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Solving Stochastic Eigenvalue Problem of Wick Type

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Abstract—In this paper we study mathematically the eigenvalue problem for stochastic elliptic partial differential equation of Wick type. Using the Wick-product and the Wiener-Itô chaos expansion, the stochastic eigenvalue problem is reformulated as a system of an eigenvalue problem for a deterministic partial differential equation and elliptic partial differential equations by using the Fredholm alternative. To reduce the computational complexity of this system, we shall use a decomposition method using the Wiener-Itô chaos expansion. Once the approximation of the solution is performed using the finite element method for example, the statistics of the numerical solution can be easily evaluated.

Keywords—Eigenvalue problem, Wick product, SPDEs, finite element, Wiener-Itô chaos expansion.

I. Introduction

THIS paper is devoted to the study of the eigenvalue problem for a stochastic partial differential equation (SPDE) of Wick type under the framework of the white noise analysis. We study the following problem: Find $(u(x,\omega),\lambda(\omega))$ solution of the linear SPDE

$$\begin{cases} -\Delta u(x,\omega)+\kappa(x,\omega)\diamond u(x,\omega)=\lambda(\omega)\diamond u(x,\omega) \\ u(x,\omega)_{|\partial D\times\Omega}=0 \end{cases} \tag{1}$$

where u is a scalar or vectorial field, κ is a stochastic coefficient, λ is the eigenvalue associated to the eigenfunction u, Ω is the set of random events, $\mathcal{D}\subset\mathbb{R}^d$ ($d=2$ or 3) an open bounded domain with a smooth boundary $\partial\mathcal{D}$.

Equation (1), is an eigenvalue problem for the evolution of a scalar field in a random medium. It arises in several physical and mathematical problems like:

- In random variations of the optical fiber guides where $u(x,\omega)$ denotes the electric field for example, κ is the refractive index of the medium with small imperfections which include the material constituting the fiber core or the extrinsic variations such as microscopic random bends and others fiber core defects and λ represents the stochastic propagation constant [2].
- In structural mechanics where eigenvalues are important properties. These eigenvalues represent resonance frequencies of the systems, e.g. vibrations and buckling [4].

We shall reformulate this stochastic problem as an infinite set of deterministic variational problems, using the properties of the Wick product. Each of these variational problems will give one of the coefficients in the Wiener-Itô chaos expansion of the solution of (1). The method we shall use is based on the ideas of Fourier analysis on Wiener space. In fact, Wiener Chaos expansion represents a stochastic function $u(x,\omega)$x as a Fourier series with respect to an orthonormal basis $\mathcal{H}_\alpha$, i.e., $u(x,\omega)=\sum_{\alpha\in\mathcal{I}} u_\alpha(x)\mathcal{H}_\alpha(\omega)$ where $\mathcal{I}$ denotes the set of multi -indices $\alpha=(\alpha_j)$ where all $\alpha_j\in\mathbb{N}$ and only finitely many $\alpha_j\neq 0$, the u_α's are deterministic coefficients and the $\mathcal{H}_\alpha$'s are the stochastic variables $\mathcal{H}_\alpha(\omega)=\Pi_{j=1}^{\infty}h_{\alpha_j}(\langle\omega,\eta_j(x)\rangle)$, $\omega\in\mathcal{S}'(\mathbb{R}^d)$ where h_n denotes the Hermite polynomial and the family $\{\eta_j\}_{j=1}^{\infty}\subset\mathcal{S}(\mathbb{R}^d)$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$.

An outline of the paper is as follows. In Section II we review notation and introduce some elements of white noise. In Section III, using the Winer-Ifo chaos expansion of the solution, we give the deterministic partial differential equations that must satisfy these chaos coefficients. Finally, in Section IV we give a finite element approximation of our problem.

II. Elements of white noise analysis

A. White noise space

Let $\mathbb{R}^d$ the set of spatial parameters equipped with the Lebesgue measure. We shall construct a Wiener process indexed by R^d, i.e. a Gaussian white noise and describe the associated Hilbert space of quadratic integrable random variables w.r.t. this process.

Let $\mathcal{S}=\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of smooth, rapidly decreasing functions on $\mathbb{R}^d$, and let $\mathcal{S}'=\mathcal{S}'(\ \mathbb{R}^d)$ be the dual space of tempered distributions. By the Bochner-Minlos theorem, cf. [3], [6], there exists a unique probability measure μ, called the white noise porbability measure, on the Borel σ-algebra on $\mathcal{S}'$ with characteristic functional

$$E[e^{i\langle\cdot,\eta\rangle}]:=\int_{\mathcal{S}'}e^{i\langle\omega,\eta\rangle}d\mu(\omega)=e^{-\frac{1}{2}\|\eta\|^2_{L^2(\mathbb{R}^d)}} \tag{2}$$

The random variable $\langle\cdot,\eta\rangle_{\mathcal{S}'}$ defined on the probability space $(\mathcal{S}',\mathcal{B}(\mathcal{S}'),\mu)$ thus follows a Gaussian distribution with mean zero and variance $\|\eta\|^2_{L^2(\mathbb{R}^d)}$, and can be interpreted as the stochastic integral w.r.t a Brownian sheet $B_{t,x}$ defined on $\mathbb{R}^d$, i.e. $\langle\omega,\eta\rangle_{\mathcal{S}'}=\int_{\mathbb{R}^d}\eta(x)dB_{t,x}(\omega),\ \omega\in\mathcal{S}',\eta\in\mathcal{S}$.

B. Chaos decomposition

A chaos decomposition is an orthonormal expansion in the Hilbert space $L^2(\mathcal{S}')$ of quadratic integrable functions

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defined on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$. For $n \in \mathbb{N}_0, x \in \mathbb{R}$ define the Hermite polynomial $h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2})$, and for $n \in \mathbb{N}$ define the Hermite functions $\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2} e^{-x^2/2} h_{n-1}(\sqrt{2}x)$.. It is well-known that $\xi_n \in \mathcal{S}(\mathbb{R})$, $\|\xi_n\|_\infty \leq 1$ $(n \in \mathbb{N})$, and that $\{\xi_n : n \in \mathbb{N}\}$ constitutes an orthonormal basis in $L^2(\mathbb{R}, dx)$. We let $\{\eta_j\}_{j\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ denote the orthonormal basis for $L^2(\mathbb{R}^d, dx)$ constructed by taking tensor-products of Hermite functions [3]:

$$\eta_j(x) = \xi_{\delta_1^{(j)}}(x_1)\xi_{\delta_1^{(j)}}(x_2)\cdots\xi_{\delta_d^{(j)}}(x_d)\ ,\ j = 1, 2, \cdots$$

where $\delta^{(j)} = (\delta_1^{(j)}, \delta_2^{(j)}, \cdots, \delta_d^{(j)})$ is the jth multi-index number in some fixed ordering of all d-dimensional multi-indices $\delta = (\delta_1, \cdots, \delta_d)$.

Let $\mathcal{I}$ denote the set of al multi-indices $\alpha = (\alpha_j)$ with $\alpha_j \in \mathbb{N}_0$ $(j \in \mathbb{N})$ with finite length $l(\alpha) = \max\{j; \alpha_j \neq 0\}$, and as usual we define $\alpha + \beta = (\alpha_j + \beta_j)$, $\alpha! = \prod_j \alpha_j!$, and $|\alpha| := \sum_j \alpha_j$. For each $\alpha \in \mathcal{I}$ we define the stochastic variable

$$H_\alpha(\omega) := \prod_{j=1}^{l(\alpha)} h_{\alpha_j}(\langle \omega, \eta_j \rangle),\ \omega \in \mathcal{S}' \tag{3}$$

The family $\{H_\alpha : \alpha \in \mathcal{I}\}$ constitutes an orthogonal basis for $L^2(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$ and it holds $E[H_\alpha H_\beta] = \alpha!\delta_{\alpha\beta}$ [3]. Thus, any f in $L^2(\mu) := L^2(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$ has a unique representation

$$f = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha \tag{4}$$

where $f_\alpha \in \mathcal{R}$ and $\|f\|^2_{L^2(\mu)} = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha!$. The expansion in (4) is often referred to as the Wiener-Itô chaos expansion. We will in the following adopt the notation f_α to denote the αth chaos coefficient of a random variable f.

C. Wiener chaos expansion of a log normal process

In this work we focus on equation (1) with random coefficient κ wich satisfy the following condition : there exist two positive constants $C_1, C_2 > 0$ such that

$$0 < C_1 \leq \kappa(x, \omega) \leq C_2 < \infty,\ \text{a.e and a.s}$$

We assume also that κ has the following Karhunen-Loeve (K-L) expansion

$$\kappa(x, \omega) = \sum_{k=0}^{\infty} \sqrt{\lambda_k}\beta_k(\omega)\phi_k(x) \tag{5}$$

where $\beta_k(\omega)$ are the uncorrelated zero mean and unite variance random variables, (λ_i, ϕ_i) is the pair of eigenvalues and eigenfunctions of the covariance function. Since each $\beta_i \in L^2(\mu)$, it my be expanded in its Wiener chaos expansion

$$\beta_k(\omega) = \sum_{\alpha\in\mathcal{I}} \beta_{k,\alpha} H_\alpha(\omega) \tag{6}$$

By substituting (7) in (6) we obtain

$$\begin{aligned} \kappa(x, \omega) &= \sum_{\alpha\in\mathcal{I}} \left(\sum_{k=0}^{\infty} \sqrt{\lambda_k}\beta_{k,\alpha}\phi_k(x) \right) H_\alpha(\omega) \\ &:= \sum_{\alpha\in\mathcal{I}} \kappa_\alpha(x) H_\alpha(\omega) \end{aligned} \tag{7}$$

D. Ordinary and Wick products

Definition 1: The Wick product $f \diamond g$ of two formal series $f = \sum_\alpha f_\alpha H_\alpha$, $g = \sum_\alpha g_\alpha H_\alpha$ is defined as

$$f \diamond g := \sum_{\alpha,\beta\in\mathcal{I}} f_\alpha g_\beta H_{\alpha+\beta}. \tag{8}$$

From [5] we have the following result:

Theorem 1: Suppose $u = \sum_{\alpha\in\mathcal{I}} u_\alpha H_\alpha, v = \sum_{\alpha\in\mathcal{I}} v_\beta H_\beta$. If $E_\mu \mid uv \mid^2) < \infty$, then the product uv has the Wiener chaos expansion

$$uv = \sum_{\theta\in\mathcal{I}} \left(\sum_{p\in\mathcal{I}} \sum_{0\leq\beta\leq\theta} C(\theta, \beta, p) u_{\theta-\beta+p} v_{\beta+p} \right) H_\theta$$

where

$$C(\theta, \beta, p) = \frac{(\theta - \beta + p)!(\beta + p)!}{\beta! p! (\theta - \beta)!}$$

Since

$$\begin{aligned} u \diamond v &= \sum_{\alpha\in\mathcal{I}} \sum_{\beta\in\mathcal{I}} u_\alpha v_\beta H_{\alpha+\beta} \\ &= \sum_{\theta\in\mathcal{I}} \sum_{0\leq\beta\leq\theta} u_{\theta-\beta} v_\beta H_\theta \end{aligned}$$

we have

Theorem 2:

$$uv = u \diamond v + \sum_{\theta\in\mathcal{I}} \left(\sum_{p\in\mathcal{I}, p\neq 0} \sum_{0\leq\beta\leq\theta} C(\theta, \beta, p) u_{\theta-\beta+p} v_{\beta+p} \right) H_\theta$$

By approaching $uv \approx u \diamond v$, we can view equation (1) as an approximation or a regularization of the general stochastic eigenvalue problem

$$\begin{cases} -\Delta u(x, \omega) + \kappa(x, \omega) u(x, \omega) = \lambda(\omega) u(x, \omega) \\ u(x, \omega)_{|\partial D\times\Omega} = 0 \end{cases} \tag{9}$$

III. Deterministic equations for the chaos coefficients

We give now a reformulation of (1) as an infinite set of deterministic PDEs, using the properties of the Wick product.

Theorem 3: Let $\kappa(x, \omega) = \sum_{\gamma\in\mathcal{I}} \kappa_\gamma(x) H_\gamma(\omega)$. Let $u(x, \omega) = \sum_{\gamma\in\mathcal{I}} u_\gamma(x) H_\gamma(\omega)$ and $\lambda(\omega) = \sum_{\gamma\in\mathcal{I}} \lambda_\gamma H_\gamma(\omega)$ be a solution of (1). Then the γ-th chaos coefficients u_γ of u and λ_γ of λ are given by the following triangular system of deterministic equations:

• If $\gamma = 0$, then (u_0, λ_0) is solution of the eigenvalue problem

$$\begin{cases} -\Delta u_0 + \kappa_0 u_0 = \lambda_0 u_0 \\ u_{0|\partial D} = 0 \end{cases} \tag{10}$$

• If $\mid \gamma \mid = 1$, then $(u_\gamma, \lambda_\gamma)$ is solution of the following problem

$$\begin{cases} -\Delta u_\gamma + (\kappa_0 - \lambda_0)u_\gamma = (\lambda_\gamma - \kappa_\gamma)u_0 \\ u_{\gamma|\partial D} = 0 \end{cases} \quad (11)$$

• If $\mid \gamma \mid > 1$, then $(u_\gamma, \lambda_\gamma)$ is solution of the following problem

$$\begin{cases} -\Delta u_\gamma + (\kappa_0 - \lambda_0)u_\gamma = (\lambda_\gamma - \kappa_\gamma)u_0 \quad + \\ \sum\limits_{0<\alpha<\gamma} (\lambda_\alpha - \kappa_\alpha)u_{\gamma-\alpha} \\ u_{\gamma|\partial D} = 0 \end{cases} \quad (12)$$

Proof: By definition of the Wick product we have

$$-\sum_{\gamma\in\mathcal{I}} \Delta u_\gamma H_\gamma(\omega) + \sum_{\gamma\in\mathcal{I}} \sum_{0\le\alpha\le\gamma} (\kappa_\alpha - \lambda_\alpha)u_{\gamma-\alpha}H_\gamma(\omega) = 0$$

Due to uniqueness of the Wiener-Itô chaos expansion, this is equivalent to

$$-\Delta u_\gamma + \sum_{0\le\alpha\le\gamma} (\kappa_\alpha - \lambda_\alpha)u_{\gamma-\alpha} = 0,\ \gamma \in \mathcal{I}$$

from which we deduce the equations (10), (11) and (12). ■

We note that we have decomposed problem (1) into a cascade of deterministic partial differential equations where the equation for the first coefficient (u_0, λ_0) is an eigenvalue problem and others coefficients involve non-eigenvalue inhomogeneous problems.

Theorem 4: For each $\gamma \in \mathcal{I}$ and knowing $(u_\alpha, \lambda_\alpha)$ for all $0 \le \alpha < \gamma$ we have

• If $\mid \gamma \mid = 1$ then

$$\lambda_\gamma = \frac{1}{\|u_0\|^2} \int_D \kappa_\gamma u_0^2\, dx \quad (13)$$

• If $\mid \gamma \mid > 1$ then

$$\begin{cases} \lambda_\gamma = \dfrac{1}{\|u_0\|^2} \displaystyle\int_D \kappa_\gamma u_0^2\, dx - \\ \dfrac{1}{\|u_0\|^2} \displaystyle\sum_{0<\alpha<\gamma} \int_D (\lambda_\alpha - \kappa_\alpha)u_{\gamma-\alpha}u_0\, dx \end{cases} \quad (14)$$

Proof: We multiply equation (12) by u_0 and integrate over D. Integrating the second derivatives term twice by parts we find

$$\begin{cases} \displaystyle\int_D (-\Delta u_0 + (\kappa_0 - \lambda_0)u_0)\, u_\gamma\, dx = \\ \lambda_\gamma \displaystyle\int_D u_0 u_0\, dx - \int_D \kappa_\gamma u_0^2\, dx + \\ \displaystyle\sum_{0<\alpha<\gamma} \int_D (\lambda_\alpha - \kappa_\alpha)u_{\gamma-\alpha}u_0\, dx \end{cases} \quad (15)$$

By using equation (10), we find (14). ■

In order to solve the eigenvalue problem (10) we assume for simplicity that $\kappa_0 \in L^\infty(D)$ and $\kappa_0 > 0$. In this case the operator $-\Delta u_0 + \kappa_0 u_0$ is symetric and uniformly elliptic and we can apply the standard result of the spectrum of a self-adjoint elliptic operator [1]:

Theorem 5: There exists an increasing sequence of real eigenvalues of finite multiplicity

$$0 < \lambda_{0,1} \le \lambda_{0,2} \le \cdots \le \lambda_{0,n} \le \cdots$$

such that $\lambda_{0,n} \longrightarrow \infty$. Moreover, there is an orthonormal basis $\{u_{0,n} : n \in \mathbb{N}\}$ of $L^2(D)$ consisting of eigenfunctions $u_{0,n} \in H_0^1(D)$ such that

$$\begin{cases} -\Delta u_{0,n} + \kappa_0 u_{0,n} = \lambda_{0,n} u_{0,n} \\ u_{0,n|\partial D} = 0 \end{cases} \quad (16)$$

From this result, we see from (11),(12) that for $\gamma > 0$ and $n \in \mathbb{N}$ and knowing $(u_{\alpha,n}, \lambda_{\alpha,n})$ for all $0 \le \alpha < \gamma$ we have to solve the following sequence of Fredholm alternative elliptic problems

• If $\mid \gamma \mid = 1$

$$\begin{cases} -\Delta u_{\gamma,n} + (\kappa_0 - \lambda_{0,n})u_{\gamma,n} = (\lambda_{\gamma,n} - \kappa_\gamma)u_{0,n} \\ u_{\gamma,n|\partial D} = 0 \end{cases} \quad (17)$$

where $\lambda_{n,\gamma}$ is given by

$$\lambda_{\gamma,n} = \frac{1}{\|u_{0,n}\|^2} \int_D \kappa_\gamma u_{0,n}^2\, dx \quad (17)$$

• If $\mid \gamma \mid = 1$

$$\begin{cases} -\Delta u_{\gamma,n} + (\kappa_0 - \lambda_{0,n})u_{\gamma,n} = \\ (\lambda_{\gamma,n} - \kappa_\gamma)u_{0,n} + \displaystyle\sum_{0<\alpha<\gamma} (\lambda_{\alpha,n} - \kappa_\alpha)u_{\gamma-\alpha,n} \\ u_{\gamma,n|\partial D} = 0 \end{cases} \quad (19)$$

where $\lambda_{n,\gamma}$ is given by

$$\begin{cases} \lambda_{\gamma,n} = \dfrac{1}{\|u_{0,n}\|^2} \displaystyle\int_D \kappa_\gamma u_{0,n}^2\, dx - \\ \dfrac{1}{\|u_{0,n}\|^2} \displaystyle\sum_{0<\alpha<\gamma} \int_D (\lambda_{\alpha,n} - \kappa_\alpha)u_{\gamma-\alpha,n}u_{0,n}\, dx \end{cases} \quad (20)$$

Finally by the Fredholm alternative theorem [1], we may state the following result:

Theorem 6: Let $\kappa_0 \in L^\infty(D)$ with $\kappa_0 > 0$. Then for each $n \in \mathbb{N}$ there exists a solution $u_{\gamma,n} \in H_0^1(D)$ of the problem (17) or (19) whis is not unique.

Proof: Let

$$F_{\gamma,n} := (\lambda_{\gamma,n} - \kappa_\gamma)u_{0,n} + \sum_{0<\alpha<\gamma} (\lambda_{\alpha,n} - \kappa_\alpha)u_{\gamma-\alpha,n}$$

By using (18), (20), we have the orthogonality relation

$$(F_{\gamma,n}, u_{0,n})_{L^2(D)} = 0$$

And the result follows from the Fredholm alternative theorem, since $\lambda_{0,n}$ is an eigenvalue of the auto-adjoint compact operator $-\Delta + \kappa_0 I$ with the associated eigenfunction $u_{0,n}$. ■

IV. The approximated solution

From the previous section, we can see that our problem (1) admit a sequence of eigenfunctions $u_n(x,\omega)$ and eigenvalues $\lambda_n(\omega)$ given by

$$u_n(x,\omega) = \sum_{\gamma\in\mathcal{I}} u_{\gamma,n}(x)H_\gamma(\omega) \quad (18)$$

$$\lambda_n(\omega) = \sum_{\gamma\in\mathcal{I}} \lambda_{\gamma,n}H_\gamma(\omega) \quad (19)$$

where $u_{\gamma,n}, \lambda_n$ are given by (17)-(20).

Since equations (17), (19) are recast in a deterministic manner, their numerical solution can be obtained using numerical methods widely used for approximating partial differential equations. Let us now introduce a discrete version of this model problem using the finite element method.

Let M_h a finite element sub-space of $H_0^1(\mathcal{D})$. For $N, K \in \mathbb{N}$, we define the subset

$$\mathcal{I}_{N,K} = \{0\} \cup \bigcup_{n=1}^{N} \bigcup_{k=1}^{K} \{\alpha \in \mathbb{N}_0^k : \quad |\alpha| = n \quad \text{and} \quad \alpha_k \neq 0\}$$

We approximate for example $(u_{0,n}, \lambda_{0,n}) \in H_0^1(\mathcal{D}) \times \mathbb{R}$ as follows: seek $(u_{0,n}^h, , \lambda_{0,n}^h) \in M_h \times \mathbb{R}$ such that:

$$\int_{\mathcal{D}} \nabla u_{0,n}^h \nabla v_h \, dx + \int_{\mathcal{D}} \kappa_0 u_{0,n}^h v_h \, dx = \lambda_{0,n}^h \int_{\mathcal{D}} u_{0,n}^h v_h \, dx \ (..)$$

Also, we assume that the set $\mathcal{I}_{N,K}$ is ordered in such a way that $\{u_{n,\alpha}^h, \lambda_{n,\alpha}^h \, \alpha \prec \gamma\}$ has been calculated when the γ-th equation in (..) is considered. This enable us to solve (19) as a sequence of $(N+K)!/(N!K!)$ problems.

Once we have calculated the chaos coefficients $\{(u_{n,\gamma}^h) : \gamma \in \mathcal{I}_{N,K}\}$, we may do stochastic simulations of the solution as follows: first, generate M independent standard Gaussian variables $X(\omega) = (X_i(\omega))$ $(i = 1, \ldots, M)$ using some random number generator, and then form the sums

$$u_n^h(x, \omega) := \sum_{\gamma \in \mathcal{I}_{N,K}} u_{n,\gamma}^h(x) H_\alpha(X(\omega)) \tag{20}$$

$$\lambda_n^h(\omega) := \sum_{\gamma \in \mathcal{I}_{N,K}} \lambda_{n,\alpha}^h H_\gamma(X(\omega)) \tag{21}$$

where $H_\alpha(X(\omega)) := \prod_{j=1}^{M} h_{\alpha_j}(X_j(\omega))$

The advantage of this approach is that it enables us to generate random samples easy and fast. For example, in situations where one is interested in repeated simulations of u_n^h and λ_n^h, one may compute the chaos coefficients in advance, store them, and produce the simulations whenever they are needed.

V. Conclusion

We presented in this paper a method to solve eigenvalue problem for a stochastic partial differential equations of Wick type. This method is based on a decomposition approach by using the Wiener-Itô expansions. We have shown that the computational cost of the original equation can be drastically reduced using this approach. This method decompose the original stochastic equation by solving only linear deterministic partial differential equations combined with a deterministic eigenvalue problem for the first chaos coefficient.

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