International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:8, No:3, 2014

Some Applications of Transition Matrices via Eigen Values

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Abstract**—**In this short paper, new properties of transition matrix were introduced. Eigen values for small order transition matrices are calculated in flexible method. For benefit of these properties applications of these properties were studied in the solution of Markov's chain via steady state vector, and information theory via channel entropy. The implemented test examples were promised for usages.

Keywords**—**Eigen value problem, transition matrix, state vector, information theory.

I. Introduction

TRANSITION matrices are important and has wide applications. They are used in solutions of differential equations, information theory, steady state vector, Markov's chain, and others. Chen et al. used transition matrix in straight wires [1], Kreinin and Sidelnikova studied the regularization algorithms [2], Bauchau studied "An Implicit Floquet Analysis for Rotorcraft Stability Evaluation" [3], Boyd studied the fastest mixing Markov chain in a graph [4], and Shieh used the Kolmogorov-Sinai Entropy for calculating eigenvalues [5].

In this paper a new proposed approach was introduced for calculating the eigen value problem of small order transition matrices, say submatrices of large matrix. For more efficiency of our approach, many applications were implemented. These applications involve Markov's chain via steady state vector, and information theory via channel entropy.

II. Basic Concepts

The basic mathematical concepts which needed in this paper are showing as follow.

Definition 1 Let A be $n\times n$ matrix and $Av=\lambda v$ for scalar λ and vector v then λ is called eigen values of a, and v be the corresponding eigen vector [6].

Definition 2 Let T be $n\times n$ matrix, and then T is called transition matrix if each column has the sum one.

Theorem 1 Let A a be $n\times n$ matrix and λ be the eigen values, then

1) $|A|=\prod\lambda$
2) $tr(A)=\sum\lambda$ [6].

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III. Properties of Transition Matrices

In this section a new propositions concerned for calculating the eigen values of transition matrices are studied.

Proposition 1: $\lambda_1(T)=1$

Proof:

Without loss of generality, let T be 3×3 matrix, then

$$I_3-T=\begin{bmatrix}1-t_{11} & -t_{12} & -t_{13}\\ -t_{21} & 1-t_{22} & -t_{23}\\ -t_{31} & -t_{32} & 1-t_{33}\end{bmatrix} \tag{1}$$

T is transition matrix, so the sum of each column is one. Equation (1) becomes:

$$I_3-T=\begin{bmatrix}1-t_{11} & -t_{12} & -t_{13}\\ -t_{21} & 1-t_{22} & -t_{23}\\ -(1-t_{11}-t_{21}) & -(1-t_{12}-t_{22}) & t_{13}+t_{23}\end{bmatrix} \tag{2}$$

$$\xrightarrow{r_1+r_2}$$

$$I_3-T=\begin{bmatrix}1-t_{11} & -t_{12} & -t_{13}\\ 1-t_{11}-t_{21} & 1-t_{12}-t_{22} & -t_{13}-t_{23}\\ -(1-t_{11}-t_{21}) & -(1-t_{12}-t_{22}) & t_{13}+t_{23}\end{bmatrix} \tag{3}$$

$$\Rightarrow r_2=-r_3 \tag{4}$$

$$\Rightarrow |I_3-T|=0 \tag{5}$$

By definition 1, for non zero vector u,

$$(I_3-T)u=0 \tag{6}$$

Then

$$Tu=u\ . \tag{7}$$

$$\Rightarrow$$

$$\lambda_1(T)=1 \tag{8}$$

Proposition 2

1) If T be 2×2 matrix, then $\lambda(T)$ are 1 and $|T|$.
2) If T be 3×3 matrix , then $\lambda(T)$ are 1 and

$$\frac{-c\pm\sqrt{c^2-4d}}{2}$$

where $c = tr(T) - 1,$ and $D = |T|$.

Proof

One can prove it by using Theorem 1 and Proposition 1.

IV. Applications of Transition Matrices

This section was concerned for applications of eigen values problem of transition matrix. The applications were represented by steady state vector of Markov chain, solutions of differential equations, and the probability of receiver in information channel matrix.

A. Solutions of Steady State Vectors

By taking

$$Tu = u \tag{9}$$

Then

$$P^{-1}Tu = P^{-1}u \tag{10}$$

$\Rightarrow$

$$P^{-1}TP(P^{-1}u) = P^{-1}u \tag{11}$$

$\Rightarrow$

$$Dv = v \tag{12}$$

where

$$v = P^{-1}u \tag{13}$$

For explaining, one can take the following two examples

Example 1

Let $T = \begin{bmatrix} 0.67 & 0.5 \\ 0.33 & 0.5 \end{bmatrix}$ be the transition matrix of the weather (rainy or dry) then the probability vector or steady state vector u is calculated generally by iterations [6].

Now u is calculated using our approach as follows $\lambda(T) = 1, 0.17$. So $D = \begin{bmatrix} 1 & 0 \\ 0 & 0.17 \end{bmatrix}$, $P = \begin{bmatrix} 0.83 & -0.7 \\ 0.55 & 0.7 \end{bmatrix}$, and $(I - D)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$. Then $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$ where r is calculated by $P_{11}r + P_{21}r = 1$ implies that $r = 0.721$. So $u = r\begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} = \begin{bmatrix} 0.6024 \\ 0.3976 \end{bmatrix}$.

Example 2

By taking

$$T = \begin{bmatrix} 0.5 & 0.6 & 0.4 \\ 0.25 & 0.3 & 0.3 \\ 0.25 & 0.1 & 0.3 \end{bmatrix}$$

be the transition matrix, then

$\lambda(T) = 1, 0.05 \pm 0.08i$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.05 + 0.08i & 0 \\ 0 & 0 & 0.05 - 0.08i \end{bmatrix}$

$P = \begin{bmatrix} 0.82 & 0.70 & 0.70 \\ 0.44 & -0.17 + 0.3i & -0.17 - 0.3i \\ 0.35 & -0.53 - 0.3i & -0.53 + 0.3i \end{bmatrix}$, $r = 0.6159$, $v = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$,

and $u = \begin{bmatrix} 0.5055 \\ 0.2747 \\ 0.2198 \end{bmatrix}$.

B. Information of Channels

One can use our proposed approach for calculating the probability of source transition of binary information system .For explaining, one can study the following example.

Example 3

For choosing

$$Q = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

be the probability of matrix channel where (x_1, x_2) represents the transition elements and (y_1, y_2) represents the receiving elements. Then

$$T = Q' = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

be the transition matrix, and $\lambda(T) = 1, 0.7$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 0.7 \end{bmatrix}$, $P = \begin{bmatrix} 0.8944 & -0.7071 \\ 0.4472 & -0.7071 \end{bmatrix}$, and $r = 0.7454$. So $u = \begin{bmatrix} p(x_1) \\ p(x_2) \end{bmatrix} = r\begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} = \begin{bmatrix} 0.6667 \\ 0.3333 \end{bmatrix}$

This represents the probability of transmission source.

V. Conclusion

It is proved that one of eigen values of transition matrix T can be calculated as 1. This idea leads to introduce a proposed new approach for calculating steady state vector u. The implemented test examples showed that our approach be promised for its flexibility. One can summarized our main approach by following:

1) Given T.
2) Calculate $\lambda(T)$.
3) Deduce v from $(I - D)v = 0$.
4) Deduce r from P.
5) $r = f(v)$.
6) $u = Pv$.

Clearly, the new results concerned on transition matrix where the latest has many and good applications in information theory. So for future one can study large transition matrix, then transform it to smaller submatrices. Finally these submatrices can solve fast.

ACKNOWLEDGEMENT

This paper was supported by the faculty of mathematics and computer science of university of Kufa, Iraq. I thank all reviewers for deep reading on this paper.

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