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Solving Fully Fuzzy Linear Systems by use of a Certain Decomposition of the Coefficient Matrix

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***Abstract*—In this paper, we give a certain decomposition of the coefficient matrix of the fully fuzzy linear system (FFLS) to obtain a simple algorithm for solving these systems. The new algorithm can solve FFLS in a smaller computing process. We will illustrate our method by solving some examples.**

***Keywords*—Fully fuzzy linear system, Fuzzy number, LU-decomposition.**

I. Introduction

ONE major application of the fuzzy number arithmetic is treating linear systems whose parameters are all or partially represented by fuzzy numbers. The term fuzzy matrix, which is the most important concept in this paper, has various meanings. For definition of a fuzzy matrix we follow the definition of Dubois and Prade, i.e. a matrix with fuzzy numbers as its elements [5]. This class of fuzzy matrices consist of applicable matrices, which can model uncertain aspects and the works on them are too limited. Some of the most interesting works on these matrices can be seen in [2], [3], [4], [7]. A general model for solving a fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy vector, was first proposed by Friedman et al. [6] and many authors is considered these models for their studies (see in [8] and [9]). Another important kind of fuzzy linear systems are including triangular fuzzy numbers in whose all parameters and is named fully fuzzy linear systems (see in [3], [4], [7]). Nevertheless, there is just a few computational methods for solving the fully fuzzy linear systems until now. For example, recently Dehghan and his colleagues in [3] and [4] proposed two numerical methods for solving these kind of systems. In [7], authors used a new method for solving these systems based on QR decomposition. Hence, in this paper we intend to solve $\widetilde{A}\otimes\widetilde{x}=\widetilde{b}$, where $\widetilde{A}$ is a fuzzy matrix and $\widetilde{x}$ and $\widetilde{b}$ are fuzzy vectors with appropriate sizes.

The structure of this paper is organized as follows:
In Section 2, we first give some basic concepts of fuzzy sets theory and then define a fully fuzzy linear system of equations. A numerical method for computing the solution of FFLS is designed in Section 3. Numerical examples are given in Section 4 to examine our method.

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II. Preliminaries

In this section, we review some necessary backgrounds and notions of fuzzy sets theory (taken from [5], [7]).
Definition 2.1. A fuzzy subset $\widetilde{A}$ of R is defined by its membership function

$$\mu_{\widetilde{A}} : R \to [0,1],$$

which assigns a realnumber $\mu_{\widetilde{A}}$ in the interval $[0,1]$, to each element $x \in R$, where the value of $\mu_{\widetilde{A}}$ at x shows the grade of membership of x in $\widetilde{A}$.
Definition 2.2. A fuzzy set with the following membership function is named a triangular fuzzy number and in this paper we will use these fuzzy numbers.

$$\mu_{\tilde{A}}(x) = \begin{cases} 1-\frac{m-x}{\alpha}, & m-\alpha \le x < m,\ \alpha > 0, \\ 1-\frac{x-m}{\beta}, & m \le x \le m+\beta,\ \beta > 0, \\ 0, & otherwise. \end{cases} \quad (1)$$

Definition 2.3. A fuzzy number $\tilde{A}$ is called positive (negative), denoted by $\tilde{A} > 0$ ($\tilde{A} < 0$), if its membership function $\mu_{\tilde{A}}(x)$ satisfies $\mu_{\tilde{A}}(x) = 0$, $\forall x \le 0$ ($\forall x \ge 0$).
Using its mean value and left and right spreads, and shape functions, such a fuzzy number $\tilde{A}$ is symbolically written

$$\tilde{A} = (m, \alpha, \beta).$$

Clearly, $\tilde{A} = (m,\alpha,\beta)$ is positive, if and only if $m - \alpha \ge 0$.

Remark 2.1. We consider $\widetilde{0} = (0, 0, 0)$ as a zero triangular fuzzy number.
Remark 2.2. We show the set of all triangular fuzzy numbers by $F(R)$.
Definition 2.4 (Equality in fuzzy numbers). Two fuzzy numbers $M = (m,\alpha,\beta)$ and $N = (n,\gamma,\delta)$ are said to be equal, if and only if $m = n$, $\alpha = \gamma$ and $\beta = \delta$.
Definition 2.5. For two fuzzy numbers $M = (m,\alpha,\beta)$ and $N = (n,\gamma,\delta)$ the formula for the extended addition becomes:

$$(m,\alpha,\beta) \oplus (n,\gamma,\delta) = (m+n, \alpha+\gamma, \beta+\delta). \quad (2)$$

The formula for the extended opposite becomes:

$$-M = -(m,\alpha,\beta) = (-m,\beta,\alpha). \quad (3)$$

The approximate formulas for the extended multiplication of two fuzzy numbers can be summarized as follows as given in [5]:
If $M > 0$ and $N > 0$, then

$$(m,\alpha,\beta) \otimes (n,\gamma,\delta) \cong (mn, m\gamma + n\alpha, m\delta + n\beta). \quad (4)$$

For scalar multiplication:

$$\lambda \otimes (m, \alpha, \beta) = \begin{cases} (\lambda m, \lambda\alpha, \lambda\beta), & \lambda \geq 0, \\ (\lambda m, -\lambda\beta, -\lambda\alpha), & \lambda < 0. \end{cases} \tag{5}$$

Definition 2.6. A matrix $\widetilde{A} = (\widetilde{a_{ij}})$ is called a fuzzy matrix, if each element of $\widetilde{A}$ is a fuzzy number.
A fuzzy matrix $\widetilde{A}$ will be positive and denoted by $\widetilde{A} > \widetilde{0}$, if each element of $\widetilde{A}$ be positive. We may represent $n \times n$ fuzzy matrix $\widetilde{A} = (\widetilde{a_{ij}})_{n\times n}$, such that $\widetilde{a_{ij}} = (a_{ij}, \alpha_{ij}, \beta_{ij})$, with the new notation $\widetilde{A} = (A, M, N)$, where $A = (a_{ij})$, $M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are three $n \times n$ crisp matrices.
Definition 2.7. A square fuzzy matrix $\widetilde{A} = (\widetilde{a_{ij}})$ will be an upper triangular fuzzy matrix, if

$$\widetilde{a_{ij}} = \widetilde{0} = (0, 0, 0), \qquad \forall i > j,$$

and a square fuzzy matrix $\widetilde{A} = (\widetilde{a_{ij}})$ will be a lower triangular fuzzy matrix, if

$$\widetilde{a_{ij}} = \widetilde{0} = (0, 0, 0), \qquad \forall i < j.$$

Definition 2.8. Consider the $n \times n$ fuzzy linear system of equations [3], [7]:

$$\begin{cases} (\widetilde{a_{11}} \otimes \widetilde{x_1}) \oplus (\widetilde{a_{12}} \otimes \widetilde{x_2}) \oplus ... \oplus (\widetilde{a_{1n}} \otimes \widetilde{x_n}) = \widetilde{b_1}, \\ (\widetilde{a_{21}} \otimes \widetilde{x_1}) \oplus (\widetilde{a_{22}} \otimes \widetilde{x_2}) \oplus ... \oplus (\widetilde{a_{2n}} \otimes \widetilde{x_n}) = \widetilde{b_2}, \\ \quad . \\ \quad . \\ \quad . \\ (\widetilde{a_{n1}} \otimes \widetilde{x_1}) \oplus (\widetilde{a_{n2}} \otimes \widetilde{x_2}) \oplus ... \oplus (\widetilde{a_{nn}} \otimes \widetilde{x_n}) = \widetilde{b_n}. \end{cases} \tag{6}$$

The matrix form of the above equations is

$$\widetilde{A} \otimes \widetilde{x} = \widetilde{b},$$

where the coefficient matrix $\widetilde{A} = (\widetilde{a_{ij}}),\ \ 1 \leq i, j \leq n$ is a $n \times n$ fuzzy matrix and $\widetilde{x_j}, \widetilde{b_j} \in F(R)$. This system is called a fully fuzzy linear system (FFLS).
In this paper we are going to obtain a positive solution of FFLS $\widetilde{A} \otimes \widetilde{x} = \widetilde{b}$, where $\widetilde{A} = (A, M, N) > \widetilde{0}$, $\widetilde{b} = (b, g, h) > \widetilde{0}$ and $\widetilde{x} = (x, y, z) > \widetilde{0}$. So we have

$$(A, M, N) \otimes (x, y, z) = (b, g, h).$$

Then by using Eq.(5) we have

$$(Ax, Ay + Mx, Az + Nx) = (b, h, g).$$

Therefore, Definition 2.4 concludes that

$$\begin{cases} Ax = b, \\ Ay + Mx = g, \\ Az + Nx = h. \end{cases} \tag{7}$$

So, by assuming that A be a nonsingular matrix we have

$$\begin{cases} Ax = b & \Rightarrow x = A^{-1}b, \\ Ay = g - Mx & \Rightarrow y = A^{-1}(g - Mx), \\ Az = h - Nx & \Rightarrow z = A^{-1}(h - Nx). \end{cases}$$

III. A NEW DECOMPOSITION METHOD FOR SOLVING FFLS

Theorem 3.1. Let A be an $n \times n$ matrix with all non-zero leading principal minors. Then A has a unique factorization:

$$A = LU,$$

where L is unit lower triangular and U is upper triangular [1]. Assume that $\widetilde{A} = (A, M, N)$, where A is a full rank crisp matrix. Then if we let

$$(L_1, 0, 0) \otimes (U_1, U_2, U_3) = (A, M, N), \tag{8}$$

then from (5) we have

$$\begin{cases} L_1U_1 = A & \Rightarrow \quad U_1 = L_1^{-1}A \\ L_1U_2 = M & \Rightarrow \quad U_2 = L_1^{-1}M \\ L_1U_3 = N & \Rightarrow \quad U_3 = L_1^{-1}N \end{cases} \tag{9}$$

where matrix L_1 is a lower triangular crisp matrix and matrix U_1 is an upper triangular crisp matrix.
Again consider the fully fuzzy linear systems. We are going to construct a new method for solving FFLS $\widetilde{A} \otimes \widetilde{x} = \widetilde{b}$, where $\widetilde{A} = (A, M, N)$, $\widetilde{x} = (x, y, z)$, $\widetilde{b} = (b, g, h)$, that is

$$(A, M, N) \otimes (x, y, z) = (b, g, h).$$

Eq.(8) implies that

$$(L_1U_1, L_1U_2, L_1U_3) \otimes (x, y, z) = (b, g, h).$$

Therefore, by using (5), we have

$$(L_1U_1x, L_1U_2x + L_1U_1y, L_1U_3x + L_1U_1z) = (b, g, h).$$

The current system by use of Definition 2.4 can be rewrite as follows:

$$\begin{cases} L_1U_1x = b, \\ L_1U_2x + L_1U_1y = g, \\ L_1U_3x + L_1U_1z = h, \end{cases}$$

and therefore

$$\begin{cases} x = U_1^{-1}L_1^{-1}b, \\ y = U_1^{-1}L_1^{-1}(g - L_1U_2x), \\ z = U_1^{-1}L_1^{-1}(h - L_1U_3x). \end{cases} \tag{10}$$

Now we are a position to present a new algorithm to solve the fully fuzzy linear system.

A. Algorithm

(Fully Fuzzy Linear Systems Solver)
Step 1: Assume that $\widetilde{A} = (A, M, N)$, where A is a full rank crisp matrix.
Compute LU-decomposition for crisp matrix A as

$$A = L_1U_1.$$

Step 2: Set

$$L_2 = 0, \quad L_3 = 0.$$

Step 3: Compute

$$U_2 = L_1^{-1} M.$$

Step 4: Compute

$$U_3 = L_1^{-1} N.$$

Step 5: Compute the solution of the fully fuzzy linear system $(A, M, N) \otimes (x, y, z) = (b, g, h)$ as follows:

$$\begin{cases} x = U_1^{-1} L_1^{-1} b, \\ y = U_1^{-1} L_1^{-1} (g - L_1 U_2 x), \\ z = U_1^{-1} L_1^{-1} (h - L_1 U_3 x). \end{cases}$$

IV. Numerical examples

In this section, we apply our algorithm for solving three fully fuzzy linear systems to illustrate the advantage of our method.

Example 4.1. Consider the following FFSL (taken from [3]):

$$\begin{pmatrix} (6,1,4) & (5,2,2) & (3,2,1) \\ (12,8,20) & (14,12,15) & (8,8,10) \\ (24,10,34) & (32,30,30) & (20,19,24) \end{pmatrix} \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix}$$

$$= \begin{pmatrix} (58,30,60) \\ (142,139,257) \\ (316,297,514) \end{pmatrix}$$

First we obtain LU-decomposition for matrix A as follows:

$$\begin{pmatrix} 6 & 5 & 3 \\ 12 & 14 & 8 \\ 24 & 32 & 20 \end{pmatrix} = L_1 U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} 6 & 5 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

So we can obtain matrices U_2 and U_3 from steps 3 and 4 in Algorithm as follows:

$$U_2 = L_1^{-1} M = \begin{pmatrix} 1 & 2 & 2 \\ 6 & 8 & 4 \\ -12 & -2 & -1 \end{pmatrix},$$

$$U_3 = L_1^{-1} N = \begin{pmatrix} 4 & 2 & 1 \\ 12 & 11 & 8 \\ -18 & -11 & -4 \end{pmatrix}.$$

Therefore, Eq.(10) concludes that

$$\widetilde{x} = (4,1,3), \quad \widetilde{y} = (5, \frac{1}{2}, 2), \quad \widetilde{z} = (3, \frac{1}{2}, 1).$$

As we see, the mentioned system has a same solution with LU decomposition method as given in [3].

Example 4.2. Consider the following FFSL (taken from [4]):

$$\begin{pmatrix} (19,1,1) & (12,1.5,1.5) & (6,0.5,0.2) \\ (2,0.1,0.1) & (4,0.1,0.4) & (1.5,0.2,0.2) \\ (2,0.1,0.2) & (2,0.1,0.3) & (4.5,0.1,0.1) \end{pmatrix} \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix}$$

$$= \begin{pmatrix} (1897,427.7,536.2) \\ (434.5,76.2,109.3) \\ (535.5,88.3,131.9) \end{pmatrix}$$

First we obtain LU-decomposition for matrix A as follows:

$$\begin{pmatrix} 19 & 12 & 6 \\ 2 & 4 & 1.5 \\ 2 & 2 & 4.5 \end{pmatrix} = L_1 U_1 =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.1052 & 1 & 0 \\ 0.1052 & 0.2693 & 1 \end{pmatrix} \begin{pmatrix} 19 & 12 & 6 \\ 0 & 2.7368 & 0.8684 \\ 0 & 0 & 3.6346 \end{pmatrix}.$$

So we can obtain matrices U_2 and U_3 from steps 3 and 4 in Algorithm as follows:

$$U_2 = L_1^{-1} M = \begin{pmatrix} 1 & 1.5 & 0.5 \\ -0.0052 & -0.0578 & 0.1473 \\ -0.0038 & -0.0423 & 0.0076 \end{pmatrix},$$

$$U_3 = L_1^{-1} N = \begin{pmatrix} 1 & 1.5 & 0.2 \\ -0.0052 & 0.2421 & 0.1789 \\ 0.0961 & 0.0769 & 0.0307 \end{pmatrix}.$$

Therefore, Eq.(10) concludes that

$$\widetilde{x} = (36.9999, 7, 13.3015),$$

$$\widetilde{y} = (61.9999, 5.5, 4.5793),$$

$$\widetilde{z} = (74.9999, 10.1999, 13.9195).$$

V. Conclusion

In this paper, we used a certain decomposition of the coefficient matrix of the fully fuzzy linear system of equations to construct a new algorithm for solving fully fuzzy linear systems. We examined our algorithm by solving three fully fuzzy linear systems. In particular, in the second and third cases, we exactly used two examples which was used in [3] and [4], respectively. We saw that both algorithms obtained the same solutions, but our algorithm (As one may compare with [3] and [4]) can solve the mentioned linear systems by a shorter calculation.

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