilon Z} \quad \ln|\frac{\lambda c_2'}{\lambda' c_2}| + \|B - B'\|_v.$$

Under assumption $\mu(a, a') < (1 - \rho)^2$ and from inequality (15) of Theorem 2, the distance between ruin probabilities is expressed as follows:

$$\|\Psi_2(u_2) - \Psi_2'(u_2)\|_v \leq \frac{\mu(a, a')}{(1 - \rho) \ \left( \ (1 - \rho)^2 - \mu(a, a') \ \right)} \qquad (23)$$

where $\rho$ is given by relation (21).

Then, we obtain an estimation for the deviation of the ruin probability $\psi_2(u_2)$ with respect to the norm $\| \ . \ \|_v$. Thus, under the condition of the reduction which is

$$(u_1, u_2) \in \mathcal{C} = \{(u_1, u_2) : u_2 \leq (\frac{\delta_2}{\delta_1})u_1\},$$

the deviation $\|\Psi_2(u_2) - \Psi_2'(u_2)\|_v$ is equal to the deviation of $\Psi(u_1, u_2)$; $\|\Psi(u_1, u_2) - \Psi'(u_1, u_2)\|_v$ associate to the two-dimensional risk process considered.

## V. Conclusion

In this work, we proved the applicability of the strong stability method to approximate one type of the ruin probabilities associates to a two dimensional risk process in the case of one dimensional reduction.

The stability bounds of ruin probability derived above contain only explicitly written parameters. The precision obtained allows us to confirm the efficiency of this method and its importance for practical problems.

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