

# Stability analysis for a multicriteria problem with linear criteria and parameterized principle of optimality "from lexicographic to Slater"

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**Abstract**—A multicriteria linear programming problem with integer variables and parameterized optimality principle "from lexicographic to Slater" is considered. A situation in which initial coefficients of penalty cost functions are not fixed but may be potentially a subject to variations is studied. For any efficient solution, appropriate measures of the quality are introduced which incorporate information about variations of penalty cost function coefficients. These measures correspond to the so-called stability and accuracy functions defined earlier for efficient solutions of a generic multicriteria combinatorial optimization problem with Pareto and lexicographic optimality principles. Various properties of such functions are studied and maximum norms of perturbations for which an efficient solution preserves the property of being efficient are calculated.

**Index Terms**—stability and accuracy, multicriteria optimization, lexicographic optimality

## I. INTRODUCTION

The frequently used tool of stability theory and post-optimal analysis is so-called stability radius of some given optimal solution. In single objective optimization, it gives an upper bound on a subset of problem parameters for which this solution remains optimal (see [4] and bibliography therein). There are already similar investigations in multiobjective case, where the stability radius defines extreme level of problem parameter perturbations preserving efficiency of the given solution. For example, in [2] one can find a large survey on sensitivity analysis of vector unconstrained integer linear programming, where the stability radius is a key object under investigation.

It is important to note that even in single objective case the stability radius does not provide us with any information about the quality of a given solution in the case when problem data are outside of the stability region. Some attempts to study quality of the problem solution in this case are connected with the concepts of stability and accuracy functions, being originally proposed in [5] and [6] for scalar combinatorial optimization problems. Later, the results were extended for the case of multicriteria combinatorial optimization problems with Pareto and generalized lexicographic optimality principles in [7]. In [11], the similar questions of stability and accuracy were investigated under the framework of game theory, more precisely accuracy and stability functions for a coalition game with bans, linear payoffs, antagonistic strategies and

parameterized principle of optimality "from Nash to Pareto" were studied. In [12] a multicriteria Boolean programming problem with linear cost functions and parameterized principle of optimality "from Condorcet to Pareto" is considered and similar questions of stability and accuracy are studied. Some aspects of robustness based on the concepts of stability and accuracy functions were studied recently in [8] and [9].

A problem of decision making where experts are grouped into coalitions and coalitions are ordered according to their importance is considered in this paper. The decision making process is based on penalty costs specified by each expert according to their own preferences. The problem can be generalized in terms of multicriteria linear programming problem with integer variables and parameterized optimality principle "from lexicographic to Slater". A situation, in which initial coefficients of penalty cost functions are not fixed but may be potentially a subject to variations, is considered. The situation represents that fact that the experts may not be completely confident with precise values of own preferences during the decision making process. For any efficient solution, specified with original information about expert preferences, appropriate measures of the quality are introduced which incorporate information about expert preferences variations. These measures correspond to the so-called stability and accuracy functions defined earlier for efficient solutions of a generic multicriteria combinatorial optimization problem with Pareto and lexicographic optimality principles. Various properties of such functions are studied and maximum norms of perturbations for which an efficient solution preserves the property of being efficient are calculated.

In this paper an extension of the concepts of stability and accuracy functions under the parameterized optimality principle "from lexicographic to Slater" is given. The paper is organized as follows. In section II a multicriteria integer linear programming problem is considered. The problem consists in finding the set of optimal solutions, i.e. alternatives which are optimal with respect to the chosen optimality principle. A problem interpretation as one evaluation and decision making problem is given. In section III, for a given solution an appropriate relative error is introduced. It characterizes the deviation of the current solution from optimality. In section IV so called stability and accuracy functions are introduced, which are defined as functions of the corresponding norm of data perturbations. Afterwards, extreme norm values of problem parameter perturbations are specified, for which the stability and accuracy functions are equal to zero. These extreme

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values are known as stability and accuracy radii. Formulae to calculate values of both functions and corresponding radii are presented. Final remarks and conclusions appear in section V. The problem of using stability and accuracy functions as efficient tools for post-optimal analysis and alternative ranking is discussed at the end.

## II. PROBLEM FORMULATION

A problem with  $m \geq 2$  objective functions is considered. Let  $X \subset \mathbf{Z}^n$ ,  $|X| \geq 2$  be a non-empty set of feasible solutions or alternatives  $x := \{x_1, \dots, x_n\}^T \neq 0_{(n)} := (0, 0, \dots, 0)^T$ , where  $n$  denotes the problem size and  $\mathbf{Z}^n$  is a set of integer vectors of size  $n$ .

For each solution  $x \in X$ , a vector cost function

$$f(C, x) := (f_1(C, x), \dots, f_m(C, x))^T$$

consists of individual cost functions  $f_i(C, x)$ ,  $i \in N_m := \{1, 2, \dots, m\}$ , which are defined as linear functions, i.e.:

$$f_i(C, x) := C_i x.$$

Here  $C_i$  is  $i$ -th row of matrix  $C = [c_{ij}] \in \mathbf{R}_+^{m \times n}$ , where  $\mathbf{R}_+^{m \times n}$  is a set of  $m \times n$  matrices with all elements being real and positive.

Without loss of generality, it is assumed that  $f_i(C, x)$  are minimized on the set of feasible solutions  $X$  for each  $i \in N_m$ .

Contrary to the single objective case where the concept of optimal solution is unique, under multicriteria framework the concept of optimality may vary and is usually based on binary relations reflecting preferability of one solutions over others (see e.g. [1] and [10]). In its turn, any binary relation generates a principle of optimality (in other terminology, sometimes referred as a choice function).

Let us define two binary relations in the space  $\mathbf{R}^m$  of arbitrary finite dimension  $m \in \mathbf{N}$ , where  $\mathbf{N}$  denotes the set of cardinal numbers:

$$y' \succ_{sl} y \Leftrightarrow y'_i > y_i, \quad i \in N_m,$$

$$y' \succ_{lex} y \Leftrightarrow y'_k > y_k,$$

where  $k = \min\{i \in N_m \mid y'_i \neq y_i\}$ ,  $y = (y_1, y_2, \dots, y_m)$  and  $y' = (y'_1, y'_2, \dots, y'_m)$ .

A solution  $x' \in X$  is called *weakly efficient* if

$$\{x \in X : f(C, x') \succ_{sl} f(C, x)\} = \emptyset.$$

The the set of all weakly efficient solutions is referred to as the *Slater set* and denoted by  $Sl^m(C)$ .

A solution  $x' \in X$  is called *lexicographically optimal* if

$$\{x \in X : f(C, x') \succ_{lex} f(C, x)\} = \emptyset.$$

The set of all lexicographically optimal solutions is referred to as the *lexicographic set* and denoted by  $L^m(C)$ .

The Slater set contains the solutions improvement in some objectives can only be obtained at the expense of some other objective together with solutions where improvement of one or several objectives is possible without deteriorating some other objectives, however all the objectives cannot be improved simultaneously.

Lexicographic optimality is generally applied to the situation where objectives have no equal importance anymore but ordered according to their significance. A rigid arrangements of partial criteria with respect to importance is often used for a wide spectrum of important optimization problems, for example problems of stochastic programming, problems of axiomatic systems of utility theory and so on. Observe also that any scalar constrained optimization problem may be transformed to an unconstrained bicriteria lexicographic problem by using as first criterion some exact penalty function for problem constraints, and an original objective function as a second constraint.

Let  $s \in N_m, \mathcal{I} = \{I_1, I_2, \dots, I_s\}$  be a partition of the set  $N_m$  into non-intersecting non-empty subsets. i.e.

$$N_m = \bigcup_{r \in N_s} I_r,$$

where  $I_r \neq \emptyset$ ,  $r \in N_s$ , and if  $p \neq q$ , then  $I_p \cap I_q = \emptyset$ .

For any such partitioning, define the binary relation  $\vdash_{\mathcal{I}}$  between two vectors  $y, y' \in \mathbf{R}^m$ ,  $m \in \mathbf{N}$ ,  $y \neq y'$ ,  $y = (y_1, y_2, \dots, y_m)$  and  $y' = (y'_1, y'_2, \dots, y'_m)$  as follows:

$$y' \vdash_{\mathcal{I}} y \Leftrightarrow y'_{I_k} >_{I_k} y_{I_k},$$

where  $k := \min\{i \in N_s : y_{I_i} \neq y'_{I_i}\}$ , and  $y_{I_k}$  and  $y'_{I_k}$  are the projections of the vectors  $y$  and  $y'$  correspondingly onto the coordinate axes of the space  $\mathbf{R}^m$  with numbers from the subset  $I_k$ . In other words,  $y_{I_k} = ((y_{I_k})_1, (y_{I_k})_2, \dots, (y_{I_k})_m)$ ,  $y'_{I_k} = ((y'_{I_k})_1, (y'_{I_k})_2, \dots, (y'_{I_k})_m)$ , and  $(y_{I_k})_i = y_i$ ,  $(y'_{I_k})_i = y'_i$  if  $i \in I_k$ , and 0 otherwise. Notice that  $y'_{I_k} >_{I_k} y_{I_k}$  means that  $(y'_{I_k})_i > (y_{I_k})_i$  for all  $i \in I_k$ .

It is easy to check that the binary relation  $\vdash_{\mathcal{I}}$  is antireflexive, asymmetric, transitive, and hence it is cyclic.

The introduced binary relation  $\vdash_{\mathcal{I}}$  determines ordering of the shaped subset of objectives such that any previous subset is significantly more important than any consequent subset. This relation induces the concept of  $\mathcal{I}$ -optimality. A solution  $x' \in X$  is called  $\mathcal{I}$ -optimal if

$$\{x \in X : f(C, x') \vdash_{\mathcal{I}} f(C, x)\} = \emptyset.$$

The set of all  $\mathcal{I}$ -optimal solutions is referred to as the  $\mathcal{I}$ -set and denoted by  $Opt_{\mathcal{I}}^m(C)$ .

It is evident that

$$Opt_{\mathcal{I}}^m(C) = Sl^m(C) \text{ if } \mathcal{I} = N_m \text{ (} s = 1\text{),}$$

and

$$Opt_{\mathcal{I}}^m(C) = L^m(C) \text{ if } \mathcal{I} = \{\{1\}, \{2\}, \dots, \{m\}\} \text{ (} s = m\text{).}$$

Since the set of feasible solutions  $X$  is finite, the set  $Opt_{\mathcal{I}}^m(C)$  is non-empty for any  $C \in \mathbf{R}_+^{m \times n}$  and  $\mathcal{I}$  due to acyclicity of  $\vdash_{\mathcal{I}}$ .

Notice that originally this type of parametrization, which is known as "from lexicographic to Slater" was proposed in [3].

The problem presented above may be considered as a generalization of many combinatorial optimization and integer programming problems. In what follows the problem interpretation in terms of one evaluation and decision making problem is given. Assume a collection of  $|X|$  projects is given and  $m$

experts make evaluation and decision about project acceptance for funding or rejection. The experts specify own penalty costs for each of  $n$  criteria, which are assumed to have negative sense. So the penalty cost matrix  $C \in \mathbf{R}_+^{m \times n}$  is formed as the result. The quantitative measure of presence of negative criteria  $i$  in project  $x$  is reflected by given value  $x_i$ . Also assume that all the experts are divided into coalitions, and the coalitions are ordered by importance, so that any previous coalition of experts is significantly more important than any consequent coalition. The division by coalitions is defined by  $\mathcal{I}$ . The  $\mathcal{I}$ -optimal solutions constitute the set of projects which may be potentially funded.

In what follows assume also that one or several  $\mathcal{I}$ -optimal solutions have been detected, and concentrate on analyzing some quality measures of these solutions with respect to small perturbations of the original matrix of penalty costs. This corresponds to the situation when the experts are not completely sure about exact values of penalty costs, and therefore some sort of uncertainty may appear in the original problem.

### III. RELATIVE ERROR

If the coefficients of the objective functions are allowed to be changed, then initially  $\mathcal{I}$ -optimal solution may lose its optimality. The quality of this solution will be evaluated from the point of view of its robustness on possible data perturbations. Namely, for a given matrix  $C \in \mathbf{R}_+^{m \times n}$  and  $x^* \in \text{Opt}_{\mathcal{I}}^m(C)$ , introduce the so-called *relative error* of this solution:

$$\varepsilon(C, x^*) := \max_{x \in X} \min_{i \in I_1} \frac{f_i(C, x^*) - f_i(C, x)}{f_i(C, x)}. \quad (1)$$

Note that  $\varepsilon(C, x^*) \geq 0$ . The numbers  $\frac{f_i(C, x^*) - f_i(C, x)}{f_i(C, x)}$ ,  $i \in I_1$ ,  $x \in X$  are well-defined, since  $x \neq (0, 0, \dots, 0)^T$  and  $c_{ij} > 0$  for all  $i \in I_1$  and  $j \in N_n$ .

Obviously if  $s = 1$ , i.e.  $\mathcal{I} = \{I_1\} = \{N_m\}$ , and (1) transforms into

$$\varepsilon(C, x^*) = \max_{x \in X} \min_{i \in N_m} \frac{f_i(C, x^*) - f_i(C, x)}{f_i(C, x)}. \quad (2)$$

If  $s = m$ , i.e.  $\mathcal{I} = \{\{1\}, \{2\}, \dots, \{m\}\}$ , and (1) transforms into

$$\varepsilon(C, x^*) = \max_{x \in X} \frac{f_1(C, x^*) - f_1(C, x)}{f_1(C, x)}. \quad (3)$$

In the scalar case, i.e. for  $m = 1$ , the lexicographic and Slater sets transform into the set of optimal solutions. Therefore the relative errors (2) and (3) convert into (see [6]):

$$\varepsilon(C, x^*) = \frac{f_1(C, x^*) - \min_{x \in X} f_1(C, x)}{\min_{x \in X} f_1(C, x)}.$$

The use of relative error is evidently advantageous to the use of simple absolute error, since the deviation from the optimal solution is measured taking into account the cost function ranges.

In the scalar case the equality  $\varepsilon(C, x^*) = 0$  gives necessary and sufficient conditions for the optimality of  $x^*$ . However, in the multicriteria case the situation is a bit different. Observe,

that for arbitrary  $C \in \mathbf{R}_+^{m \times n}$  it is true that  $\varepsilon(C, x^*) \geq 0$ . The equality  $\varepsilon(C, x^*) = 0$  formulates in general only necessary condition for  $x^*$  to be  $\mathcal{I}$ -optimal, i.e.  $\varepsilon(C, x^*) = 0$  does not guarantee that  $x^* \in \text{Opt}^m(C)$ . Indeed, consider the following example.

**Example 1.** Let  $m = 4$ ,  $n = 2$ ,  $s = 2$ ,  $\mathcal{I} = \{\{1, 2\}, \{3, 4\}\}$  and

$$C^0 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Assume also that  $X = \{x^1, x^2\}$ ,  $x^1 = (1, 0)^T$ ,  $x^2 = (0, 1)^T$ . Then  $f(C^0, x^1) = (2, 2, 2, 1)^T$ ,  $f(C^0, x^2) = (2, 2, 1, 2)^T$ , i.e.  $\text{Opt}_{\mathcal{I}}^4(C^0) = \{x^1, x^2\}$ . If consider matrix

$$C = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix},$$

then  $f(C, x^1) = (2, 2, 1, 1)^T$ ,  $f(C, x^2) = (2, 2, 2, 2)^T$ , i.e.  $\text{Opt}_{\mathcal{I}}^4(C) = \{x^1\}$ . Then it can be deduced that  $x^2 \in \text{Opt}^4(C^0)$  and  $\varepsilon(C^0, x^2) = 0$ , but  $x^2 \notin \text{Opt}_{\mathcal{I}}^4(C)$  and  $\varepsilon(C, x^2) = 0$ . This ends the example.

Later it will be shown, that if the equality  $\varepsilon(C, x^*) = 0$  is valid for any matrix in some open neighborhood of  $C^0$ , i.e. there exists  $\phi > 0$  such that  $\varepsilon(C, x^*) = 0$  for any  $C$ ,  $\|C - C^0\| < \phi$ , where  $\|\cdot\|$  denotes a norm in  $\mathbf{R}^{m \times n}$ , then this equality provides also a sufficient condition for the solution  $x^*$  to belong  $\text{Opt}_{\mathcal{I}}^m(C^0)$ .

Now assume that the set of feasible solutions  $X$  is fixed, but the matrix of objective costs  $C$  may vary or be estimated with errors. Moreover, assume also that for some originally specified matrix  $C^0 = [c_{ij}^0] \in \mathbf{R}_+^{m \times n}$ , an  $\mathcal{I}$ -optimal solution  $x^*$  is known, which is an element of the set of  $\mathcal{I}$ -optimal solutions  $\text{Opt}_{\mathcal{I}}^m(C^0)$ .

### IV. STABILITY AND ACCURACY FUNCTIONS

In the following the focus is given to the maximum value of the error  $\varepsilon(C, x^*)$  when the matrix  $C$  belongs to some specified set which describes possible absolute perturbations of the original matrix  $C^0$ . Two particular cases are considered: In the first case the research interest is related to *absolute perturbations* of the weights of elements, and the quality of a given solution  $x^*$  is described by the so-called stability function. For a given  $\rho \geq 0$ , the value of the stability function is equal to the maximum relative error of a given situation under the assumption that no penalty costs are increased or decreased by more than  $\rho$ .

In the second case, the focus is given to *relative perturbations* of weights. This leads to the concept of accuracy function. The value of the accuracy function for a given  $\delta \in [0, 1]$  is equal to the maximum relative error of the solution  $x^*$  under the assumption that the penalty costs are perturbed by no more than  $\delta \cdot 100\%$  of their original values.

The two types of perturbations, absolute and relative, reflect different types of input data uncertainty that may appear in the problem. While the absolute perturbations are usually specified by some global parameter which reflects admissible perturbation range valid for all the elements, the relative perturbations incorporate discrepancy in element ranges, i.e. the range of actual admissible perturbations depends on the nominal element range.

Observe that if two  $\mathcal{I}$ -optimal solutions are compared from the point of view of their robustness on data perturbations or inaccuracy, then smaller value of the stability or accuracy function for a given norm of data perturbation is more preferable. Thus, both defined functions may be used to evaluate the quality of solutions, which are  $\mathcal{I}$ -optimal in the original problem.

For a given  $\rho \in [0, q(C^0))$ , where  $q(C^0) = \min\{c_{ij}^0 : i \in N_m, j \in N_n\}$ , consider a set

$$\Omega_\rho(C^0) := \{C \in \mathbf{R}_+^{m \times n} : |c_{ij} - c_{ij}^0| \leq \rho, i \in N_m, j \in N_n\}.$$

For  $x^* \in \text{Opt}_T^m(C^0)$ , and  $\rho \in [0, q(C^0))$ , the value of the *stability function* is defined as follows:

$$S(C^0, x^*, \rho) := \max_{C \in \Omega_\rho(C^0)} \varepsilon(C, x^*).$$

In a similar way, for a given  $\delta \in [0, 1)$ , consider a set

$$\Theta_\delta(C^0) := \{C \in \mathbf{R}_+^{m \times n} : |c_{ij} - c_{ij}^0| \leq \delta \cdot c_{ij}^0, i \in N_m, j \in N_n\}.$$

For  $x^* \in \text{Opt}_T^m(C^0)$  and  $\delta \in [0, 1)$ , the value of the *accuracy function* is defined as follows:

$$A(C^0, x^*, \delta) := \max_{C \in \Theta_\delta(C^0)} \varepsilon(C, x^*).$$

It is easy to check that  $S(C^0, x^*, \rho) \geq 0$  for any  $\rho \in [0, q(C^0))$  as well as  $A(C^0, x^*, \delta) \geq 0$  for each  $\delta \in [0, 1)$ .

Let

$$\begin{aligned} \Omega'_\rho(C^0) &:= \{C \in \mathbf{R}_+^{m \times n} : |c_{ij} - c_{ij}^0| < \rho, i \in N_m, j \in N_n\}, \\ \Theta'_\delta(C^0) &:= \{C \in \mathbf{R}_+^{m \times n} : |c_{ij} - c_{ij}^0| < \delta \cdot c_{ij}^0, i \in N_m, j \in N_n\}. \end{aligned}$$

Note that  $\text{cl}(\Omega'_\rho(C^0)) = \Omega_\rho(C^0)$  as well as  $\text{cl}(\Theta'_\delta(C^0)) = \Theta_\delta(C^0)$ .

Here two results will be formulated which motivate the use of stability and accuracy functions as possible quantitative measures to estimate solution robustness.

**Proposition 1:** For  $x^* \in \text{Opt}_T^m(C^0)$  and  $\rho \in [0, q(C^0))$ , it is true that  $x^* \in \text{Opt}_T^m(C)$  for any  $C \in \Omega'_\rho(C^0)$  if and only if  $S(C^0, x^*, \rho) = 0$ .

**Proposition 2:** For  $x^* \in \text{Opt}_T^m(C^0)$  and  $\delta \in [0, 1)$ , it is true that  $x^* \in \text{Opt}_T^m(C)$  for any  $C \in \Theta'_\delta(C^0)$  if and only if  $A(C^0, x^*, \delta) = 0$ .

For any  $z \in \mathbf{R}^m$ , denote *linear norm* in standard way:

$$\|z\|_1 := \sum_{i \in N_m} |z_i|,$$

The following theorem gives a formula for calculating value of the stability function.

**Theorem 1:** For  $x^* \in \text{Opt}_T^m(C^0)$  and  $\rho \in [0, q(C^0))$ , the stability function can be expressed by the formula:

$$S(C^0, x^*, \rho) = \max_{x \in X} \min_{i \in I_1} \frac{C_i^0(x^* - x) + \rho \|x^* - x\|_1}{C_i^0 x - \rho \|x\|_1}. \quad (4)$$

**Theorem 2:** For  $x^* \in \text{Opt}_T^m(C^0)$  and  $\delta \in [0, 1)$ , the accuracy function can be expressed by the formula:

$$A(C^0, x^*, \delta) = \max_{x \in X} \min_{i \in I_1} \frac{C_i^0(x^* - x) + \delta \sum_{j \in I_n} c_{ij}^0 |x_j^* - x_j|}{C_i^0 x (1 - \delta)}.$$

For shortening purposes the proofs will be omitted.

Propositions 1 and 2 suggest, that it is of special interest to know the extreme values of  $\rho$  and  $\delta$ , for which  $S(C^0, x^*, \rho) = 0$  and  $A(C^0, x^*, \delta) = 0$ , respectively, because these values determine maximum norms of perturbations which preserve the property of a given solution to be  $\mathcal{I}$ -optimal. These values are analogous to the so-called stability and accuracy radii introduced earlier for single/multiple objective combinatorial optimization problems (see e.g. [2]). Formally, the stability radius  $R^S(C^0, x^*)$  and the accuracy radius  $R^A(C^0, x^*)$  are defined in the following way:

$$R^S(C^0, x^*) := \sup \{\rho \in [0, q(C^0)) : S(C^0, x^*, \rho) = 0\},$$

$$R^A(C^0, x^*) := \sup \{\delta \in [0, 1) : A(C^0, x^*, \delta) = 0\}.$$

If these radii are equal to zero, then it means that there exist arbitrary small perturbations of the original matrix  $C^0$  such that the initially  $\mathcal{I}$ -optimal solution  $x^*$  loses its  $\mathcal{I}$ -optimality. Otherwise, the solution  $x^*$  remains  $\mathcal{I}$ -optimal for any problem with matrix  $C \in \Omega_\rho(C^0)$ ,  $\rho < R^S(C^0, x^*)$  or  $C \in \Theta_\delta(C^0)$ ,  $\delta < R^A(C^0, x^*)$ .

The following theorems give formulae for calculating value of the stability and accuracy radii.

**Theorem 3:** (c.f. [3]) For  $x^* \in \text{Opt}_T^m(C^0)$ , the stability radius can be expressed by the formula:

$$R^S(C^0, x^*) = \min \left\{ q(C^0), \min_{x \in X \setminus \{x^*\}} \max_{i \in I_1} \frac{C_i^0(x - x^*)}{\|x - x^*\|_1} \right\}. \quad (5)$$

**Theorem 4:** For  $x^* \in \text{Opt}_T^m(C^0)$  and  $\delta \in [0, 1)$ , the accuracy radius can be expressed by the formula:

$$R^A(C^0, x^*, \delta) = \min \left\{ 1, \min_{x \in X \setminus \{x^*\}} \max_{i \in I_1} \frac{C_i^0(x - x^*)}{\sum_{j \in N_n} c_{ij}^0 |x_j - x_j^*|} \right\}.$$

As corollaries from theorem 3, the following well-known results (c.f. [3]) can be formulated:

**Corollary 1:** The stability radius of  $x^* \in \text{Sl}^m(C^0)$  can be expressed by the formula

$$R^S(C^0, x^*) = \min \left\{ q(C^0), \min_{x \in X \setminus \{x^*\}} \max_{i \in N_m} \frac{C_i^0(x - x^*)}{\|x - x^*\|_1} \right\}.$$

**Corollary 2:** The stability radius of  $x^* \in L^m(C^0)$  can be expressed by the formula

$$R^S(C^0, x^*) = \min \left\{ q(C^0), \min_{x \in X \setminus \{x^*\}} \frac{C_1^0(x - x^*)}{\|x - x^*\|_1} \right\}.$$

## V. CONCLUSION

Now it becomes evident that small changes or inaccuracies in estimating objective function coefficients may have significant influence on the set of  $\mathcal{I}$ -optimal solutions. Moreover, some solutions being initially optimal, cannot be considered as stable, because very small changes of input data destroy their properties of being optimal.

The simplest measure of the stability of the optimum is its stability or accuracy radius. But frequently, this measure is not sufficient to rank the solutions, among multiple optimal solutions, which so often occur in multicriteria optimization. Therefore, calculating stability radii only may be not sufficient to make a conclusion about solution stability, thus it is necessary to calculate some complementary measures reflecting more information about solution behavior under uncertainty.

The accuracy and stability functions describe the quality of  $\mathcal{I}$ -optimal solutions in the problem with uncertain coefficients of objective functions. The definitions of these functions are directly connected with a given optimality principle. Such common optimality principles, as weak and lexicographic optimality, may not fully cover all of the decision maker preferences. Sometimes, introducing a parameterized version of optimality principles may reflect the desirable preference specific much better.

The accuracy and stability functions can be potentially used as an efficient tool for ranking multiple optimal solutions. The problem of solution ranking is especially important in multi-objective optimization where a large number of solutions can be optimal, and therefore there is a need to find some methods which could help to select the most preferred solutions. Most previously known selection procedures are mainly based on asking decision maker about his or her specific preferences, and therefore very subjective and personal. Provided some efficient algorithms how to calculate the exact or approximated values of stability and accuracy functions could be found, they might become a very efficient tool for optimal alternative ranking in multicriteria optimization. The concepts of stability and accuracy functions would provide more flexible approach to the alternative ranking which is less sensitive to the decision maker personality.

As possible continuation of the research within this topic, it would be interesting to study whether the approach presented in this paper may be extended for the case of non-linear objective functions. Finding efficient strategies to compute stability and accuracy functions for the problems with larger dimension may be also interesting to study.

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