Spin One Hawking Radiation from Dirty Black Holes

Petarpa Boonserm, Tritos Ngampitipan, Matt Visser

Abstract—A "clean" black hole is a black hole in vacuum such as the Schwarzschild black hole. However in real physical systems, there are matter fields around a black hole. Such a black hole is called a "dirty black hole". In this paper, the effect of matter fields on the black hole and the greybody factor is investigated. The results show that matter fields make a black hole smaller. They can increase the potential energy to a black hole to obstruct Hawking radiation to propagate. This causes the greybody factor of a dirty black hole to be less than that of a clean black hole.

Keywords—A dirty black hole, Greybody factor, Hawking radiation, Matter fields.

I. INTRODUCTION

$\mathbf{I}$N general relativity, the spherically symmetric vacuum solution is the Schwarzschild metric. A Schwarzschild black hole, which is described by the Schwarzschild metric, is a black hole in empty space. It is referred to a "clean" black hole. However, in real situations a black hole is surrounded by various types of matter fields and referred to a "dirty" black hole [1]-[3]. We are interested in the presence of matter fields around a black hole. In this paper, the effect of matter fields on the black hole and the greybody factor is studied.

II. THE SCHWARZSCHILD BLACK HOLE

The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2, \qquad (1)$$

where M is the Schwarzschild black hole mass. In this paper, we are interested in a spin one particle radiated from the Schwarzschild black hole. We define the tortoise coordinate r_* by

$$dr_* = \frac{dr}{1 - 2M/r}. \qquad (2)$$

The Schwarzschild metric can be rewritten as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\left(-dt^2 + dr_*^2\right) + r^2 d\Omega^2. \qquad (3)$$

The Schrödinger-like equation is given by

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$$\frac{d^2\psi}{dr_*^2} + [\omega^2 - V(r)]\psi = 0, \qquad (4)$$

where the potential for a vector field is [4]

$$V(r) = \left(1 - \frac{2M}{r}\right)\frac{l(l+1)}{r^2}. \qquad (5)$$

The potential is plotted with $l = 1$ and $M = 3$ as shown in Fig. 1. In one-dimensional scattering problem, there are very general and rigorous bounds on the transmission probability [5]. Application for the generic systems can be found in [6]-[8]. For developments in applying to black hole greybody see [9], [10]. The lower bounds on the transmission probabilities are given by [5] (see also [11], [12])

$$T \geq \mathrm{sech}^2\left(\int_{-\infty}^{\infty} \vartheta \, dr_*\right), \qquad (6)$$

where

$$\vartheta = \frac{\sqrt{(h')^2 + (\omega^2 - V - h^2)^2}}{2h}, \qquad (7)$$

for any positive function h. We set $h = \omega$, then

$$T \geq \mathrm{sech}^2\left(\frac{1}{2\omega}\int_{-\infty}^{\infty} V \, dr_*\right). \qquad (8)$$

From (2), the tortoise coordinate can be written as

$$r_* = r + 2M \ln\left(\frac{r - 2M}{2m}\right). \qquad (9)$$

when $r_* \to -\infty, r \to 2M$ and when $r_* \to \infty, r \to \infty$. Therefore,

$$T_{\mathrm{sch}} \geq \mathrm{sech}^2\left(\frac{1}{2\omega}\int_{2M}^{\infty} \frac{l(l+1)}{r^2} dr\right) = \mathrm{sech}^2\left(\frac{l(l+1)}{4\omega M}\right) \quad (10)$$

The transmission probability and the energy of the emitted particle for the Schwarzschild black hole are plotted as shown in Fig. 2.
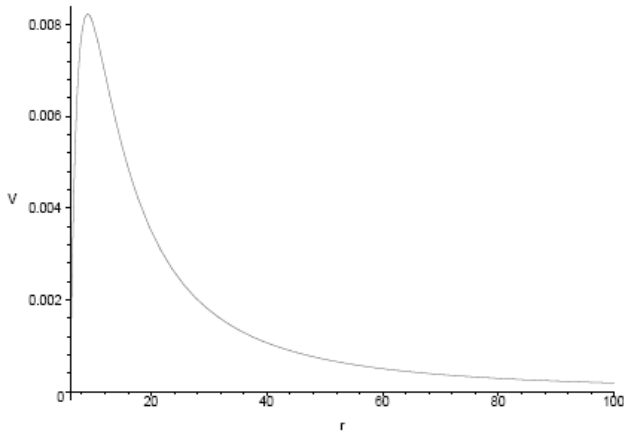
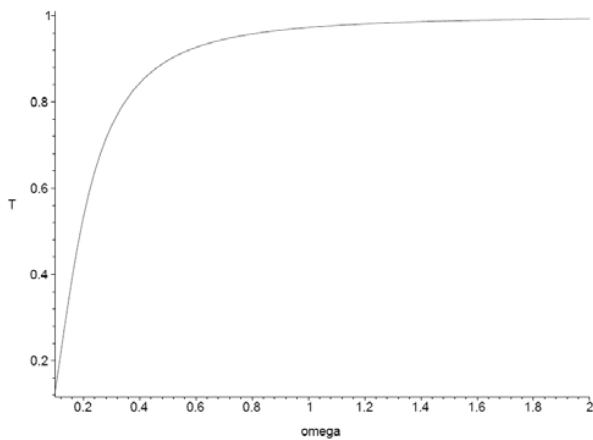Fig. 1 The Schwarzschild potential with $l = 1$ and $M = 3$

Fig. 2 The transmission probability for the Schwarzschild black hole with $l = 1$ and $M = 3$

III. THE DIRTY BLACK HOLE

The metric of a generic static spherically symmetric spacetime is given by [1]

$$ds^2 = -e^{-2\phi(r)}\left(1 - \frac{2m(r)}{r}\right)dt^2 + \frac{dr^2}{1-2m(r)/r} + r^2 d\Omega^2 \quad (11)$$

where $m(r)$ and $\phi(r)$ are arbitrary functions. If the black hole event horizon exists, the function $m(r)$ will satisfy the boundary condition $2m(r_H) = r_H$, where r_H is the horizon radius. Asymptotic flatness of spacetime is assumed so that $\phi(\infty) = 0$ and $m(\infty)$ is finite. The Einstein equations are

$$\frac{dm}{dr} = 4\pi\rho r^2 \text{ and } \frac{d\phi}{dr} = -\frac{4\pi(\rho+p_r)r}{1-2m(r)/r}. \quad (12)$$

In this paper, we find an appropriate choice for $m(r)$ and $\phi(r)$ in order to derive the specific results. The Weak Energy Condition (WEC) states that $\rho \geq 0$. Thus the function $m(r)$ is an increasing function. Because of finiteness of $m(r)$ at spatial infinity, this motivate us to assume $\rho r^4 = c_1$, where c_1 is a constant. Therefore,

$$m(r) = -\frac{4\pi c_1}{r} + c_2 \quad (13)$$

where c_2 is a constant of integration. Applying the boundary condition $2m(r_H) = r_H$, we obtain

$$c_2 = \frac{r_H}{2} + \frac{4\pi c_1}{r_H}. \quad (14)$$

Therefore,

$$m(r) = \frac{r_H}{2} + 4\pi c_1 \left(\frac{1}{r_H} - \frac{1}{r}\right). \quad (15)$$

Applying the boundary condition $m(\infty) = M$, we obtain

$$4\pi c_1 = \frac{(2M-r_H)r_H}{2}. \quad (16)$$

thus

$$m(r) = M - \frac{(2M-r_H)r_H}{2}. \quad (17)$$

That is

$$1 - \frac{2m(r)}{r} = 1 - \frac{2M}{r} - \frac{(2M-r_H)r_H}{r^2}. \quad (18)$$

Since $m(r)$ is an increasing function, then

$$m(r_H) \leq m(\infty) \quad (19)$$

or

$$r_H \leq 2M. \quad (20)$$

It can be seen that the horizon radius of the dirty black hole is smaller than that of the Schwarzschild black hole. Thus, matter fields around the black hole can shorten its horizon radius. The dirty black hole is, therefore, smaller than the Schwarzschild black hole for the same mass. From (12), we obtain

$$\frac{d\phi}{dr} = -\frac{4\pi(\rho+p_r)r}{1-2M/r-(2M-r_H)r_H/r^2}. \quad (21)$$

so

$$\phi(r) = \ln\left(\frac{r^{1/2}}{x^{a_2/2}y^{a_3/2}}\right) + c_3, \quad (22)$$

where

$$x \equiv r - M - \sqrt{M^2 + A}$$

$$y \equiv r - M + \sqrt{M^2 + A}$$

$$a_2 \equiv \frac{A}{\left(\sqrt{M^2 + A} + M\right)\left(2\sqrt{M^2 + A}\right)}$$

$$a_3 \equiv \frac{A}{\left(\sqrt{M^2 + A} - M\right)\left(2\sqrt{M^2 + A}\right)}$$

$$A \equiv (2M - r_H)r_H$$

and c_3 is a constant of integration. Derivation of (22) can be seen in Appendix A. Applying the boundary condition $\phi(\infty) = 0$, we obtain $c_3 = 0$. Therefore,

$$\phi(r) = \ln\left(\frac{r^{1/2}}{x^{a_2/2}y^{a_3/2}}\right). \tag{23}$$

The Schrödinger-like equation is given by

$$\frac{d^2\psi}{dr_*^2} + [\omega^2 - V(r)]\psi = 0, \tag{24}$$

where

$$dr_* = \frac{e^{\phi(r)}}{1 - 2m(r)/r}dr. \tag{25}$$

and the potential for a vector field is [13]

$$V(r) = e^{-2\phi(r)}\left(1 - \frac{2m(r)}{r}\right)\frac{l(l+1)}{r^2}. \tag{26}$$

The dirty potential is plotted with $l = 1$, $M = 3$, and $r_H = 1$ as shown in Fig. 3. The comparison between the dirty potential and the Schwarzschild potential is plotted in Fig. 4. It can be seen that the dirty potential is higher than the Schwarzschild potential. Increment of the potential energy in case of the dirty black hole comes from matter fields. The lower bounds on the transmission probabilities are given by [13]

$$T_d \geq \mathrm{sech}^2\left(\frac{1}{2\omega}\int_{-\infty}^{\infty}Vdr_*\right) = \mathrm{sech}^2\left(\frac{1}{2\omega}\int_{r_H}^{\infty}e^{-\phi(r)}\frac{l(l+1)}{r^2}dr\right) \geq \mathrm{sech}^2\left(\frac{l(l+1)}{2\omega r_H}\right) \tag{27}$$

Because of (20), we derive

$$T_d \leq T_{\mathrm{sch}}. \tag{28}$$

Fig. 5 shows the plot between the transmission probability and the energy of the emitted particle for the dirty black hole with $l = 1$, $M = 3$, and $r_H = 1$. Fig. 6 shows the comparison between the transmission probabilities of the Schwarzschild black hole and the dirty black hole. The graph confirms the validity of (28).
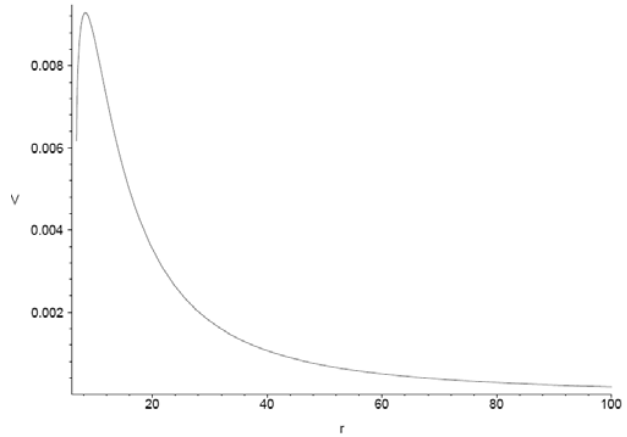

Fig. 3 The dirty potential with $l = 1$, $M = 3$, and $r_H = 1$
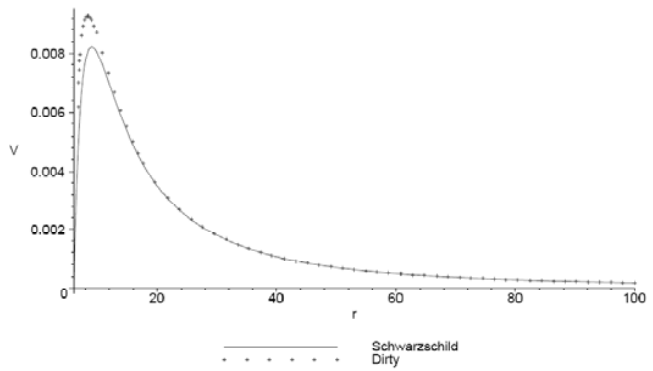

Fig. 4 The comparison between the dirty potential and the Schwarzschild potential with $l = 1$, $M = 3$, and $r_H = 1$
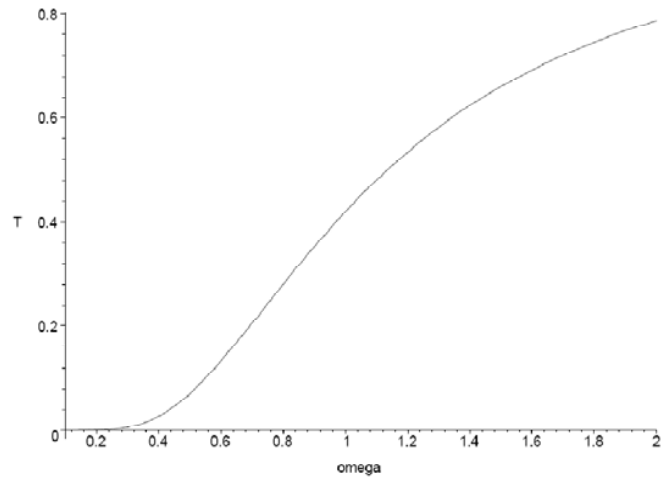

Fig. 5 The transmission probability for the dirty black hole with $l = 1$, $M = 3$, and $r_H = 1$
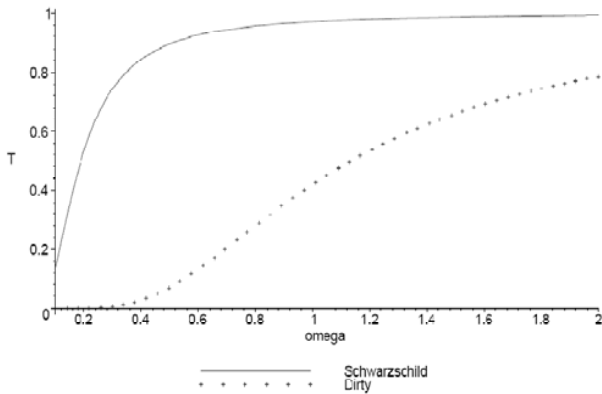
Fig. 6 The comparison between the transmission probabilities of the Schwarzschild black hole and the dirty black hole with $l = 1$, $M = 3$, and $r_H = 1$

IV. Conclusion

In this paper, we have investigated what effect of matter fields surrounding a black hole on the black hole and the greybody factor. It has been found that a black hole with the presence of matter fields is smaller than a clean black hole with the same mass. Matter fields around a black hole can shorten its horizon radius. Increment of the potential energy in case of the dirty black hole is an effect of matter fields. This causes the greybody factor of a dirty black hole to be less than that of a clean black hole. Matter fields can obstruct a spin one Hawking radiation to propagate.

Appendix

A. Explicit Form of $\phi(r)$

For the simple model, assume that we are now in the matter-dominated universe. So $p_r = 0$. From (21), we obtain

$$\frac{d\phi}{dr} = -\frac{4\pi\rho r}{1 - \frac{2M}{r} - \frac{(2M - r_H)r_H}{r^2}}. \quad (29)$$

Using $\rho r^4 = c_1$, we derive

$$\frac{d\phi}{dr} = -\frac{4\pi\rho r^4}{r^3 - 2Mr^2 - (2M - r_H)r_H r} = -\frac{4\pi c_1}{r[r^2 - 2Mr - (2M - r_H)r_H]}. \quad (30)$$

From (16), then

$$\frac{d\phi}{dr} = -\frac{(2M - r_H)r_H}{2r[r^2 - 2Mr - (2M - r_H)r_H]}. \quad (31)$$

Let $A \equiv (2M - r_H)r_H$. The denominator of (31) can be factorized

$$r^2 - 2Mr - A = r^2 - 2Mr + M^2 - M^2 - $$
$$= (r - M)^2 - (M^2 + A) = \left(r - M - \sqrt{M^2 + A}\right)\left(r - M + \sqrt{M^2 + A}\right) \quad (32)$$

Using the partial faction, the right hand side of (31) becomes

$$\frac{A}{r(r^2 - 2Mr - A)} = \frac{a_1}{r} + \frac{a_2}{r - M - \sqrt{M^2 + A}} + \frac{a_3}{r - M + \sqrt{M^2 + A}} \quad (33)$$

$$= \frac{a_1}{r} + \frac{a_2 r\left(r - M + \sqrt{M^2 + A}\right)}{r(r^2 - 2Mr - A)} + \frac{a_3 r\left(r - M - \sqrt{M^2 + A}\right)}{r(r^2 - 2Mr - A)} \quad (34)$$

equating the coefficients of powers of r gives

$$a_1 = -1$$
$$a_2 = \frac{A}{\left(\sqrt{M^2 + A} + M\right)\left(2\sqrt{M^2 + A}\right)}$$
$$a_3 = \frac{A}{\left(\sqrt{M^2 + A} - M\right)\left(2\sqrt{M^2 + A}\right)}.$$

We define $x \equiv r - M - \sqrt{M^2 + A}$ and $y \equiv r - M + \sqrt{M^2 + A}$. Using (33), we can integrate (31)

$$\phi(r) = -\frac{1}{2}\int\left(-\frac{1}{r} + \frac{a_2}{x} + \frac{a_3}{y}\right)dr = \frac{1}{2}\ln r - \frac{a_2}{2}\ln x - \frac{a_3}{2}\ln y + c_3 = \ln\left(\frac{r^{1/2}}{x^{a_2/2}y^{a_3/2}}\right) + c_3 \quad (35)$$

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