Stability Analysis of Mutualism Population Model with Time Delay

Rusliza Ahmad and Harun Budin

Abstract—This paper studies the effect of time delay on stability of mutualism population model with limited resources for both species. First, the stability of the model without time delay is analyzed. The model is then improved by considering a time delay in the mechanism of the growth rate of the population. We analyze the effect of time delay on the stability of the stable equilibrium point. Result showed that the time delay can induce instability of the stable equilibrium point, bifurcation and stability switches.

Keywords—Bifurcation, Delay margin, Mutualism population model, Time delay

I. INTRODUCTION

TRADITIONAL Lotka-Volterra type predator-prey models or competing models have been widely studied by many authors (for example, see [5],[6],[8],[11],[12]). But there are few papers considering mutualism system. Mutualism is an interaction in which species help one another. Mutualism can be classified into four types: seed-dispersal mutualisms, pollination mutualisms, digestive mutualisms and protection mutualisms. Some examples of each type of mutualisms can be seen in Kot [10].

Ravindra Reddy et al. [1] have studied two mutually interacting species with limited resources for both species. They investigated the global stability analysis of the model by constructing a suitable Liapunov's function. A model of two mutually interacting species with limited resources for first species and unlimited resources for second species have been studied by Ravindra Reddy et al. [3]. Other research on mutualism model was studied by Fay et al. [5].

A Lotka-Volterra type mutualism system with several delays was studied by Xia et al. [15]. Some new and interesting sufficient conditions are obtained for the global existence of positive periodic solutions of the mutualism system. Xia [14] has studied global existence of positive periodic solutions for a class of mutualism systems with several delays and obtains a new and interesting criterion. Stability analysis of a two species mutualistic model with limited resources and a time delay was investigated by Ravindra Reddy et al. [2]. Mutualism population model with time delay has also been considered by Dang and Cheng [4] and He and Gopalsamy [7].

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In this paper, we discuss mutualism population model based on Lotka-Volterra model which is extended by incorporating a time delay into the model. The stability of the model without time delay is analyzed and the effect of time delay on the stability of stable equilibrium point will be inspected. Bifurcation and time delay margin will be determined.

II. MUTUALISM POPULATION MODEL

We consider a two population ecosystem in which both populations interact with each other in a mutually beneficial way. Since the food source is limited, a logistic growth model is proposed for each population in the absence of the other. The considered model of mutualism between the two populations is based on Lotka-Volterra model. The effect of their interaction will increase the size of the two populations. The model is

$$
\begin{aligned}
\frac{dx}{dt} &= rx\left(1 - \frac{x}{K_x}\right) + \alpha xy, \\
\frac{dy}{dt} &= sy\left(1 - \frac{y}{K_y}\right) + \beta xy.
\end{aligned}
\tag{1}
$$

The symbols x and y denote the population sizes at time t, the constants K_x and K_y denote the carrying capacities of the populations x and y. The parameter r and s are the intrinsic growth rates of populations x and y, the constants α and β denote the coefficients interaction between the two populations that increase the size of the populations x and y respectively.

For simplification, model (1) is written in the form

$$
\begin{aligned}
\frac{dx}{dt} &= rx - bx^2 + \alpha xy, \\
\frac{dy}{dt} &= sy - ey^2 + \beta xy.
\end{aligned}
\tag{2}
$$

where $b = \dfrac{r}{K_x}$ and $e = \dfrac{s}{K_y}$. The constants b and e are also well known as the coefficient of self interaction or the coefficient of intraspecific interaction for populations x and y respectively.

The equilibrium points of model (2) are $E_1 = (0,0)$, $E_2 = \left(\dfrac{r}{b}, 0\right)$, $E_3 = \left(0, \dfrac{s}{e}\right)$ and $E^* = \left(\dfrac{er + \alpha s}{be - \alpha\beta}, \dfrac{bs + \beta r}{be - \alpha\beta}\right)$.

Only the equilibrium point E^* can occur in the positive quadrant. The Jacobian matrix of model (2) takes the form

$$J = \begin{pmatrix} r - 2bx + \alpha y & \alpha x \\ \beta y & s - 2ey + \beta x \end{pmatrix}.$$

To analyze the equilibrium point E^*, we refer to the following result.

Theorem 1. Toaha, S. and Budin, H. [13].
Let E^* be the positive equilibrium point of model (2). If $be - \alpha\beta > 0$, E^* is asymptotically stable.

Example 1. Consider model (2) with the parameters $r = 0.405$, $b = 0.03375$, $\alpha = 0.015$, $s = 0.34$, $e = 0.02833$, and $\beta = 0.020$. With these parameters, Theorem 1 is satisfied. The equilibrium point is $E^* = (25.26, 29.83)$ and the eigenvalues associated with this equilibrium point are -1.32421 and -0.37336. This means that the equilibrium point E^* is asymptotically stable. Some trajectories and vector field are given in Fig. 1.
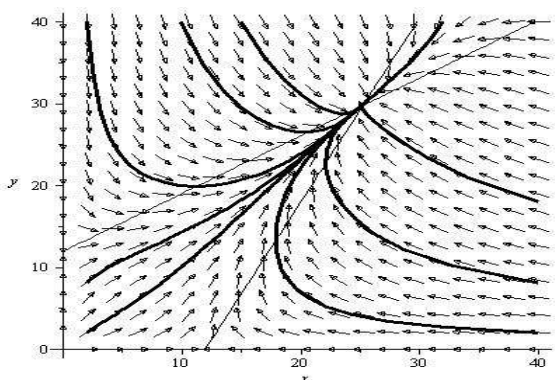

Fig. 1 Some trajectories around the stable equilibrium point E^*

III. MUTUALISM POPULATION MODEL WITH TIME DELAY

We consider the mutualism population model (2) where the two populations are incorporated with time delay. The model becomes

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{bx(t - \tau)}{r}\right) + \alpha x(t)y(t),$$

$$\frac{dy(t)}{dt} = sy(t)\left(1 - \frac{ey(t - \tau)}{s}\right) + \beta x(t)y(t). \qquad (3)$$

where r, b, s, e, α and β are positive constants. The terms $\left(1 - \dfrac{bx(t - \tau)}{r}\right)$ and $\left(1 - \dfrac{ex(t - \tau)}{s}\right)$ in model (3) denote the density dependent feedback mechanism which takes τ units of time to respond to changes in the population densities of populations x and y respectively.

The equilibrium point of model (3) that possibly occurs in the first quadrant is $E^* = \left(x^*, y^*\right) = \left(\dfrac{er + \alpha s}{be - \alpha\beta}, \dfrac{bs + \beta r}{be - \alpha\beta}\right)$. For $\tau = 0$, this equilibrium point is asymptotically stable when the conditions $be - \alpha\beta > 0$ is satisfied, Theorem 1. To linearize the model about the equilibrium point E^* of model (3), let $u(t) = x(t) - x^*$ and $v(t) = y(t) - y^*$. Then substitute into model (3) to get

$$\dot{u}(t) = r\big(u(t) + x^*\big) - b\big(u(t) + x^*\big)\big(u(t - \tau) + x^*\big)$$
$$+ \alpha\big(u(t) + x^*\big)\big(v(t) + y^*\big)$$
$$\dot{v}(t) = s\big(v(t) + y^*\big) - e\big(v(t) + y^*\big)\big(v(t - \tau) + y^*\big)$$
$$+ \beta\big(u(t) + x^*\big)\big(v(t) + y^*\big).$$

After simplifying and neglecting the product terms, we have the linearized model

$$\dot{u}(t) = -bx^* u(t - \tau) + \alpha x^* v(t)$$
$$\dot{v}(t) = -ey^* v(t - \tau) + \beta y^* u(t). \qquad (4)$$

Analyzing the local stability of the equilibrium point E^* in the model with time delay is equivalent to analyzing the stability of the zero equilibrium point in the linearized model. From the linearized model we have the characteristic equation

$$f(\lambda) = \begin{vmatrix} bx^* e^{-\lambda\tau} + \lambda & -\alpha x^* \\ -\beta y^* & ey^* e^{-\lambda\tau} + \lambda \end{vmatrix} = 0,$$

$$\Delta(\lambda, \tau) = \lambda^2 + a_{00} + a_{11} e^{-\lambda\tau}\lambda + a_{02} e^{-2\lambda\tau} = 0,$$

where $a_{00} = -\alpha\beta x^* y^*$, $a_{11} = bx^* + ey^*$, $a_{02} = bex^* y^*$.

Since x^* and y^* are both positive we have $a_{00} < 0$, $a_{11} > 0$, and $a_{02} > 0$. For $\tau = 0$, the characteristic equation becomes

$$\lambda^2 + a_{11}\lambda + a_{00} + a_{02} = 0 \qquad (5)$$

which has the roots

$$\lambda_{1,2} = \frac{-a_{11} \pm \sqrt{a_{11}^2 - 4(a_{00} + a_{02})}}{2}. \qquad (6)$$

Under the condition $be - \alpha\beta > 0$, the characteristic equation (5) has negative real roots.

In order to analyze the stability of the equilibrium point E^* of model (3), we transform model (4) into a second order delay differential equation. Then we have

$$\ddot{u}(t)+a_{00}u(t)+a_{02}u(t-2\tau)+a_{11}\dot{u}(t-\tau)=0,$$

and for determining the delay margin of model (4) we apply theorem 2.

Theorem 2. J. Chiasson, [9]
Suppose that the equation

$$\frac{d^2}{dt^2}y(t)+\sum_{i=0}^{1}\sum_{j=0}^{2}a_{ij}\frac{d^i}{dt^i}y(t-j\tau)=0 \text{ is stable for } \tau=0. \text{ The}$$

characteristic equation with $d=e^{-s\tau}$ is

$$a(s,d)=s^2+\sum_{i=0}^{1}\sum_{j=0}^{2}a_{ij}s^i d^j \qquad (7)$$

$$=s^2+a_1(d)s+a_0(d).$$

where $a_i(d)=\sum_{j=0}^{2}a_{ij}d^j$ for $i=0$, 1.

The polynomial $a(s,d)$ is said to be asymptotically stable independent of delay iff
$a(s,e^{-s\tau})\neq 0$ for $\mathrm{Re}(s)\geq 0$, $\tau\geq 0$.

Definition 1: Let $a(s,d)$ be a two – variable polynomial of the form (2). Then define
$$\tilde{a}(s,d)=d^2 a(-s,1/d),$$

$$=(-1)^2 d^2 s^2+\sum_{i=0}^{1}\sum_{j=0}^{2}a_{ij}(-s)^i d^{2-j}.$$

Definition 2: τ^* - the range of stability for τ.
Let $\{(s_i,d_i) \quad i=1,\ldots,k\}$ be the common zeros of $\{a(s,d),\tilde{a}(s,d)\}$ for which $\mathrm{Re}(s_i)=0$, $s_i\neq 0$ and $|d_i|=1$, $d_i\neq 1$. For each such pair (s_i,d_i), let $\tau_i=\min_{\tau>0}\{\tau\in\Re|d_i=e^{-\tau i}\}$. Such an τ_i exists since $|d_i|=1$ and $s_i\neq 0$. Define $\tau^*=\min\{\tau_i\}>0$.

Example 2. Consider model (2) with the parameters $r=0.405$, $b=0.03375$, $\alpha=0.015$, $s=0.34$, $e=0.02833$, and $\beta=0.020$. For $\tau=0$, these parameters satisfy Theorem 1. The equilibrium point is $E^*=(25.26, 29.83)$ and the eigenvalues associated with this equilibrium point are -1.32421 and -0.37336. This means that E^* is asymptotically stable. Then we have $a_{00}=-0.22601$, $a_{02}=0.72042$, and $a_{11}=1.69757$. Further we have

$$\ddot{u}(t)-0.22601u(t)+0.72042u(t-2\tau)+1.69757\dot{u}(t-\tau).$$

Referring to theorem 2, then from the delay differential equation we have

$$a(s,d)=s^2+1.69757ds-0.22601+0.72042d^2$$

and an auxiliary polynomial $\tilde{a}(s,d)$ in term of $a(s,d)$ is

$$\tilde{a}(s,d)=d^2 s^2-1.69757ds-0.22601d^2+0.72042$$

Solving simultaneously $a(s,d)=0$ and $\tilde{a}(s,d)=0$ we get
$d_1=-i$, $d_2=i$, $d_3=0.56011+0.82842i$,
$d_4=0.56011-0.82842i$, $d_5=-0.56011+0.82842i$ and
$d_6=-0.56011-0.82842i$.

Next we compute τ according to Definition 2.
(s_1,d_1) and (s_2,d_2) are to be neglected since $\mathrm{Re}(s_i)\neq 0$.
The common zeros of $a(s,d)=0$ and $\tilde{a}(s,d)=0$ that satisfy
$\mathrm{Re}(s_i)=0$ and $|d_i|=1$ are
$(s_3,d_3)=(-0.70314, 0.56011+0.82842i)$,
$(s_4,d_4)=(0.70314, 0.56011-0.82842i)$,
$(s_5,d_5)=(-0.70314, -0.56011+0.82842i)$,
$(s_6,d_6)=(0.70314, -0.56011-0.82842i)$.

Then

$$\tau_3=\min_{\tau>0}\{\tau\in\Re|d_3=^{-\tau_3}\}=1.389$$

$$\tau_4=\min_{\tau>0}\{\tau\in\Re|d_4=^{-\tau_4}\}=1.389$$

$$\tau_5=\min_{\tau>0}\{\tau\in\Re|d_5=^{-\tau_5}\}=3.079$$

$$\tau_6=\min_{\tau>0}\{\tau\in\Re|d_6=^{-\tau_6}\}=3.079$$

Finally, we find time delay margin $\tau^*=\min\{\tau_3,\tau_4,\tau_5,\tau_6\}=1.388$. Some trajectories of $(x(t),y(t))$ with various time delays are given in Fig. 2, 3, 4 and 5.

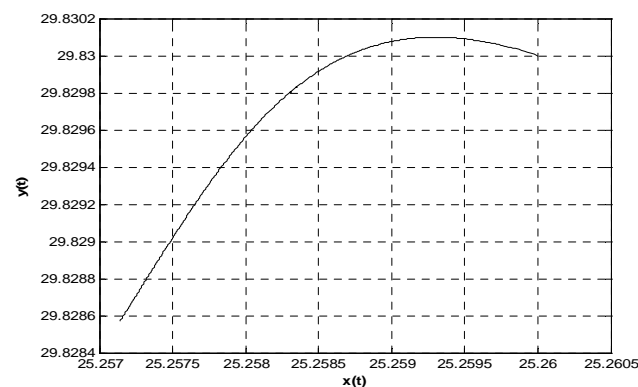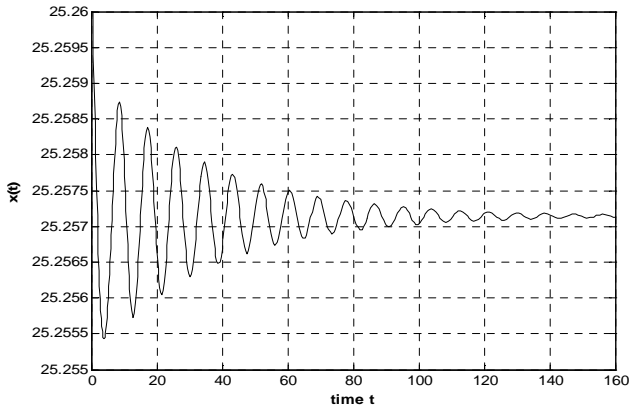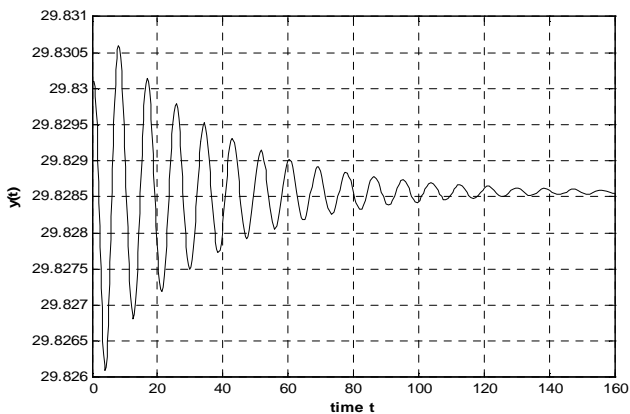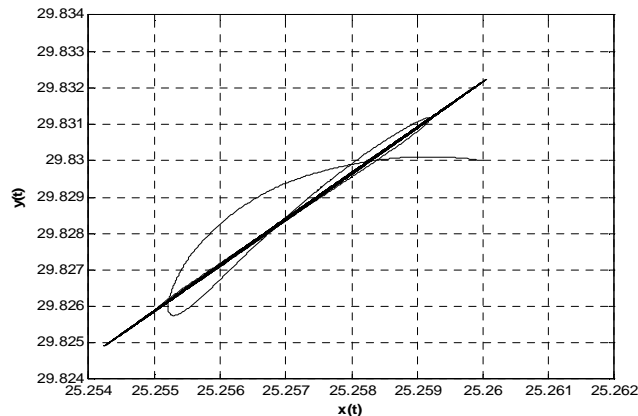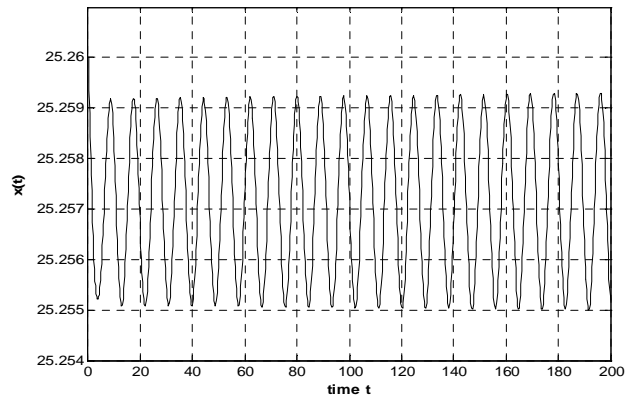Fig. 2 Trajectory of $(x(t),y(t))$ with $x(0)=25.26$, $y(0)=29.83$, and $\tau=0.1$

Fig. 3a Trajectory of $x(t)$ with $x(0) = 25.26$ and $\tau = 1.33$

Fig. 3b Trajectory of $y(t)$ with $y(0) = 29.83$ and $\tau = 1.33$

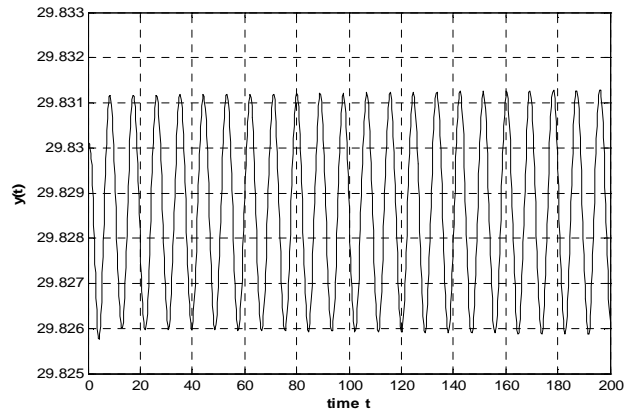Fig. 4 Trajectory of $(x(t), y(t))$ with $x(0) = 25.26$, $y(0) = 29.83$, and $\tau = 1.39$

Fig. 5a Trajectory of $x(t)$ with $x(0) = 25.26$ and $\tau = 1.389$

Fig. 5b Trajectory of $y(t)$ with $y(0) = 29.83$ and $\tau = 1.389$

We know that when the time delay $\tau = 0$, the eigenvalues associated with the characteristic equation are both negative. This means that the trajectory tends to the equilibrium point asymptotically without oscillation. It is reasonable that when we put time delay quite small, the trajectory still tends to the equilibrium point without oscillation, figure 2. When we increase the value of time delay, the trajectory still tends to the equilibrium point and oscillates around the equilibrium point, figure 3a and 3b. Figure 4 with time delay $\tau = 1.39$ shows that the equilibrium point $E^* = (25.26, 29.83)$ is not stable. The equilibrium point is stable when $0 \leq \tau \leq 1.389$. Figure 5a and 5b show that after a short time the trajectories periodically oscillate around the equilibrium point. When we disturb the value of time delay, the trajectories will either converge to the equilibrium point or diverge. A bifurcation occurs when $\tau = 1.389$, figure 5. In this case the time delay margin is $\tau = 1.389$.

IV. CONCLUSION

In the mutualism population model, there exist four equilibrium points. One of them is in the first quadrant and asymptotically stable, Theorem 1. This means that the two populations can live in coexistence. When the time delay is

incorporated into the mutualism model, the positive equilibrium point may also remain stable. By an example and using Theorem 2, we can see that there exists a time delay margin for stability of the equilibrium point. This means that if we take the value of time delay quite small, the positive equilibrium point remains stable and the two populations can coexist.

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