Some Characterizations of Isotropic Curves In the Euclidean Space

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Abstract**—**The curves, of which the square of the distance between the two points equal to zero, are called minimal or isotropic curves [4]. In this work, first, necessary and sufficient conditions to be a Pseudo Helix, which is a special case of such curves, are presented. Thereafter, it is proven that an isotropic curve's position vector and pseudo curvature satisfy a vector differential equation of fourth order. Additionally, In view of solution of mentioned equation, position vector of pseudo helices is obtained.

Keywords**—**Classical Differential Geometry, Euclidean space, Minimal Curves, Isotropic Curves, Pseudo Helix.

I. INTRODUCTION

IN the existing literature, we cannot see much works based on the definition of minimal curves. The notion of a minimal curve was due to E. Cartan (for details see [2]). Thereafter, such kind curves were deeply studied by F. Semin [4]. In recent years, U. Pekmen [3] wrote some characterizations of minimal curves of constant breadth by means of E. Cartan equations.

A minimal curve with constant pseudo curvature is called a *Pseudo Helix* or *Isotropic Helix*. In this paper, first, we present some characterizations of pseudo helices. Then, we prove that every minimal (isotropic) curve's E. Cartan elements satisfy a vector differential equation by the method of [6]. By this way, we write a parameterization for pseudo helices in the Euclidean space E^3.

II. PRELIMINARIES

Definition 1. Let x_p be a complex analytic function of a complex variable t. Then the vector function

$$\vec{x} = \sum_{p=0}^{3} x_p(t)\vec{k}_p \tag{1}$$

is called an imaginary curve, where $\vec{x} : C \to C^3$ and $\vec{k}_p$ are standard basis unit vectors of E^3 [3], [4].

Definition 2. The curves, of which the square of the distance between the two points equal to zero, are called minimal or isotropic curves [3].

Definition 3. Let s denote the arclength (see [4], [5]). A curve is a minimal (isotropic) curve if and only if $ds^2 = 0$.

Let $\vec{x} = \vec{x}(t)$ be a minimal (isotropic) curve in space with t complex variable. Then above definitions follow that $ds^2 = dx^2 = 0$. For every regular point, we know that $\frac{d\vec{x}}{dt} = \vec{x}'(t) \neq 0$. Via this, it is safe to report that, isotropic curves in space $\vec{x} = \vec{x}(t)$ satisfy the vector differential equation

$$\left[\vec{x}'(t)\right]^2 = 0. \tag{2}$$

By differentiation, we have $\vec{x}'(t).\vec{x}''(t) = 0$. And by trivial calculus, it can be written that $\left[\vec{x}'(t) \wedge \vec{x}''(t)\right]^2 = 0$. This means that it is also an isotropic vector which is perpendicular to itself. Then

$$\vec{x}'(t) \wedge \vec{x}''(t) = \Psi.\vec{x}'(t), \text{ whereby } \Psi \neq 0 \tag{3}$$

can be written. By vector product with $\vec{x}''(t)$, we know $\Psi^2 = -\left[\vec{x}''\right]^2$ and therefore

$$\vec{x}' = \frac{\vec{x}' \wedge \vec{x}''}{\sqrt{-\left[\vec{x}''\right]^2}}. \tag{4}$$

For another complex variable t^*, $t = f(t^*)$, $\frac{df}{dt^*} = f^{\cdot} \neq 0$ we may write

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The second author would like to thank TUBITAK-BIDEB for their financial supports during his Ph.D. studies.

$$d\vec{x} = \frac{\vec{x}' \wedge \vec{x}''}{\sqrt{-[\vec{x}'']^2}} dt = \frac{\vec{x}^{\cdot} \wedge \vec{x}^{\cdot\cdot}}{\sqrt{-[\vec{x}^{\cdot\cdot}]^2}} dt^*, \tag{5}$$

where $\vec{x}^{\cdot} = \vec{x}'.f^{\cdot}$, $\vec{x}^{\cdot\cdot} = \vec{x}''.(f^{\cdot})^2 + \vec{x}'.f^{\cdot\cdot}$. The equality $(\vec{x}^{\cdot\cdot})^2 = \vec{x}''.(f^{\cdot})^4$ can be written in the form $-\left[(\vec{x}^{\cdot\cdot})^2\right]^{\frac{1}{4}} dt^* = -\left[(\vec{x}'')^2\right]^{\frac{1}{4}} dt.$ If we choose t^* such that $(\vec{x}^{\cdot\cdot})^2 = -1,$ the by integration

$$t^* = s = \int_{t_0}^{t} -\left[(\vec{x}'')^2\right]^{\frac{1}{4}} dt \tag{6}$$

is obtained. It is called the pseudo arclength of the minimal curve which is invariant with respect to parameter t (see [4]).

For each point $\vec{x}$ of the minimal curve, E. Cartan frame is defined (for well-known complex number $i^2 = -1$) as follows (see [1], [4]):

$$\begin{cases} \vec{e}_1 = \vec{x}^{\cdot}, \\ \vec{e}_2 = i\vec{x}^{\cdot\cdot}, \\ \vec{e}_3 = -\dfrac{\beta}{2}\vec{x}^{\cdot} + \vec{x}^{\cdot\cdot\cdot}, \text{ where } \beta = (\vec{x}^{\cdot\cdot\cdot})^2, \\ \vec{e}_j.\vec{e}_k = \begin{bmatrix} 1, & j+k=4, \\ 0, & j+k \neq 4. \end{bmatrix} \\ \vec{e}_j \wedge \vec{e}_k = i.\vec{e}_{j+k-2}, \\ \det(\vec{e}_1, \vec{e}_2, \vec{e}_3) = i \end{cases} \tag{7}$$

for $j,k = 1,2,3.$ $s = \int_{t_0}^{t} -\left[(\vec{x}'')^2\right]^{\frac{1}{4}} dt$ is a pseudo arclength, also invariant with respect to parameter t. Suffice it to say that, differentiation of () yields that

$$\begin{cases} \vec{e}_1^{\cdot} = -i\vec{e}_2 \\ \vec{e}_2^{\cdot} = i(k\vec{e}_1 + \vec{e}_3) \\ \vec{e}_3^{\cdot} = -ik\vec{e}_2 \end{cases} \tag{8}$$

where $k = \dfrac{\beta}{2}$ is called pseudo curvature of the minimal curve $\vec{x} = \vec{x}(s)$. These equations can be used if the minimal curve is at least of class C^4.

III. Characterizations of Isotropic Helices in the Euclidean Space

Definition 4. An isotropic curve, whose pseudo curvature is constant, is called a *Pseudo Helix* or *Isotropic Helix* in E^3 (see [4]).

Theorem 1. Let $\vec{x} = \vec{x}(s)$ be an isotropic curve with a pseudo arclength in E^3. $\vec{x}(s)$ is a pseudo helix (isotropic helix) if and only if

i) $\det(\dfrac{d^2\vec{x}}{ds^2}, \dfrac{d^3\vec{x}}{ds^3}, \dfrac{d^4\vec{x}}{ds^4}) = 0.$

ii) $\det(\dfrac{d\vec{e}_1}{ds}, \dfrac{d^2\vec{e}_1}{ds^2}, \dfrac{d^3\vec{e}_1}{ds^3}) = 0.$

iii) $\det(\dfrac{d\vec{e}_3}{ds}, \dfrac{d^2\vec{e}_3}{ds^2}, \dfrac{d^3\vec{e}_3}{ds^3}) = 0.$

Proof. First, let us form following differentiations with respect to s:

$$\begin{cases} \vec{x}^{\cdot} = \vec{e}_1, \\ \vec{x}^{\cdot\cdot} = -i\vec{e}_2, \\ \vec{x}^{\cdot\cdot\cdot} = k\vec{e}_1 + \vec{e}_3, \\ \vec{x}^{\cdot\cdot\cdot\cdot} = k^{\cdot}\vec{e}_1 - 2ik\vec{e}_2. \end{cases} \tag{9}$$

If $\vec{x} = \vec{x}(s)$ is a pseudo helix, then we write

$$\det(\frac{d^2\vec{x}}{ds^2}, \frac{d^3\vec{x}}{ds^3}, \frac{d^4\vec{x}}{ds^4}) = \begin{vmatrix} 0 & -i & 0 \\ k & 0 & 1 \\ 0 & -2ik & 0 \end{vmatrix} = 0. \tag{10}$$

Conversely, let the statement (i) holds. Then

$$\det(\frac{d^2\vec{x}}{ds^2}, \frac{d^3\vec{x}}{ds^3}, \frac{d^4\vec{x}}{ds^4}) = -ik^{\cdot} = 0. \tag{11}$$

Thus, $\dfrac{dk}{ds} = 0$ and $k = \text{constant}.$ This result completes the proof of (i). If we take $\vec{x}^{\cdot} = \vec{e}_1$, proof of (ii) can be easily deduced. Now, let us form

$$\begin{cases} \vec{e}_3^{\cdot} = -ik\vec{e}_2, \\ \vec{e}_3^{\cdot\cdot} = k^2\vec{e}_1 - ik^{\cdot}\vec{e}_2 + k\vec{e}_3, \\ \vec{e}_3^{\cdot\cdot\cdot} = 3kk^{\cdot}\vec{e}_1 - (2ik^2 + ik^{\cdot\cdot})\vec{e}_2 + 2k^{\cdot}\vec{e}_3. \end{cases} \tag{12}$$

If $\vec{x} = \vec{x}(s)$ is a pseudo helix, we have

$$\det\left(\frac{d\vec{e}_3}{ds}, \frac{d^2\vec{e}_3}{ds^2}, \frac{d^3\vec{e}_3}{ds^3}\right) = 0. \qquad (13)$$

Conversely, let us form the determinant

$$\det\left(\frac{d\vec{e}_3}{ds}, \frac{d^2\vec{e}_3}{ds^2}, \frac{d^3\vec{e}_3}{ds^3}\right) = \begin{vmatrix} 0 & -ik & 0 \\ k^2 & -ik^{\cdot} & k \\ 3kk^{\cdot} & -2ik^2 - ik^{\cdot\cdot} & 2k^{\cdot} \end{vmatrix} \qquad (14)$$

$$= -ik^3 k^{\cdot} = 0.$$

By virtue of (14), we arrive $\frac{dk}{ds} = 0$ and $k = \text{constant}$. Since, $\vec{x} = \vec{x}(s)$ is a pseudo helix.

IV. Vector Differential Equation of Fourth Order Satisfied by Isotropic Curves

Theorem 2. Let $\vec{x} = \vec{x}(s)$ be an isotropic curve with a pseudo arclength in E^3. Position vector of and pseudo curvature $\vec{x}$ satisfy a vector differential equation of fourth order.

Proof. Let $\vec{x} = \vec{x}(s)$ be an isotropic curve with a pseudo arclength and pseudo curvature k in E^3. Considering E. Cartan equations, we write that

$$\vec{e}_2 = -\frac{\vec{e}^{\cdot}}{i} \qquad (15)$$

and

$$\vec{e}_3 = k\vec{e}_1 - \frac{\vec{e}_2^{\cdot}}{i}. \qquad (16)$$

Differentiating (16) with respect to s and using (15), we have

$$\vec{e}_1^{\cdot\cdot\cdot} - k^{\cdot}\vec{e}_1 = 0. \qquad (17)$$

Taking $\vec{x}^{\cdot} = \vec{e}_1$, it follows that

$$\vec{x}^{\cdot\cdot\cdot\cdot} - k^{\cdot}\vec{x}^{\cdot} = 0. \qquad (18)$$

Formula (18) proves the theorem as desired.

This differential equation is a characterization for the isotropic curve $\vec{x} = \vec{x}(s)$. It is well-known that, solving equation (18) with elementary methods is not easy. Let us suppose that $\vec{x} = \vec{x}(s) = (x_1(s), x_2(s), x_3(s))$ is a pseudo helix. Then

$$\vec{x}^{\cdot\cdot\cdot\cdot} = 0. \qquad (19)$$

Therefore, we have parameterization of the pseudo helix as

$$\vec{x} = \vec{x}(s) = \begin{pmatrix} \lambda_1 \frac{s^3}{3} + \lambda_2 \frac{s^2}{2} + \lambda_3 s + \lambda_4, \\ \eta_1 \frac{s^3}{3} + \eta_2 \frac{s^2}{2} + \eta_3 s + \eta_4, \\ \mu_1 \frac{s^3}{3} + \mu_2 \frac{s^2}{2} + \mu_3 s + \mu_4 \end{pmatrix} \qquad (20)$$

where λ_i, η_i and μ_i for $1 \le i \le 4$ are real numbers. Now, we give the following theorem.

Theorem 3. Position vector of a pseudo helix with pseudo arclength and pseudo curvature k in E^3 can be composed by the parameterization (20).

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