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Some Algebraic Properties of Universal and Regular Covering Spaces

Ahmet Tekcan

Abstract**—**Let $\widetilde{X}$ be a connected space, X be a space, let $p:\widetilde{X}\longrightarrow X$ be a continuous map and let $(\widetilde{X},p)$ be a covering space of X. In the first section we give some preliminaries from covering spaces and their automorphism groups. In the second section we derive some algebraic properties of both universal and regular covering spaces $(\widetilde{X},p)$ of X and also their automorphism groups $A(\widetilde{X},p)$.

Keywords**—**covering space, universal covering, regular covering, fundamental group, automorphism group.

I. Preliminaries

Let $\widetilde{X}$ be a connected space, X be a space and let

$$p:\widetilde{X}\longrightarrow X$$

be a continuous map. If for every $x\in X$ has a path connected open neighborhood U such that $p^{-1}(U)$ is open in $\widetilde{X}$ and each component of $p^{-1}(U)$ is mapped topologically onto U by p then p is called a covering map (projection). In this case the pair $(\widetilde{X},p)$ is called a covering space of X.

Let $p:E\to B$ and $f:X\to B$ be two maps. The lifting problem for f is to determine whether there is a continuous map

$$f^{\imath}:X\to E$$

such that

$$pf^{\imath}=f.$$

If there is such a map $f^{\imath}$, then f can be lifted to E, and we call $f^{\imath}$ a lifting of f. A map $p:E\to B$ is said to have homotopy lifting property with respect to the space X if given maps $f^{\imath}:X\to E$ and $F:X\times I\to B$ such that $F(x,0)=pf^{\imath}(x)$ for $x\in X$, there exists a map $F^{\imath}:X\times I\to E$ such that $F^{\imath}(x,0)=f^{\imath}(x)$ for $x\in X$ and $pF^{\imath}=F$. A map $p:E\to B$ is called a fibration if p has the homotopy lifting property with respect to every space. E is called the total space and B the base space of the fibration. For $b\in B$, $p^{-1}(b)$ is called the fiber over b (see [1]).

Let $(\widetilde{X},p)$ be a covering space of X, $\widetilde{x}\in\widetilde{X}$, and $p(\widetilde{x})=x$ for $x\in X$. Let $\pi(X,x)$ be the fundamental group of X based x. For any point $\widetilde{x}\in p^{-1}(x)$ and any $\alpha\in\pi(X,x)$ we define $\widetilde{x}\alpha\in p^{-1}(x)$ as follows: There exists a unique path class $\widetilde{\alpha}$ in $\widetilde{X}$ such that $p_*(\widetilde{\alpha})=\alpha$ and the initial point of $\widetilde{\alpha}$ is the point $\widetilde{x}$. Define $\widetilde{x}\alpha$ to be the terminal point of the path class $\widetilde{\alpha}$. Then it is easily verify that $(\widetilde{x}\alpha)\beta=\widetilde{x}(\alpha\beta)$ and $\widetilde{x}e=\widetilde{x}$. Thus $\pi(X,x)$ be a group of right operators on the set $p^{-1}(x)$. Moreover, $\pi(X,x)$ acts transitively on the set $p^{-1}(x)$, and hence $p^{-1}(x)$ is a homogeneous right $\pi(X,x)-$ space.

Ahmet Tekcan is with the Uludag University, Department of Mathematics, Faculty of Science, Bursa-TURKEY, email: tekcan@uludag.edu.tr, http://matematik.uludag.edu.tr/AhmetTekcan.htm.

From definition, we see that for any point $\widetilde{x}\in p^{-1}(x)$, the isotropy subgroup corresponding to this point is precisely the subgroup $p_*\pi(\widetilde{X},\widetilde{x})$ of $\pi(X,x)$. Hence $p^{-1}(x)$ is isomorphic to the space of cosets $\pi(X,x)/p_*\pi(\widetilde{X},\widetilde{x})$, and the number of sheets of the covering is equal to the index of the subgroup $p_*\pi(\widetilde{X},\widetilde{x})$ in $\pi(X,x)$.

Lemma 1.1: Let E be a homogen $G-$space and let H be a isotropy subgroup of $e\in E$. Then the automorphism group of E is isomorphic to the quotient space $N[H]/H$, where $N[H]$ denotes the normalization of H. [4]

Let $(\widetilde{X},p)$ be a covering space of X. If X is simply connected, then the fundamental group $\pi(X,x)$ is trivial and the index of $p_*\pi(\widetilde{X},\widetilde{x})$ in $\pi(X,x)$ is 1. So $(\widetilde{X},p)$ is an one-sheeted covering of X and therefore p is a homeomorphism. Similarly, if $\widetilde{X}$ is simply connected, then $\pi(\widetilde{X},\widetilde{x})$ is trivial and the index of $p_*\pi(\widetilde{X},\widetilde{x})$ in $\pi(X,x)$ is equal to the order of $\pi(X,x)$.

Let $(\widetilde{X}_1,p_1)$ and $(\widetilde{X}_2,p_2)$ be two covering spaces of X. Then a homomorphism from $(\widetilde{X}_1,p_1)$ into $(\widetilde{X}_2,p_2)$ is a continuous map

$$h:\widetilde{X}_1\longrightarrow\widetilde{X}_2$$

such that

$$p_2h=p_1.$$

Let h be a homomorphism from $(\widetilde{X}_1,p_1)$ into $(\widetilde{X}_2,p_2)$. Then h is called an isomorphism, if there exists a homomorphism k from $(\widetilde{X}_2,p_2)$ into $(\widetilde{X}_1,p_1)$ such that both compositions hk and kh are identity maps. A homomorphism of covering spaces is an isomorphism if and only if it is a homeomorphism in the usual sense.

Lemma 1.2: Let $(\widetilde{X}_1,p_1)$ and $(\widetilde{X}_2,p_2)$ be two simply connected covering spaces of X. Then there exists a unique homeomorphism $h:(\widetilde{X}_1,\widetilde{x}_1)\to(\widetilde{X}_2,\widetilde{x}_2)$ such that $p_2h=p_1$.[2]

A covering transformation of a covering space $(\widetilde{X},p)$ of X is a homeomorphism

$$h:\widetilde{X}\longrightarrow\widetilde{X}$$

such that

$$ph=p.$$

The set of all covering transformations of $(\widetilde{X},p)$ form a group denoted by $A(\widetilde{X},p)$ called the automorphism group.

Lemma 1.3: Let $(\widetilde{X}_1,p_1)$ and $(\widetilde{X}_2,p_2)$ be two covering spaces of X such that $p_1(\widetilde{x}_1)=p_2(\widetilde{x}_2)=x$. Then there exists a homomorphism φ from $(\widetilde{X}_1, p_1)$ into $(\widetilde{X}_2, p_2)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$ if and only if $p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1) \subset p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2)$. [4]

Let $(\widetilde{X}, p)$ be a covering space of X and $p(\widetilde{x}_1) = p(\widetilde{x}_2) = x$ for $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$ and $x \in X$. Let consider the homomorphisms

$$p_* : \pi(\widetilde{X}, \widetilde{x}_1) \to \pi(X, x)$$

and

$$p_* : \pi(\widetilde{X}, \widetilde{x}_2) \to \pi(X, x).$$

Let $\{\gamma_i : i \in I\}$ be a path class in $\widetilde{X}$ with initial point $\widetilde{x}_1$ and terminal point $\widetilde{x}_2$. Define

$$u_* : \pi(\widetilde{X}, \widetilde{x}_1) \to \pi(\widetilde{X}, \widetilde{x}_2)$$

to be

$$u_*(\alpha) = \gamma^{-1}\alpha\gamma$$

for $\gamma \in \{\gamma_i : i \in I\}$. Then

$$v_* : \pi(X, x) \to \pi(X, x)$$

is defined by

$$v_*(\beta) = (p_*\gamma)^{-1}\beta(p_*\gamma).$$

Since $p_*\gamma$ is a closed it is a path in $\pi(X, x)$. So the images of the fundamental groups $\pi(\widetilde{X}, \widetilde{x}_1)$ and $\pi(\widetilde{X}, \widetilde{x}_2)$ under p_* are conjugate subgroups of $\pi(X, x)$ (for further details on covering spaces see [3,4]).

Lemma 1.4: Let $(\widetilde{X}, p)$ be path connected covering space of a locally pathwise connected space X. Then p is a homeomorphism if and only if $p_*\pi(\widetilde{X}, \widetilde{x}) = \pi(X, x)$. [5]

From Lemmas 1.3 and 1.4, we say that if $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ are two covering spaces of a locally pathwise connected space X, then these two coverings are isomorphic if and only if $p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1)$ and $p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2)$ are conjugate subgroups of $\pi(X, x)$.

Let X be a connected space. The category of connected spaces of X has objects which are covering projections $p : \widetilde{X} \to X$, where $\widetilde{X}$ is connected, and morphisms $p_1 : \widetilde{X}_1 \to X, p_2 : \widetilde{X}_2 \to X$ and $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 f = p_1$.

Let X be a connected space and $\widetilde{X}$ be a locally path connected space. A universal covering space of X is an object $p : \widetilde{X} \to X$ of the category of connected covering spaces of X such that for any object $p_1 : \widetilde{X}_1 \to X$ there is a morphism $f : \widetilde{X} \to \widetilde{X}_1$ such that $p_1 f = p$.

Lemma 1.5: If $(\widetilde{X}, p)$ is a universal covering space of X, then the automorphism group $A(\widetilde{X}, p)$ is isomorphic to $\pi(X, x)$. Moreover, the order of the fundamental group $\pi(X, x)$ is equal to the number of the sheets of the covering $(\widetilde{X}, p)$ of X. [4]

Let $\widetilde{X}$ be be a pathwise connected space and let $(\widetilde{X}, p)$ be a covering space of a locally pathwise connected space X. Then $(\widetilde{X}, p)$ is a regular covering space of X if and only if $p_*\pi(\widetilde{X}, \widetilde{x})$ is a normal subgroup of $\pi(X, x)$.

Lemma 1.6: Let $p : \widetilde{X} \to X$ be a covering map such that $p(\widetilde{x}_1) = p(\widetilde{x}_2)$ for $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$. Then p is regular if and only if $p_*\pi(\widetilde{X}, \widetilde{x}_1) = p_*\pi(\widetilde{X}, \widetilde{x}_2)$. [5]

The connection between universal and regular covering is given by the following Lemma.

Lemma 1.7: Universal covering is regular. [6]

Lemma 1.8: Let $\widetilde{X}$ be a pathwise connected space and let $(\widetilde{X}, p)$ is a regular covering space of a locally pathwise connected space X. Then X homeomorphic to the quotient space $\widetilde{X}/A(\widetilde{X}, p)$. [3]

Let G be a group of homeomorphisms of X. If for every $x \in X$, there exists a neighborhood V of x such that $gV \cap V = \theta$, for all $g \in G$ different from the unity of G, then we say G acts discontinuosly on X.

Lemma 1.9: Let G be a discontinuous proper group of homeomorphisms of a locally pathwise connected space X and let $q : X \to X/G$ be a naturel projection defined by $q(x) = [x]$. Then $(\widetilde{X}, q)$ is a regular covering space of X/G and $A(\widetilde{X}, q)$ is isomorphic to G. [5]

II. Universal and Regular Covering Spaces.

In [6], we consider covering spaces $(\widetilde{X}, p)$ of a pathwise connected space X and also consider the automorphism group $A(\widetilde{X}, p)$ of this covering. In the present paper we will derive some algebraic properties of universal and regular covering spaces $(\widetilde{X}, p)$ of a pathwise connected space X.

Theorem 2.1: Let $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ be two covering spaces of X such that $p_1(\widetilde{x}_1) = p_2(\widetilde{x}_2) = x$ for $\widetilde{x}_1 \in \widetilde{X}_1, \widetilde{x}_2 \in \widetilde{X}_2$. Then there exists an isomorphism φ from $(\widetilde{X}_1, p_1)$ into $(\widetilde{X}_2, p_2)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$ if and only if $p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1) = p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2)$.

Proof: We know from Lemma 1.3 that if $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ are two covering spaces of X such that $p_1(\widetilde{x}_1) = p_2(\widetilde{x}_2) = x$, then there exists a homomorphism φ from $(\widetilde{X}_1, p_1)$ into $(\widetilde{X}_2, p_2)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$ if and only if $p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1) \subset p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2)$. Therefore there exists another homomorphism ψ from $(\widetilde{X}_2, p_2)$ into $(\widetilde{X}_1, p_1)$ such that $\psi(\widetilde{x}_2) = \widetilde{x}_1$ if and only if $p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2) \subset p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1)$. Hence ψ is the invers homomorphism of φ since both compositions $\varphi\psi$ and $\psi\varphi$ are identity maps. Therefore there exists an isomorphism φ from $(\widetilde{X}_1, p_1)$ into $(\widetilde{X}_2, p_2)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$ if and only if $p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1) = p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2)$. ■

From Theorem 2.1 the following two corollaries can be given.

Corollary 2.2: If $(\widetilde{X}, p)$ is a covering space of X such that $p(\widetilde{x}_1) = p(\widetilde{x}_2) = x$ for $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$, then there exists an automorphism $\varphi \in A(\widetilde{X}, p)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$ if and only if $p_*\pi(\widetilde{X}, \widetilde{x}_1) = p_*\pi(\widetilde{X}, \widetilde{x}_2)$.

Corollary 2.3: If $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ are two covering spaces of X such that $p_1(\widetilde{x}_1) = p_2(\widetilde{x}_2) = x$ for $\widetilde{x}_1 \in \widetilde{X}_1, \widetilde{x}_2 \in \widetilde{X}_2$, then these two coverings are isomorphic if and only if $p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1)$ and $p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2)$ are belong the same conjuge class in $\pi(X, x)$.

Theorem 2.4: Let $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ be two universal covering spaces of X such that $p_1(\widetilde{x}_1) = p_2(\widetilde{x}_2) = x$ for $\widetilde{x}_1 \in \widetilde{X}_1, \widetilde{x}_2 \in \widetilde{X}_2$. Then $p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1) = p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2)$.

Proof: $\widetilde{X}_1$ and $\widetilde{X}_2$ are simply connected since $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ are universal covering space of X. From Lemma 1.2, there exists a homeomorphism $h : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 h = p_1$. Therefore $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ are isomorphic and hence $p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1) = p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2)$ by Theorem 2.1. ■

From Theorem 2.4, the following corollary can be given.

Corollary 2.5: If $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ are two universal covering spaces of X such that $p_1(\widetilde{x}_1) = p_2(\widetilde{x}_2) = x$ for $\widetilde{x}_1 \in \widetilde{X}_1, \widetilde{x}_2 \in \widetilde{X}_2$, then

1) these coverings are isomorphic;
2) $p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1)$ and $p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2)$ are conjugate subgroups of $\pi(X, x)$ if and only if there exists an automorphism $\varphi \in A(\widetilde{X}, p)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$.

Theorem 2.6: If $(\widetilde{X}, p)$ is a universal covering space of X, then the index of the subgroup $p_*\pi(\widetilde{X}, \widetilde{x})$ in $\pi(X, x)$ is equal to the order of the automorphism group $A(\widetilde{X}, p)$.

Proof: Let $(\widetilde{X}, p)$ be a universal covering space of X. Then the automorphism group $A(\widetilde{X}, p)$ is isomorphic to $\pi(X, x)$ by Lemma 1.5, and the number of the sheets of the covering of X is equal to the order of the automorphism group $A(\widetilde{X}, p)$. On the other hand, we know that the number of sheets of covering is equal to the index of $p_*\pi(\widetilde{X}, \widetilde{x})$ in $\pi(X, x)$. Therefore the index of the subgroup $p_*\pi(\widetilde{X}, \widetilde{x})$ in $\pi(X, x)$ is equal to the order of the automorphism group $A(\widetilde{X}, p)$. ■

Let $(\widetilde{X}, p)$ be a covering space of X and let

$$N[p_*\pi(\widetilde{X}, \widetilde{x})] = \left\{ \begin{array}{r} \alpha \in \pi(X, x) : \alpha p_*\pi(\widetilde{X}, \widetilde{x})\alpha^{-1} \\ = p_*\pi(\widetilde{X}, \widetilde{x}) \end{array} \right\} \quad (1)$$

be the normalizator of $p_*\pi(\widetilde{X}, \widetilde{x})$ in $\pi(X, x)$. Then the automorphism group $A(\widetilde{X}, p)$ is isomorphic to the quotient group $N\left[p_*\pi(\widetilde{X}, \widetilde{x})\right] / p_*\pi(\widetilde{X}, \widetilde{x})$ by Lemma 1.1.

Theorem 2.7: If $(\widetilde{X}, p)$ is a regular covering space of X, then the automorphism group $A(\widetilde{X}, p)$ is isomorphic to the quotient group $\pi(X, x)/p_*\pi(\widetilde{X}, \widetilde{x})$.

Proof: Let $(\widetilde{X}, p)$ be a regular covering space of X. Then from definition $p_*\pi(\widetilde{X}, \widetilde{x})$ is a normal subgroup of $\pi(X, x)$ and therefore the normalizator of $p_*\pi(\widetilde{X}, \widetilde{x})$ is equal to $\pi(X, x)$, i.e. $N\left[p_*\pi(\widetilde{X}, \widetilde{x})\right] = \pi(X, x)$. Therefore $A(\widetilde{X}, p)$ is isomorphic to the quotient group $\pi(X, x)/p_*\pi(\widetilde{X}, \widetilde{x})$ by (1) and Lemma 1.1. ■

Theorem 2.8: If $(\widetilde{X}, p)$ is a regular covering space of X such that $p(\widetilde{x}_1) = p(\widetilde{x}_2) = x$, for $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$ and $x \in X$, then $p_*\pi(\widetilde{X}, \widetilde{x}_1)$ and $p_*\pi(\widetilde{X}, \widetilde{x}_2)$ are conjugate subgroups of $\pi(X, x)$.

Proof: Let $(\widetilde{X}, p)$ is a regular covering space of X. Then $p_*\pi(\widetilde{X}, \widetilde{x}_1) = p_*\pi(\widetilde{X}, \widetilde{x}_2)$ by Lemma 1.6. Hence from Corollary 2.2, there exists an automorphism $\varphi \in A(\widetilde{X}, p)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$. Therefore $p_*\pi(\widetilde{X}, \widetilde{x}_1)$ and $p_*\pi(\widetilde{X}, \widetilde{x}_2)$ are conjugate subgroups of $\pi(X, x)$ by Corollary 2.3. ■

Theorem 2.9: If $(\widetilde{X}, p)$ is a universal covering space of X, then the automorphism group of the set $p^{-1}(x)$, which is a right $\pi(X, x)-$ space, is isomorphic to the normalizator of $p_*\pi(\widetilde{X}, \widetilde{x})$ in $\pi(X, x)$.

Proof: We know that the automorphism group of the set $p^{-1}(x)$ is isomorphic to the automorphism group $A(\widetilde{X}, p)$. On the other hand this covering is regular since universal covering is regular by Lemma 1.7, and from definition $p_*\pi(\widetilde{X}, \widetilde{x})$ is a normal subgroup of $\pi(X, x)$. Therefore

$$N\left[p_*\pi(\widetilde{X}, \widetilde{x})\right] = \pi(X, x).$$

From Lemma 1.5, $A(\widetilde{X}, p)$ is isomorphic to $\pi(X, x)$. Hence the automorphism group of the set $p^{-1}(x)$ is isomorphic to the normalizator of $p_*\pi(\widetilde{X}, \widetilde{x})$ in $\pi(X, x)$. ■

Theorem 2.10: If $(\widetilde{X}, p)$ is a universal covering spaces of X, then $N\left[p_*\pi(\widetilde{X}, \widetilde{x})\right]$ is isomorphic to $\pi(X, x)/p_*\pi(\widetilde{X}, \widetilde{x})$.

Proof: Let $(\widetilde{X}, p)$ be a universal covering space of X. Then this covering is regular by Lemma 1.7. Therefore from definition $p_*\pi(\widetilde{X}, \widetilde{x})$ is a normal subgroup of $\pi(X, x)$, and $N\left[p_*\pi(\widetilde{X}, \widetilde{x})\right] = p_*\pi(\widetilde{X}, \widetilde{x})$. From Lemma 1.5, $A(\widetilde{X}, p)$ is isomorphic to $\pi(X, x)$ since this covering is universal. Therefore $A(\widetilde{X}, p)$ is isomorphic to $N\left[p_*\pi(\widetilde{X}, \widetilde{x})\right]$. Moreover $A(\widetilde{X}, p)$ is isomorphic to $\pi(X, x)/p_*\pi(\widetilde{X}, \widetilde{x})$ by Theorem 2.7. Hence $N\left[p_*\pi(\widetilde{X}, \widetilde{x})\right]$ and $\pi(X, x)/p_*\pi(\widetilde{X}, \widetilde{x})$ are isomorphic. ■

Let X be a locally pathwise connected space and let G be the discontinuous group of homeomorphisms of X. Now consider the naturel map $q : X \to X/G$ defined by $q(x) = [x]$. Then (X, q) is a regular covering space of X/G and the automorphism group $A(X, q)$ is isomorphic to G by Lemma 1.9.

Theorem 2.11: Let X be a locally pathwise connected, and let G be the discontinuous group of homeomorphisms of X. Let $q : X \to X/G$ be the naturel map such that $q(x_1) = q(x_2)$ for $x_1, x_2 \in X$. Then there exists an automorphism

$\varphi \in A(X,q)$ such that $\varphi(x_1) = x_2$. Moreover $A(X,q)$ is isomorphic to the quotient group $\pi(X/G,[x])/q_*\pi(X,x)$.

Proof: The map $q : X \to X/G$ is a regular map since G is the discontinuous group of homeomorphisms of X by Lemma 1.9. Moreover $q_*\pi(X,x_1) = q_*\pi(X,x_2)$ for $x_1, x_2 \in X$ by Lemma 1.6, and hence there exists an automorphism $\varphi \in A(X,q)$ such that $\varphi(x_1) = x_2$ by Theorem 2.1. It is clear from Theorem 2.7 that $A(X,q)$ is isomorphic to the quotient group $\pi(X/G,[x])/q_*\pi(X,x)$ since (X,q) is a regular covering of X/G. ∎

Theorem 2.12: Let $(\widetilde{X},p)$ be the universal covering space of a locally pathwise connected space X and let $A(\widetilde{X},p)$ be the discontinuous group of homeomorphisms of $\widetilde{X}$. Then the fundamental groups $\pi(X,x)$ and $\pi\left(\widetilde{X}/A(\widetilde{X},p),[\widetilde{x}]\right)$ are isomorphic for the naturel map $q : \widetilde{X} \to \widetilde{X}/A(\widetilde{X},p)$ defined by $q(\widetilde{x}) = [\widetilde{x}]$.

Proof: Note that

$$q : \widetilde{X} \to \widetilde{X}/A(\widetilde{X},p)$$

is a regular map since $A(\widetilde{X},p)$ is the discontinuous group of homeomorphisms of $\widetilde{X}$ by Lemma 1.9, and the automorphism group $A(\widetilde{X},q)$ is isomorphic to the automorphism group $A(\widetilde{X},p)$, i.e. $A(\widetilde{X},q) = A(\widetilde{X},p)$. Moreover $(\widetilde{X},q)$ is a universal covering space of $\widetilde{X}/A(\widetilde{X},p)$ since $(\widetilde{X},p)$ is a universal covering space of X, i.e. $\widetilde{X}$ is simply connected. Therefore from Lemma 1.5, the automorphism group $A(\widetilde{X},p)$ is isomorphic to the fundamental group $\pi(X,x)$ and the automorphism group $A(\widetilde{X},q)$ is isomorphic to the fundamental group $\pi\left(\widetilde{X}/A(\widetilde{X},p),[\widetilde{x}]\right)$. Hence $\pi(X,x)$ and $\pi\left(\widetilde{X}/A(\widetilde{X},p),[\widetilde{x}]\right)$ are isomorphic. ∎

From Theorem 2.12, the following corollary can be obtained.

Corollary 2.13: If $(\widetilde{X},p)$ is a universal covering space of a locally pathwise connected space X and $A(\widetilde{X},p)$ is the discontinuous group of homeomorphisms of $\widetilde{X}$, then $\pi\left(\widetilde{X}/A(\widetilde{X},p),[\widetilde{x}]\right)$ and $A(\widetilde{X},p)$ are isomorphic.

Theorem 2.14: If $(\widetilde{X},p)$ is the universal covering space of X, then the order of the automorphism group $A(\widetilde{X},p)$ is equal to the number of the elements in the orbit $[\widetilde{x}]$ of $\widetilde{x} \in p^{-1}(x)$.

Proof: Let $(\widetilde{X},p)$ be the universal covering space of X. Then the automorphism group $A(\widetilde{X},p)$ is isomorphic to the fundamental group $\pi(X,x)$ by Lemma 1.5, and the order of $\pi(X,x)$ is equal to the number of the sheets of covering. We proved in [4] that the number of the elements in the orbit $[\widetilde{x}]$ of the point $\widetilde{x} \in p^{-1}(x)$ is equal to the number of the sheets of the covering $(\widetilde{X},p)$ of X. Therefore the order of the automorphism group $A(\widetilde{X},p)$ is equal to the number of the elements in the orbit $[\widetilde{x}]$ of $\widetilde{x} \in p^{-1}(x)$. ∎

From Theorem 2.14, the following corollary can be given.

Corollary 2.15: If $(\widetilde{X},p)$ is a universal covering space of a locally pathwise connected space X, then the number of the elements in the orbit $[\widetilde{x}]$ is equal to the number of the sheets of the covering $(\widetilde{X},p)$ of X for the naturel map $q : \widetilde{X} \to \widetilde{X}/A(\widetilde{X},p)$ defined by $q(x) = [x]$.

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