International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:10, No:7, 2016

Spectral Investigation for Boundary Layer Flow over a Permeable Wall in the Presence of Transverse Magnetic Field

Saeed Sarabadan, Mehran Nikarya, Kouroah Parand

Abstract—The magnetohydrodynamic (MHD) Falkner-Skan equations appear in study of laminar boundary layers flow over a wedge in presence of a transverse magnetic field. The partial differential equations of boundary layer problems in presence of a transverse magnetic field are reduced to MHD Falkner-Skan equation by similarity solution methods. This is a nonlinear ordinary differential equation. In this paper, we solve this equation via spectral collocation method based on Bessel functions of the first kind. In this approach, we reduce the solution of the nonlinear MHD Falkner-Skan equation to a solution of a nonlinear algebraic equations system. Then, the resulting system is solved by Newton method. We discuss obtained solution by studying the behavior of boundary layer flow in terms of skin friction, velocity, various amounts of magnetic field and angle of wedge. Finally, the results are compared with other methods mentioned in literature. We can conclude that the presented method has better accuracy than others.

Keywords—MHD Falkner-Skan, nonlinear ODE, spectral collocation method, Bessel functions, skin friction, velocity.

I. Introduction

A large number of theoretical investigations dealing with MHD flows of viscous fluids have been performed during the last decades due to their rapidly increasing applications in many fields of technology and engineering, such as MHD power generation, MHD flow meters, MHD pumps, etc. [1], [2]. Many mathematical models have been proposed to explain the behaviors of the viscous MHD flow under different conditions. Generally, the fundamental equations governing the flow of a viscous electrically conducting fluid are very complicated in the form. Therefore, the fluid motion is split into two parts: Near the boundary in which the viscosity has an important effect, and the fluid is regarded as a viscous fluid; far away from the boundary in which the fluid viscosity effect is negligible, and the fluid may be treated as an inviscid fluid. Based on this theory, the governing equations of the flows of incompressible viscous MHD fluids can be reduced drastically as passing in the vicinity of solid boundaries, and many mathematical results agree well with the experimental observations. Falkner and Skan [3] considered two-dimensional wedge flows. They developed a similarity transformation method in which the partial differential boundary-layer equation was reduced to a nonlinear third-order ordinary differential equation which could then be solved numerically. Recently, several researches have been conducted to employ the boundary layer theory in order to analyze the MHD Falkner-Skan flow of viscous fluids. A summary on these studies is as: Soundalgekar et al. [4] studied the MHD Falkner-Skan flow and heat transfer characteristics in an incompressible fluid. Ashwini and Eswara [5] solved the MHD Falkner-Skan flow in presence of heat and magnetic field via finite difference method and analyzed the skin friction and velocity. Su and Zheng obtained an approximation solution of MHD Falkner-Skan flow over permeable wall in presence of transverse magnetic field, by using the DTM-Padé [6]. The authors in [7], [8] used an implicit finite difference scheme known as the Keller-box method, Abbasbandy and Hayat [9] have given the skin friction coefficients by using Hankel-Padé. Parand et al. utilized spectral and pseudo-spectral methods to solve this problem [10]. Many other researchers have investigated the MHD-boundary layer flow and solved it by different numerical methods [2], [11]-[14].

In this paper, we aim to use the collocation method [15]-[17] based on Bessel functions [18] of the first kind [19] to solve the MHD Falkner-Skan equation. The Spectral methods are one of the powerful methods for solving problems and equations, e.g. [20], [16]. Due to characteristics of Bessel functions specially in the infinite, we have decided to use these functions for solving MHD Falkner-Skan equation.

The remainder of this paper is organized as: In Section II, the MHD Falkner-Skan equation and its transformation to a nonlinear ordinary differential equation is presented. The basics of the first kind of Bessel function and its properties are expressed in Section III. In Section IV, we describe the spectral methods and discuss how to approximate the functions, as well as the Collocation algorithm. In Section V, the presented algorithm is applied to solve Falkner-Skan equation. The results and accuracy of presented method is discussed in Section VI. Finally, in Section VII, we have described concluding remarks.

II. Modeling the Falkner-Skan Equation

Let us consider the problem related to the effects of viscosity and stress on the steady two-dimensional laminar magneto-hydrodynamic flow of an electrically conducting viscous fluid over a wedge in the presence of a transverse magnetic field, Fig. 1. The momentum boundary layer equations of motion are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (1)$$

Saeed Sarabadan and Mehran Nikarya are with the Department of Mathematics, Imam Hossein University, Tehran, Iran (e-mail: mehran.nikarya@gmail.com).

International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:10, No:7, 2016

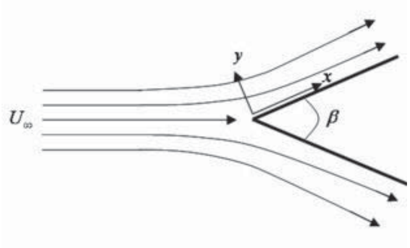

Fig. 1 The flow over a wedge in presence of a transverse magnetic field

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2}{\rho}(u-U) \qquad (2)$$

with the boundary conditions:

$$u=0\ , \qquad v=0 \quad at \quad y=0\ , \qquad u=U\ , \quad as \quad y\to\infty,$$

where

$$B(x)=B_0x^{\frac{m-1}{2}} \quad \texttt{and} \qquad U=ax^m$$

that B_0 is the magnetic field in the y-direction. Here, u and v are velocity components, U is the inherent characteristic velocity, ν is a kinematic viscosity, σ is the electrical conductivity, ρ is the fluid density and a is a constant. By using stream function definition $\psi(x,y)$ as:

$$u=\frac{\partial\psi}{\partial y}\ , \qquad v=-\frac{\partial\psi}{\partial x} \qquad (3)$$

Equation (1) will be satisfied and by similarity solution [3], [21]-[23], the stream function and similarity variable become [24]:

$$\eta=\sqrt{\frac{U(m+1)}{2\nu x}}y. \qquad \psi=\sqrt{\frac{2\nu xU}{m+1}}f(\eta) \qquad (4)$$

By this definition, the velocity components u and v are as:

$$u=\frac{\partial\psi}{\partial y}=Uf'(\eta) \qquad (5)$$

$$v=-\frac{\partial\psi}{\partial x}=-\sqrt{\frac{(m+1)\nu U}{2x}}(f+\frac{m-1}{m+1}\eta f'(\eta)) \qquad (6)$$

Now substituting and embedding obtained results of (3)-(6) into (2) yields:

$$f'''+ff''+\beta(1-f'^2)-M^2(f'-1)=0, \qquad (7)$$

and boundary conditions are:

$$f(0)=f'(0)=0 \quad \texttt{and} \quad f'(\infty)=1. \qquad (8)$$

where $\beta=\frac{2m}{m+1}$ and $M^2=2\sigma B_0/\rho a(1+m)$. Equation (7) with conditions (8) is known as the MHD Falkner-Skan equation. For more detail about these arguments and formulations refer to [24]-[26].

The $f(\eta)$ must be obtained to describe stream function, $f'(\eta)$ must be derived to denote the velocity components u and v and the $f''(\eta)$ must be obtained to denote the shear stress:

$$\tau=\mu\frac{du}{dy} \qquad (9)$$

(μ is viscosity and $\nu=\frac{\mu}{\rho}$ is kinematic viscosity). The skin friction coefficient C_f is:

$$C_f\equiv\frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$$

where τ_w is wall shear stress, that occurs at $y=0$ in (9). Therefore, to obtain the skin friction coefficient, the $f''(0)$ must be calculated. Also, (7) describes the accelerated flow as $\beta>0$, while decelerated flow as $\beta<0$ with separation. Note that, when $m>0$ then, $\beta>0$ and we have accelerated flow, while when $-1<m<0$ then $\beta<0$ and we have decelerated flow with separation. For $m=0$, we have the famous Blasius flow. When $m>1$, the magnetic field increases in x, while when $m<1$, the magnetic field dies off in x, for positive values of x. These are in conformance with [5]-[7], [9], [27].

III. Bessel Functions and Its Properties

In this section, we describe the first kind of Bessel functions and its properties which will be used to construct the Bessel functions collocation (BFC) method.

The Bessel's equation of order n, is [19], [28]:

$$x^2y''(x)+xy'(x)+(x^2-n^2)y(x)=0, \quad \text{for} \quad x\in(-\infty,\infty), \quad (n\in\mathbb{R}). \qquad (10)$$

One of the solutions of this equation is [19]:

$$\sum_{r=0}^{\infty}a_0\frac{(-1)^r\Gamma(n+1)}{2^{2r}r!\Gamma(n+r+1)}(\frac{x}{2})^{2r+n},$$

for any value of a_0; where $\Gamma(\lambda)$ is the gamma function which is defined as:

$$\Gamma(\lambda)=\int_0^\infty e^{-t}t^{\lambda-1}dt.$$

Let us choose $a_0=\frac{1}{2^n\Gamma(n+1)}$. Accordingly, we obtain the solution denoted by $J_n(x)$ and call it the Bessel function of the first kind of order n:

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n+r+1)}(\frac{x}{2})^{2r+n}, \qquad (11)$$

where series (11) is convergent for all $-\infty<x<\infty$.

Some properties of the first kind of Bessel functions are as: [19]:

$$\frac{d}{dx}(x^nJ_n(x))=x^nJ_{n-1}(x), \qquad (12)$$

$$J_n'(x)=J_{n-1}(x)-\frac{n}{x}J_n(x), \qquad (13)$$

$$J_n'(x)=\frac{n}{x}J_n(x)-J_{n+1}(x), \qquad (14)$$

$$\lim_{x\to\infty}J_n(x)=0 \quad \texttt{for all } n, \qquad (15)$$

$$\lim_{x\to\infty}J_n^{(m)}(x)=0 \quad \texttt{for all n,m} \qquad (16)$$

IV. Spectral Methods and Function Approximation

Spectral methods, in the context of numerical schemes for differential equations, generically belong to the family of weighted residual methods (WRMs) [15]. WRMs represent a particular group of approximation techniques, in which the residuals (or errors) are minimized in a certain way and thereby leading to specific methods including Galerkin, Petrov-Galerkin, collocation and tau formulations. WRMs are traditionally regarded as the foundation and cornerstone of the finite element, spectral, finite volume, boundary element and some other methods. Now in this section, we describe WRMs for collocation method to solve differential equations [15], [20], [16].

Prior to introducing spectral methods, we first give a brief introduction to the WRM. Consider the approximation of the following problem:

$$\mathcal{L}u(x) + \mathcal{N}u(x) = f(x) \quad , \quad x \in \Omega, \tag{17}$$

where $\mathcal{L}$ is differential or integral operation, and $\mathcal{N}$ is a lower-order linear and/or nonlinear operator involving only derivatives (if exist), and $f(x)$ is a function of variable x, with enough initial conditions. The starting point of the WRM is to approximate the solution u by a finite sum:

$$u(x) \approx u_N(x) = \sum_{i=0}^{N} a_i \phi_i(x) \quad , \quad x \in \Omega, \tag{18}$$

where $\phi_i(x)$ are the *basis functions*, and the expansion coefficients a_i must be determined. Substituting u_N for u in (17) leads to the residual function:

$$\mathcal{R}_N(x) = \mathcal{L}u_N(x) + \mathcal{N}u_N(x) - f(x). \tag{19}$$

The notion of the WRM is to force the residual to zero by requiring:

$$< \mathcal{R}_N, \psi >_\omega = \int_\Omega \mathcal{R}_N(x)\psi_j(x)\omega(x)dx = 0 \, , \qquad 0 \le j \le N, \tag{20}$$

where $\{\psi_j(x)\}$ are *test functions*, and ω is positive weight function. If, we choose the Lagrange basis polynomials as test functions in (20), such that $\psi_j(x_k) = \delta_{jk}$, where $\{x_k\}$ are preassigned collocation points, hence the residual is forced to zero at x_j, i.e., $R_N(x_j) = 0$, [15], [16].

V. Collocation Algorithm for Solving Falkner-Skan Equation

A method for forcing the residual function (19) to zero, is collocation algorithm. In this method, by substituting the finite series (18) to residual function (19), and collocating it on $\{x_k\}_{k=0}^N$, where these $N+1$ points are distinct, we have a set of $N+1$ equations and $N+1$ unknowns (spectral coefficients). In all of spectral methods, the purpose is finding these coefficients.

Now, we aim to use the collocation method for solving Falkner-Skan equation. For solving (7) subject to the conditions (8) by using collocation method based on Bessel functions, we must construct the series (18) (By substituting $\Phi_i(x) = J_i(x)$), since for any n, $J_n(x)$ are differentiable at the point $x = 0$, and the $\lim_{x\to\infty} J_n(x) = 0$). Therefore, we can simply satisfy the conditions (8) by multiplying the operator (18) with $\frac{\eta^2}{\eta^3+1}$ and adding it to $\frac{\eta^2}{\eta+1}$. For approximating $f(\eta)$ in algorithmic form, we do:

BEGIN

1) Input N.
2) Construct the series (18) by using Bessel functions (11) as:
$$\hat{f}_N(\eta) = \frac{\eta^2}{\eta+1} + \frac{\eta^2}{\eta^3+1}\sum_{i=0}^{N} a_i J_i(\eta). \tag{21}$$
3) Insert the constructed series of step 2, into (7).
4) Construct the Residual function as:
$$Res(\eta) = \frac{d^3}{d\eta^3}\hat{f}_N(\eta) + \alpha \hat{f}_N(\eta)\frac{d^2}{d\eta^2}\hat{f}_N(\eta) +$$
$$\beta(1 - \frac{d}{d\eta}\hat{f}_N(\eta)^2) - M^2(f' - 1). \tag{22}$$
Now, we have $N+1$ unknown $\{a_n\}_{n=0}^N$. To obtain these unknown coefficients, we need $N+1$ equations, thus:
5) By choosing $N+1$ points $\{x_i\}$, $i = 0, 1, ..., N$, in the domain of (17) as collocation points and substituting them in $Res(\eta)$, we construct a system containing $N+1$ equations.
Note: In this paper, we used the roots of Rational Chebyshev of order $N+1$ [16] as $N+1$ collocation points.
6) By solving constructed system of equations in step 5 via Newton's method, the a_n, $n = 0, 1, ..., N$ are obtained.

End.

Now, we have approximation function $\hat{f}_N(\eta)$ of $f(\eta)$.

VI. Results and Discussion

The $\hat{f}(\eta)$ denotes the dimensionless stream function that is shown in Figs. 2 and 3 for $m = 2$ and $m = -3/5$, respectively. The $u(x, y)/U = \hat{f}'(\eta)$ is the velocity component which is dimensionless in x-direction of fluid flow and is shown in Fig. 4 for $m = 2$ and Fig. 5 for $m = -3/5$. According to Figs. 4 and 5, by increasing magnetic field quantity, the velocity u faster tends to U. In other words, with increase of the magnetic field quantity, in a constant x, u faster tends to U. Also by substituting the $\eta = y\sqrt{U(m+1)/2\nu x}$ in $\hat{f}'(\eta)$, $u(x, y)$ is obtained. The graphs of u are shown in Figs. 8 and 9 for $m = 2, M = 10$ and $M = 100$, respectively. As shown, in $y = 0$, u is non zero for $x < l$ while by increasing x, u tends to zero. The parameter l varies for different values of ν and U.

Figs. 6 and 7 explain the shear stress and show that by distancing from the wedge, the shear stress decreases. The obtained values of skin friction are shown in Tables I and II for $m = 2$ and $m = -\frac{3}{5}$, respectively. Then, the solution of presented method is compared with the results of Shooting method and Homotopy-padé [9] for several N and several magnetic field quantity M values. The obtained values of $f(\eta)$ via BFC method are shown in Tables (III) and IV. Figs. 4 and 5 show the graphs of the solution of Falkner-Skan equation for $m = 2$, $m = -3/5$ and different values of M.

Thickness of Boundary Layer: The boundary layer thickness δ has been defined as the locus of points where the

International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:10, No:7, 2016

TABLE I
COMPARISON OF OBTAINED VALUES OF $f''(0)$ FOR $m = 2$, $N = 12$, 22 AND SEVERAL M BY PRESENT METHOD, SHOOTING METHOD AND HAM [9], AND CONVERGENCE RATE

M	BFC $N = 12$	BFC $N = 22$	Shooting method	HAM[9]
1	1.71940810	1.719465399	1.719465400	1.71947219
5	5.19095444	5.190959458	5.190959450	5.19095980
10	10.09666904	10.09677544	10.09677545	10.09677575
50	50.0194408	50.01944071	50.01944071	50.01944084
100	100.009720	100.0097217	100.0097217	100.0097217

TABLE II
COMPARISON OF OBTAINED VALUES OF $f''(0)$ FOR $m = -3/5$, $N = 12$, 22 AND SEVERAL M PRESENT METHOD, SHOOTING METHOD AND HAM [9], AND CONVERGENCE RATE

M	BFC $N = 12$	BFC $N = 22$	shooting method	HAM[9]
5	4.60075352	4.60075495	4.60075494	4.60075228
10	9.80646408	9.80646420	9.80646420	9.80646300
15	14.8716351	14.8716748	14.8716748	14.87167401
20	19.9039313	19.9039370	19.9039370	19.90393626
50	49.9616138	49.9616523	49.9616523	49.96165198

velocity u parallel to the plate reaches %99 of the external velocity U. In other hand, where $\hat{f}'(\eta) = 0.99$, the thickness is defined as:

$$\delta \approx \frac{\eta_{0.99}\sqrt{2\nu x}}{\sqrt{U(m+1)}} \quad \text{or} \quad \frac{\delta}{x} \approx \frac{\sqrt{2}\eta_{0.99}}{\left[(m+1)Re_x\right]^{1/2}} \tag{23}$$

where $10^3 < Re_x = Ux/\nu < 10^6$ is called the local Reynolds number of the flow along the plate surface [23]. Equation (23) shows that the boundary-layer thickness is proportional to $\sqrt{\nu}$ and to $\sqrt{x}$. It is clear that δ increases proportionately to $\sqrt{x}$. On the other hand, the relative boundary layer thickness δ/x decreases with increasing Reynolds number, so that in the limiting case of frictionless flow, with $Re \to \infty$, the boundary layer thickness vanishes.

By increasing magnetic field quantity the $\eta_{0.99}$ decreases and as a result, the thickness decreases, as shown in Figs. 12 and 13. Furthermore, in order to assess the rate of accuracy and also, to show the convergence rate of the proposed method, we present the graph of $\|Res\|_2^2$ for different N values in Figs. 10 and 11. As shown, by increasing N (collocation points) the residual functions tend to zero, i.e. the convergence rate of the presented method.

VII. CONCLUSIONS

In this paper, we considered MHD Falkner-Skan equation, that is the study of laminar boundary layers flow over a permeable wall in the presence of a transverse magnetic field. Then, we attempted to solve MHD Falkner-Skan equation via one of the powerful methods, namely Spectral collocation based on Bessel functions of the first kind. Because the behavior of this functions is similar to conditions of MHD equations in the infinite. After solving MHD Falkner-Skan equation, we discussed the behavior of boundary layer flow for various physical parameters and influences of magnetic field on the thickness, skin friction and velocity of fluid flow. according to the results, our presented method shows better accuracy than other methods.

TABLE III
VALUES OF BFC METHOD OF $f(\eta)$ FOR $m = 2$ AND SEVERAL M

η	$M = 1$	$M = 5$	$M = 10$	$M = 50$	$M = 100$
0.2	0.031396	0.0759394	0.1142881	0.1799916	0.1900063
0.4	0.114644	232918883	0.3031120	0.3799907	0.3900063
0.6	0.235696	0.4183185	0.5016293	0.5799907	0.5900063
0.8	0.383549	0.6134673	0.7010370	0.7799908	0.7900063
1.0	0.549904	0.8118901	0.8980784	0.9843096	0.9838265

TABLE IV
VALUES OF BFC METHOD OF $f(\eta)$ FOR $m = -3/5$ AND SEVERAL M

η	$M = 5$	$M = 10$	$M = 15$	$M = 20$	$M = 50$
0.2	0.0688505	0.11223425	0.1361388	0.15067290	0.1799851
0.4	.21590504	0.29984306	0.3328738	0.34975451	0.3799842
0.6	.39461386	0.49810634	0.5326109	0.54973162	0.5799842
0.8	0.5861588	0.69778695	0.7307871	0.54973162	0.7799842
1.0	0.7828309	0.89720817	0.8981779	0.93445112	0.9792472

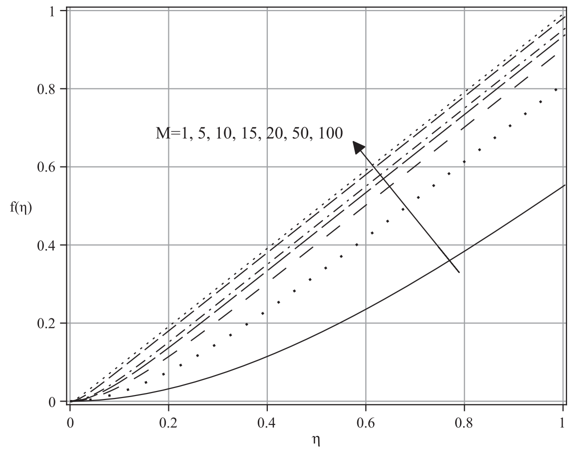

Fig. 2 Approximation Function of $f(\eta)$ of MHD Falkner-Skan Equation Solution for $N = 25,\ m = 2$

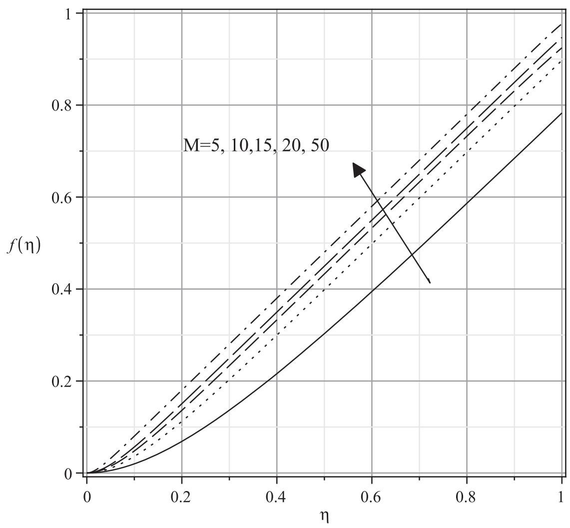

Fig. 3 Approximation Function of $f(\eta)$ of MHD Falkner-Skan Equation Solution for $N = 25,\ m = -3/5$

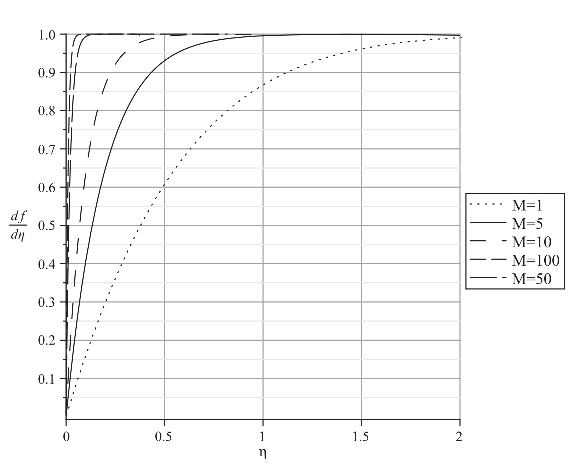

Fig. 4 Approximation Function of $f'(\eta)$ of MHD Falkner-Skan Equation Solution for $N = 25,\ m = 2$

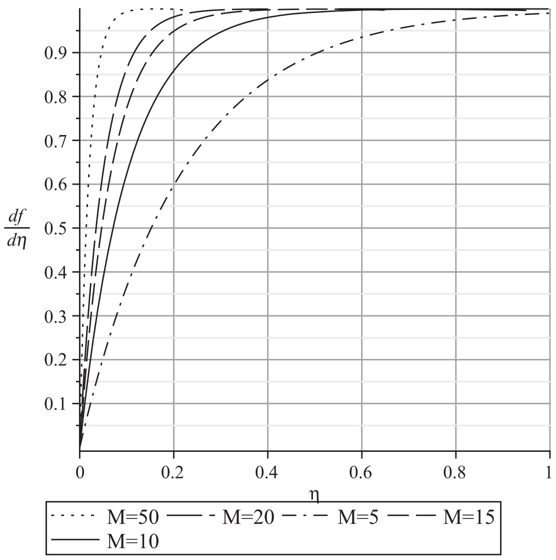

Fig. 5 Approximation Function of $f'(\eta)$ of MHD Falkner-Skan Equation Solution for $N = 25,\ m = -3/5$

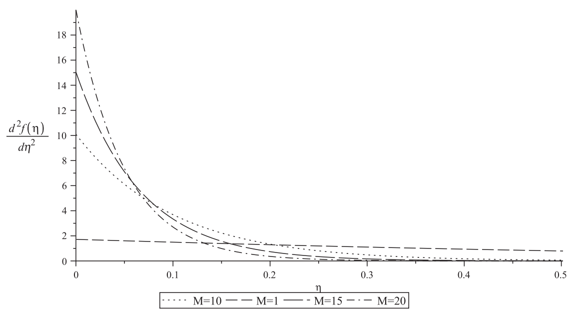

Fig. 6 Approximation Function of $f''(\eta)$ of MHD Falkner-Skan Equation Solution for $N = 25,\ m = 2$ and Expressing Acceleration

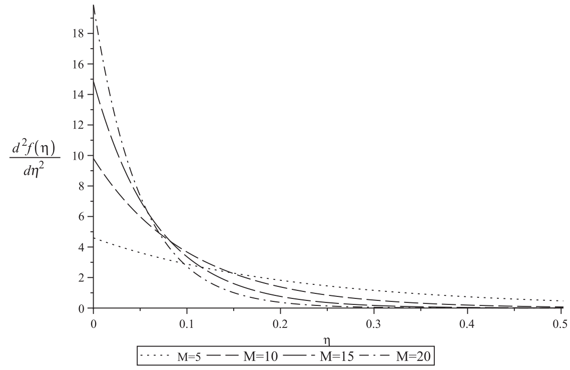

Fig. 7 Approximation Function of $f''(\eta)$ of MHD Falkner-Skan Equation Solution for $N = 25,\ m = -3/5$ and Expressing Acceleration

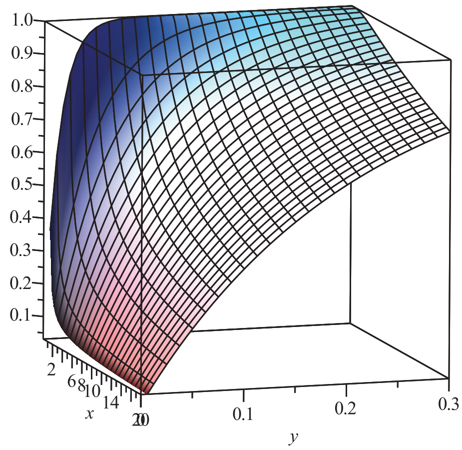

Fig. 8 Approximation Function of $u(x, y)$ of MHD Falkner-Skan Equation Solution for $N = 25,\ m = 2$ and $M = 10$ Expressing Acceleration

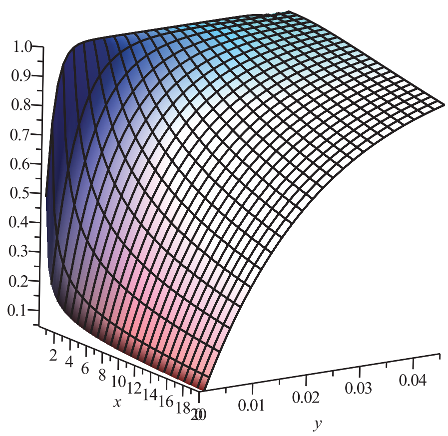

Fig. 9 Approximation Function of $u(x, y)$ of MHD Falkner-Skan Equation Solution for $N = 25,\ m = 2$ and $M = 100$ Expressing Acceleration

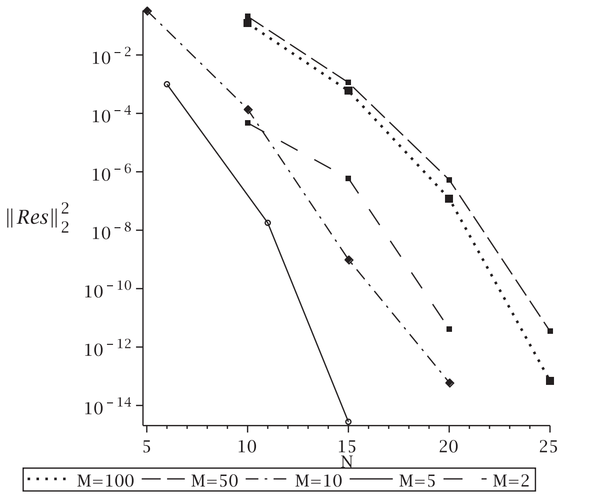

Fig. 10 Logarithmic Graph of $\|Res(\eta)\|_2^2$ of BFC Method to Solve MHD Falkner-Skan Equation with $m = 2$ and for Several N

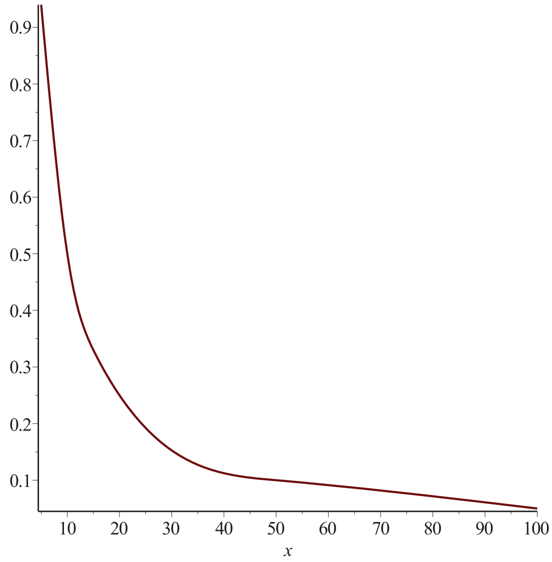

Fig. 12 $\eta_{0.99}$ for Thickness of Boundary Layer by Increasing M for $m = 2$

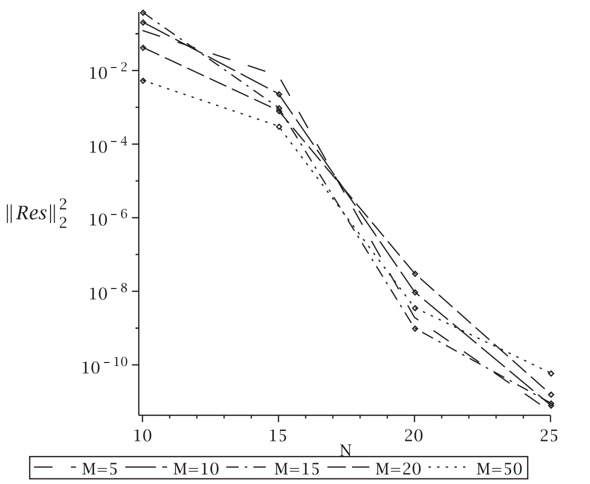

Fig. 11 Logarithmic Graph of $\|Res(\eta)\|_2^2$ of BFC Method to Solve MHD Falkner-Skan Equation with $m = -3/5$ and for Several N

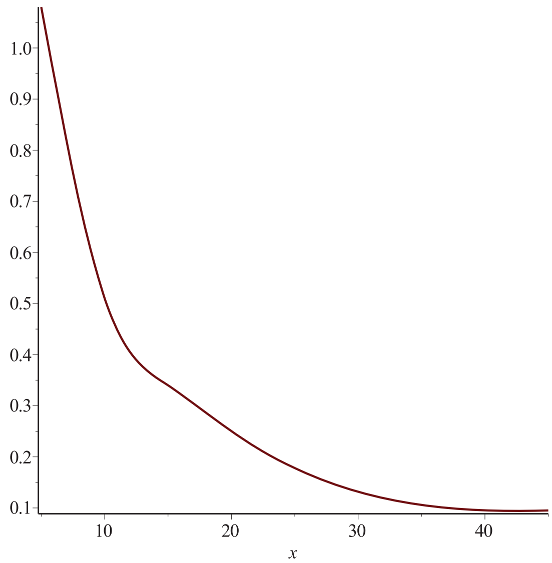

Fig. 13 $\eta_{0.99}$ for Thickness of Boundary Layer by Increasing M for $m = -3/5$

REFERENCES

[1] T. Hayat, M. Imtiaz, A. Alsaedi, Melting heat transfer in the mhd flow of cu–water nanofluid with viscous dissipation and joule heating, Advanced Powder Technology.

[2] T. Hayat, S. Qayyum, M. Imtiaz, A. Alsaedi, Mhd flow and heat transfer between coaxial rotating stretchable disks in a thermally stratified medium, PloS one 11 (5) (2016) e0155899.

[3] V. M. Falkner, S. Skan, Some approximate solutions of the boundary-layer equations, J. Math. Phy. 12 (1931) 865–896.

[4] V. M. Soundalgekar, H. S. Takhar, M. Singh, Velocity and temperature field in MHD Falkner-Skan flow, J. Phys. Soci. of Japan 50 (1981) 3139–3143.

[5] G. Ashwini, A. Eswara, MHD Falkner-Skan boundary layer flow with internal heat generation or absorption, World Aca. Sci. Engin. Tech. 65 (2012) 687–690.

[6] X. H. Su, L. C. Zheng, Approximate solutions to MHD Falkner-Skan flow over permeable wall, Appl. Math. Mech. -Engl. Ed. 32 (2011) 401–408.

[7] A. Ishak, R. Nazar, I. Pop, MHD boundary-layer flow of a micropolar fluid past a wedge with constant wall heat flux, Commun. Nonlinear Sci. Num. Simul. 14 (2009) 109–118.

[8] N. Bachoka, A. Ishak, I. Pop, Boundary layer stagnation-point flow and heat transfer over an exponentially stretching/shrinking sheet in a nanofluid, I. J. Heat Mass Trans. 55 (2012) 8122–8128.

[9] S. Abbasbandy, T. Hayat, Solution of the MHD Falkner-Skan flow by Homotopy analysis method, Nonlinear Anal. Real. World Appl. 14 (2009) 3591–3598.

[10] K. Parand, M. Dehghan, A. Pirkhedri, The use of sinc-collocation method for solving Falkner-Skan boundary-layer equationl, Int. J. Num. Meth. Fluids 69 (2004) 353–357.

[11] A. Asaithambi, A second-order finite-difference method for the FalknerSkan equation, appl. math. comput. 156 (2004) 779–786.

[12] M. Fathizadeh, M. Madani, Y. Khan, N. Faraz, A. Yildirim, S. Tutkun, An effective modification of the homotopy perturbation method for mhd viscous flow over a stretching sheet, J. King Saud Uni. Sci. 2 (2013) 107113.

[13] M. M. Rashidi, The modified differential transform method and pade approximats for solving MHD boundary-layer equationsl, Comput. Phys. Commun. 180 (2009) 2210–2217.

[14] R. Ellahi, E. Shivanian, S. Abbasbandy, T. Hayat, R. Lewis, Numerical study of magnetohydrodynamics generalized couette flow of eyring-powell fluid with heat transfer and slip condition, International Journal of Numerical Methods for Heat & Fluid Flow 26 (5).

[15] J. Shen, T. Tang, L. L. Wang, Spectral Methods Algorithms, Analysics And Applicattions, first edition, Springer, 2001.

[16] J. P. Boyd, Chebyshev and Fourier spectral methods. 2nd ed, New York Dover, New York, 2000.

[17] K. Parand, J. Rad, M. Nikarya, A new numerical algorithm based on the first kind of modified bessel function to solve population growth in a closed system, International Journal of Computer Mathematics 91 (6) (2014) 1239–1254.

[18] K. Parand, M. Nikarya, Solving the unsteady isothermal gas through a micro-nano porous medium via bessel function collocation method, Journal of Computational and Theoretical Nanoscience 11 (1) (2014) 131–136.

[19] W. W. Bell, Special Functions For Scientists And Engineers, Published simultaneously in Canada by D. Van Nostrand Company, (Canada), Ltd, 1967.

[20] G. Ben-yu, Spectral methods and their applications, World Scientific, Shelton Street, Covent Garden, London WC2H 9HE, 1998.

[21] J. D. C. G. W. Bluman, Similarity Methods for Differential Equations, Springer, New York, 1974.

[22] L. Dresner, Similarity solutions of nonlinear partial differential equations, Longman Group, London, 1983.

[23] F. M. White, Fluid Mechanics, Seventh Edition, McGraw-Hill, New York, 2009.

[24] H. Schlichting, K. Gersten, Boundary layer theory. 6th ed, Oxford, New York, 1979.

[25] T. Y. Na, Computational Method in Engineering Boundray Value Problem, Academic Pressl, New Yorkl, 1979.

[26] L. Rosenhead, Laminar Boundary layers, Clarendon Pressl, McGraw-Hill, 1963.

[27] T. Hayat, Q. Hussain, T. Javed, The modified decomposition method and padè aapproximants for theMHD flow over a non-linear stretching sheet, Commun. Nonlinear Sci. Num. Simul. 10 (2009) 966–973.

[28] G. Watson, A treatise on the theory of Bessel Functions, 2nd edition, Cambridge University Press, Cambridge (England), 1967.