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Spin Coherent State Path Integral for the Interaction of Two-Level System with Time Dependent Non-Uniform Magnetic Field

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Abstract—We study the movement of a two-level atom in interaction with time dependent nonuniform magnetic filed using the path integral formalism. The propagator is first written in the standard form by replacing the spin by a unit vector aligned along the polar and azimuthal directions. Then it is determined exactly using perturbation methods. Thus the Rabi formula of the system are deduced.

Keywords—Path integral, Formalism, Propagator, Transition probability.

I. Introduction

UP to now, a whole class of potentials have been treated successfully within the path-integral formalism, thanks to the use of certain transformations [1]. However, it is known that the most relativistic interactions are those where the spin is taken into account which is a very useful and very important notion in physics. From a practical point of view, the explicit calculus of propagators for such interactions by the path-integral formalism, are very scarce([2], [3], [4]).

For this reason we are devoted to this type of interaction; by considering a problem treats according to usual quantum mechanics [5]. It acts of an atom which has two levels and which interacts with a time dependent nonuniform magnetic filed.

$$\mathbf{B}\left(t\right)=\left(B_1\cos\omega t,\quad B_1\sin\omega t,\ B_0\right), \tag{1}$$

Its dynamics is described by the Hamiltonian

$$H=-\gamma\mathbf{SB} \tag{2}$$

where γ is the gyromagnetic ratio. In the above Hamiltonian we have neglected the exterior motion.

Another form of the Hamiltonian suitable for our calculations is

$$H=-u\left(t\right)\sigma_{+}-u^{*}\left(t\right)\sigma_{-}-\Omega\left(t\right)\sigma_z \tag{3}$$

where

$$u\left(t\right)=\frac{\gamma}{2}\left(B_x(t)-iB_y(t)\right)=\omega_1e^{-i\omega t}, \tag{4}$$

with

$$\omega_1=\frac{\gamma}{2}B_1\quad\text{and}\qquad\Omega\left(t\right)=\frac{\gamma}{2}B_0=\omega_0 \tag{5}$$

The Pauli matrices are the following:

$$\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \sigma_{+}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ \sigma_{-}=\begin{pmatrix}0&0\\1&0\end{pmatrix}. \tag{6}$$

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Considering this problem by the path integral approach, our motivation is the following. We show that for interaction with the coupling of spin-field type, the propagator is first, by construction, written in the standard form $\sum_{\text{path}}\exp\left(iS\left(path\right)/\hbar\right)$, where S is the action that describes the system. The discrete variable relative to spin being inserted as the (continuous) path using coherent states. With this approach, the formulation that uses the concept of trajectory is more suitable for a discussion of the semiclassical case which is based on the determination of classical paths [6].

The paper is organized as follows. In Section II we give some notation and the spin coherent state path integral for spin $\frac{1}{2}$ system for our further computations. In Section III, after setting up a path integral formalism for the propagator, we perform the direct calculations. The integration over the spin variables is easy to carry out and the result is given as a perturbation series. These are summed up exactly and the explicit result of the propagator is directly computed and the Rabi formula is then deduced. Finally, in Section IV, we present our conclusions.

II. Path-integral formulation

There are several ways to represent the spin in the path integral formalism ([2], [7], [8], [9]). We use the simplest way ([4], [10], [6]), which consists of:

- replacing σ by a unit vector $\mathbf{n}$ directed according to (θ,φ);
- associating a coherent state $|\Omega\rangle$

  $$|\Omega\rangle=|\theta,\varphi\rangle=e^{-i\varphi S_z}e^{-i\theta S_y}\left|\uparrow\right\rangle, \tag{7}$$

  obtained from two rotations of the angles θ and ϕ around z and y axes over the state $|\uparrow\rangle$, and whose scalar product and projector are respectively:

  $$\langle\Omega|\,\Omega'\rangle=\cos\frac{\theta}{2}\cos\frac{\theta'}{2}e^{\frac{i}{2}(\varphi-\varphi')}+\sin\frac{\theta}{2}\sin\frac{\theta'}{2}e^{-\frac{i}{2}(\varphi-\varphi')}, \tag{8}$$

  $$\frac{1}{2\pi}\int d\varphi d\cos(\theta)\left|\Omega\right\rangle\left\langle\Omega\right|=\mathbf{I}. \tag{9}$$

Now we move to the description of the system via path integral. For this we consider the quantum state $|\theta,\varphi\rangle$, where the polar angles (θ,φ) are the spin-related variables.

The transition amplitude from the initial state $|\theta_i,\varphi_i\rangle$ at $t_i=0$ to the final state $|\theta_f,\varphi_f\rangle$ at $t_f=T$ is defined with the matrix elements of the evolution operator:

$$K\left(f,i;T\right)=\langle\theta_f,\varphi_f\mid\mathbf{T}_D\exp(-i\int_0^T Hdt)\mid\theta_i,\varphi_i\rangle, \tag{10}$$

where $\mathbf{T}_D$is the Dyson chronological operator.

To move to path integral representation, we first subdivide the time interval $[0,T]$ into $N+1$ intervals of length ε, intermediate moments, by using the Trotter's formula and we then introduce the projectors according to these intermediate instants N regularly divided distributes between 0 and T in (9) .

Thus the propagator takes the following form:

$$K\left(f,i;T\right)=\lim_{N\longrightarrow\infty}\int\prod_{n=1}^{N}\frac{d\cos(\theta_n)d\varphi_n}{2\pi} \tag{11}$$

$$\times\prod_{n=1}^{N+1}\left[\left\langle\Omega_n\mid\Omega_{n-1}\right\rangle-i\varepsilon\left\langle\Omega_n\right|H\left|\Omega_{n-1}\right\rangle\right].$$

where

$$\Omega_{N+1}=\Omega_f\quad\text{and}\qquad\Omega_0=\Omega_i. \tag{12}$$

It is easy to find that the following matrix elements can be calculated:

$$\left\langle\Omega\right|\sigma_z\left|\Omega'\right\rangle=$$

$$\cos\frac{\theta}{2}\cos\frac{\theta'}{2}e^{+\frac{i}{2}\left(\varphi-\varphi'\right)}-\sin\frac{\theta}{2}\sin\frac{\theta'}{2}e^{-\frac{i}{2}\left(\varphi-\varphi'\right)}, \tag{13}$$

$$\left\langle\Omega\right|\sigma_+\left|\Omega'\right\rangle=\cos\frac{\theta}{2}\sin\frac{\theta'}{2}e^{+\frac{i}{2}\left(\varphi+\varphi'\right)}, \tag{14}$$

$$\left\langle\Omega\right|\sigma_-\left|\Omega'\right\rangle=\sin\frac{\theta}{2}\cos\frac{\theta'}{2}e^{-\frac{i}{2}\left(\varphi+\varphi'\right)}. \tag{15}$$

and the propagator related to our problem (10) takes the form of Feynman path integral

$$K=\int Dpath\exp(iAction), \tag{16}$$

which means in our case:

$$K\left(f,i;T\right)=\int\prod_{n=1}^{N}\frac{d\cos(\theta_n)d\varphi_n}{2\pi}$$

$$\exp\left\{\sum_{n=1}^{n=N+1}\left[\log<\Omega_n\left|\Omega_{n-1}\right\rangle-i\varepsilon\frac{\left\langle\Omega_n\right|H\left|\Omega_{n-1}\right\rangle}{<\Omega_n\left|\Omega_{n-1}\right\rangle}\right]\right\} \tag{17}$$

After having obtained the conventional form, it remains to integrate it, in order to extract the interesting physical properties. We thus proceed to the calculation of $K\left(f,i;T\right)$.

III. The propagator calculation

We note that (16) is written in the following form

$$\begin{aligned}K\left(f,i;T\right) &= \lim_{N\longrightarrow\infty}\int\prod_{n=1}^{N}\frac{d\cos(\theta_n)d\varphi_n}{2\pi}\\ &\times\prod_{n=1}^{N+1}\begin{pmatrix}\cos\frac{\theta_n}{2}e^{+\frac{i}{2}\varphi_n}, & \sin\frac{\theta_n}{2}e^{-\frac{i}{2}\varphi_n}\end{pmatrix}\\ &\times R(t_n)\begin{pmatrix}\cos\frac{\theta_{n-1}}{2}e^{-\frac{i}{2}\varphi_{n-1}}\\ \sin\frac{\theta_{n-1}}{2}e^{+\frac{i}{2}\varphi_{n-1}}\end{pmatrix},\end{aligned} \tag{18}$$

with

$$R(t_n)=\left[e^{i\varepsilon\Omega\sigma_z}+i\varepsilon\begin{pmatrix}0 & u\left(t_n\right)\\ u^*(t_n) & 0\end{pmatrix}\right]. \tag{19}$$

Let us integrate over all angular variables θ_n and φ_n according to [4] then (17) becomes

$$\begin{aligned}K\left(f,i;T\right) &= \begin{pmatrix}\cos\frac{\theta_f}{2}e^{\frac{i}{2}\varphi_f}, & \sin\frac{\theta_f}{2}e^{-\frac{i}{2}\varphi_f}\end{pmatrix}\\ &\times\lim_{N\longrightarrow\infty}\left(-1\right)^N\prod_{\substack{n=1\\ \leftarrow}}^{N+1}R(t_n)\begin{pmatrix}\cos\frac{\theta_i}{2}e^{-\frac{i}{2}\varphi_i}\\ \sin\frac{\theta_i}{2}e^{\frac{i}{2}\varphi_i}\end{pmatrix}.\end{aligned} \tag{20}$$

We develop the product (of 2×2 matrix) which appear in the expression (19) according to [9] we can see that the propagator takes the following form

$$\begin{aligned}K\left(f,i;T\right) &= \begin{pmatrix}\cos\frac{\theta_f}{2}e^{\frac{i}{2}\varphi_f}, & \sin\frac{\theta_f}{2}e^{-\frac{i}{2}\varphi_f}\end{pmatrix}\\ &\times\begin{pmatrix}R_{11}\left(T\right) & -R_{12}\left(T\right)\\ -R_{21}\left(T\right) & R_{22}\left(T\right)\end{pmatrix}\begin{pmatrix}\cos\frac{\theta_i}{2}e^{-\frac{i}{2}\varphi_i}\\ \sin\frac{\theta_i}{2}e^{\frac{i}{2}\varphi_i}\end{pmatrix},\end{aligned} \tag{21}$$

where R_{ij} are the elements of the matrix R. These are numbers which are represent the possible propagators between two given spin states.

The angles θ, φ are allowed to vary only in the limited domains $[0,2\pi]$ and $[0,4\pi]$. Our propagator is the following:

$$\begin{aligned}K\left(f,i;T\right) &= \sum_{n=-\infty}^{+\infty}K\left(\theta_f+2n\pi,\varphi_f+4n\pi;\ \theta_i,\varphi_i;T\right)\\ &= K\left(\theta_f,\varphi_f;\theta_i,\varphi_i;T\right).\end{aligned} \tag{22}$$

A simple calculation shows that the expression of the elements R_{ij} are the following [9]:

$$\begin{aligned}R_{11}\left(T\right) &= e^{i\int_0^T\Omega(s)ds}+\sum_{n=1}^{\infty}\left[i^{2n}\int_0^T ds_1\int_0^{s_1}ds_2\cdots\right.\\ &\int_0^{s_{2n-1}}ds_{2n}e^{i\int_{s_1}^T\Omega(s)ds}u\left(s_1\right)e^{-i\int_{s_2}^{s_1}\Omega(s)ds}\\ &\left.\ldots.e^{-i\int_{s_{2n}}^{s_{2n-1}}\Omega(s)ds}u^*\left(s_{2n}\right)e^{i\int_0^{s_{2n}}\Omega(s)ds}\right]\end{aligned} \tag{23}$$

and

$$R_{12}\left(T\right)=i\int_0^T ds_1e^{i\int_{s_1}^T\Omega(s)ds}u\left(s_1\right)R_{22}\left(s_1\right) \tag{24}$$

From the above expressions we can see that the elements R_{ij} verify

$$R_{22}\left(T\right)=R_{11}^*\left(T\right),\qquad R_{21}\left(T\right)=-\ R_{12}^*\left(T\right) \tag{25}$$

Thus the transition amplitude of the system between an initial state of spin m_i and a final state of spin m_f is related to $K\left(f,i;T\right)$ by

$$\begin{aligned}K\left(m_f,,m_i;T\right) &= \int\frac{d\cos(\theta_f)d\varphi_f}{2\pi}\frac{d\cos(\theta_i)d\varphi_i}{2\pi}\\ &\times\left\langle m_f\left|\Omega_f\right\rangle K\left(f,i;T\right)\left\langle\Omega_i\right|m_i\right\rangle,\end{aligned} \tag{26}$$

where

$$\begin{aligned}\left\langle m\left|\theta,\varphi\right\rangle\right. &= \sqrt{\frac{(2s)!}{(s+m)!\,(s-m)!}}\left(\sin\frac{\theta}{2}\right)^{s-m}\\ &\times\left(\cos\frac{\theta}{2}\right)^{s+m}e^{-im\varphi}.\end{aligned} \tag{27}$$

IV. The transition probability

If we fix the initial state of the atom to be $|m_i\rangle = |\downarrow\rangle$, and the finale state to be $|m_f\rangle = |\uparrow\rangle$, thus from (26) we obtain

$$\langle\uparrow|\Omega_f\rangle = \cos\left(\frac{\theta_f}{2}\right)e^{-\frac{i}{2}\varphi_f} \text{ and } \quad \langle\Omega_i|\downarrow\rangle = \sin\left(\frac{\theta_i}{2}\right)e^{+\frac{i}{2}\varphi_i}. \tag{28}$$

After integration over polar coordinates we obtain for instance the propagator $K(\uparrow,,\downarrow;T)$ between an up-state and a down-state of spin which coincides with element $R_{12}(T)$

$$\begin{aligned} K(\uparrow,,\downarrow;T) &= R_{12}(T) \\ &= \sum_{n=0}^{\infty}\left[(i\omega_1)^{2n+1}\int_0^T ds_1 e^{-i\Delta s_1}\int_0^{s_1} ds_2 e^{+i\Delta s_2}\cdots \right. \\ &\left. \int_0^{s_{2n-1}} ds_{2n+1}e^{-i\Delta s_{2n+1}}\right]e^{i\omega T} \end{aligned} \tag{29}$$

where $\Delta = \omega - \omega_0$.

Now we pass to its Laplace's transformation and apply the convolution theorem we obtain the result

$$K(\uparrow,,\downarrow;p) = \int_0^\infty dT e^{-pT}K(\uparrow,,\downarrow;T) = \frac{i\omega_1}{p(p+i\Delta)-\omega_1^2} \tag{30}$$

Taking the inverse Laplace transform we have

$$K(\uparrow,,\downarrow;T) = \frac{i\omega_1 e^{i(\omega-\Delta/2)T}}{\lambda}\sin\lambda T, \tag{31}$$

with $\lambda = \frac{1}{2}\sqrt{(\omega-\omega_0)^2+4\omega_1^2}$. The transition probability is then given by

$$\begin{aligned} P_{-1/2,1/2} &= |K(\uparrow,,\downarrow;T)|^2 \\ &= \frac{4\omega_1^2}{(\omega-\omega_0)^2+4\omega_1^2}\sin^2\frac{T}{2}\sqrt{(\omega-\omega_0)^2+4\omega_1^2} \end{aligned} \tag{32}$$

This formula is exactly the well-known Rabi formula given in the literature [5].

V. Conclusion

By using the formalism of the path integral and the spin coherent states approach, we showed how to determine the Rabi formula. The propagator relative to a system have been given in the series form, which for this case, these series are summed up exactly. The Rabi formula, relative to our model in this case were exactly deduced.

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