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Solving a System of Nonlinear Functional Equations Using Revised New Iterative Method

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Abstract—In the present paper, we present a modification of the New Iterative Method (NIM) proposed by Daftardar-Gejji and Jafari [J. Math. Anal. Appl. 2006;316:753–763] and use it for solving systems of nonlinear functional equations. This modification yields a series with faster convergence. Illustrative examples are presented to demonstrate the method.

Keywords—Caputo fractional derivative, System of nonlinear functional equations, Revised new iterative method.

I. Introduction

Nonlinear differential equations play very important role in modelling numerous problems in Physics, Chemistry, Biology and Engineering Science [1], [2]. Many problems can be modelled as systems of differential equations/integral equations/integro-differential equations/partial differential equations/fractional order differential equations. Since most realistic functional equations are nonlinear and do not possess exact analytical solutions, iterative and numerical methods are widely used to solve these equations. Adomian decomposition method (ADM) [3], variational iteration method (VIM) [4], homotopy perturbation method (HPM) [5] are some of the standard methods. Recently Daftardar-Gejji and Jafari [6] have introduced a new iterative method (NIM) to solve general functional equation: $y = f + N(y)$, where f is specified function and N a given nonlinear function of y. NIM is simple in its principles and easy to implement on computer using symbolic computation packages such as Mathematica. This method is better than numerical method as it is free from rounding off errors and does not require large computer power. NIM has proven successful over other methods in many cases [7], [8].

In the present paper we present a modification of the NIM to solve the following system of functional equations with improved convergence:

$$y_i = f_i + N_i(y_1, y_2, \cdots, y_n) \quad i = 1, 2, \cdots, n.$$

The revised method has been applied to solve various examples, some of which have already been solved by other methods. A comparison with other solutions reveals the usefulness of this modification.

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II. Preliminaries and notations

We review some basic definitions from fractional calculus [1], [9].

Definition 2.1: A real function f(x), $x > 0$ is said to be in space $C_\alpha, \alpha \in \Re$ if there exists a real number p $(> \alpha)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$.

Definition 2.2: A real function f(x), $x > 0$ is said to be in space $C_\alpha^m, m \in I\!N \bigcup \{0\}$ if $f^{(m)} \in C_\alpha$.

Definition 2.3: Let $f \in C_\alpha$ and $\alpha \geq -1$, then the (left-sided) Riemann-Liouville integral of order μ is given by

$$I_t^\mu f(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(x,\tau)\, d\tau, \quad t > 0. \tag{1}$$

Definition 2.4: The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^m, m \in I\!N \bigcup \{0\}$, is defined as:

$$\begin{aligned} D_t^\mu f(x,t) &= \frac{\partial^m}{\partial t^m} f(x,t), \quad \mu = m, \\ &= I_t^{m-\mu} \frac{\partial^m f(x,t)}{\partial t^m}, \end{aligned} \tag{2}$$

where $m - 1 < \mu < m, m \in I\!N$. Note that

$$I_t^\mu D_t^\mu f(x,t) = f(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k f}{\partial t^k}(x,0) \frac{t^k}{k!}, m-1 < \mu \leq m, \ m \in I\!N, \tag{3}$$

$$I_t^\mu t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}. \tag{4}$$

III. New Iterative Method for a system of nonlinear functional equations

Consider the system of nonlinear functional equations:

$$y_i = f_i + N_i(y_1, y_2, \cdots, y_n) \quad (i = 1, 2, \cdots, n), \tag{5}$$

where f_i are known functions and N_i are nonlinear operators. Let $\bar{y} = (y_1, \cdots, y_n)$ be a solution of system (5) where y_i having the series form:

$$y_i = \sum_{j=0}^{\infty} y_{i,j}, \quad i = 1, 2, \cdots, n. \tag{6}$$

We decompose the nonlinear operator N_i as

$$\begin{aligned} N_i(\bar{y}) &= N_i\left(\sum_{j=0}^{\infty} y_{1,j}, \cdots, \sum_{j=0}^{\infty} y_{n,j}\right) \\ &= N_i(y_{1,0}, \cdots, y_{n,0}) \\ &+ \sum_{k=1}^{\infty}\left\{N_i\left(\sum_{j=0}^{k} y_{1,j}, \cdots, \sum_{j=0}^{k} y_{n,j}\right)\right. \\ &\left.-N_i\left(\sum_{j=0}^{k-1} y_{1,j}, \cdots, \sum_{j=0}^{k-1} y_{n,j}\right)\right\}. \qquad (7) \end{aligned}$$

By virtue of equations (6) and (7), system (5) is equivalent to

$$\begin{aligned} \sum_{j=0}^{\infty} y_{i,j} &= f_i + N_i(y_{1,0}, \cdots, y_{n,0}) \\ &+ \sum_{k=1}^{\infty}\left\{N_i\left(\sum_{j=0}^{k} y_{1,j}, \cdots, \sum_{j=0}^{k} y_{n,j}\right)\right. \\ &\left.-N_i\left(\sum_{j=0}^{k-1} y_{1,j}, \cdots, \sum_{j=0}^{k-1} y_{n,j}\right)\right\} \qquad (8) \end{aligned}$$

$(i = 1, 2, \cdots, n)$.

For $i = 1, 2, \cdots, n$, we define the recurrence relation:

$$\begin{aligned} & y_{i,0} = f_i,\ y_{i,1} = N_i(y_{1,0}, \cdots, y_{n,0}), \\ & y_{i,m+1} = N_i\left(\sum_{j=0}^{m} y_{1,j}, \cdots, \sum_{j=0}^{m} y_{n,j}\right) \\ & -N_i\left(\sum_{j=0}^{m-1} y_{1,j}, \cdots, \sum_{j=0}^{m-1} y_{n,j}\right), \quad m = 1, 2, \cdots. \end{aligned}$$

Then $y_i = \sum_{j=0}^{\infty} y_{i,j}$. The k-th order approximation to y_i is given by $y_i = \sum_{j=0}^{k-1} y_{i,j}$.

IV. Revised NIM

In this section we suggest a modification to NIM for solving system of nonlinear functional equations. To illustrate the method we consider the system of equations (5).

Initial step:

$$y_{i,0} = f_i, \quad i = 1, 2, \cdots, n.$$

First iteration:

$$\begin{aligned} y_{1,1} &= N_1\left(y_{1,0}, y_{2,0}, \cdots, y_{n,0}\right) \\ y_{2,1} &= N_2\left(y_{1,0} + y_{1,1}, y_{2,0}, \cdots, y_{n,0}\right) \\ y_{3,1} &= N_3\left(y_{1,0} + y_{1,1}, y_{2,0} + y_{2,1}, y_{3,0},\right. \\ &\left.\cdots, y_{n,0}\right) \\ &\vdots \\ y_{n,1} &= N_n\left(y_{1,0} + y_{1,1}, y_{2,0} + y_{2,1},\right. \\ &\left.\cdots, y_{n-1,0} + y_{n-1,1}, y_{n,0}\right). \end{aligned}$$

k th iteration $(k = 2, 3, \cdots.)$

$$\begin{aligned} y_{1,k} &= N_1\left(\sum_{i=0}^{k-1} y_{1,i}, \cdots, \sum_{i=0}^{k-1} y_{n,i}\right) \\ &-N_1\left(\sum_{i=0}^{k-2} y_{1,i}, \cdots, \sum_{i=0}^{k-2} y_{n,i}\right) \\ y_{2,k} &= N_2\left(\sum_{i=0}^{k} y_{1,i}, \sum_{i=0}^{k-1} y_{2,i}, \cdots, \sum_{i=0}^{k-1} y_{n,i}\right) \\ &-N_2\left(\sum_{i=0}^{k-1} y_{1,i}, \sum_{i=0}^{k-2} y_{2,i}, \cdots, \sum_{i=0}^{k-2} y_{n,i}\right) \\ &\vdots \\ y_{j,k} &= N_j\left(\sum_{i=0}^{k} y_{1,i}, \cdots, \sum_{i=0}^{k} y_{j-1,i}, \sum_{i=0}^{k-1} y_{j,i}, \cdots, \sum_{i=0}^{k-1} y_{n,i}\right) \\ &-N_j\left(\sum_{i=0}^{k-1} y_{1,i}, \cdots, \sum_{i=0}^{k-1} y_{j-1,i}, \sum_{i=0}^{k-2} y_{j,i}, \cdots, \sum_{i=0}^{k-2} y_{n,i}\right) \\ &\vdots \\ y_{n,k} &= N_n\left(\sum_{i=0}^{k} y_{1,i}, \cdots, \sum_{i=0}^{k} y_{n-1,i}, \sum_{i=0}^{k-1} y_{n,i}\right) \\ &-N_n\left(\sum_{i=0}^{k-1} y_{1,i}, \cdots, \sum_{i=0}^{k-1} y_{n-1,i}, \sum_{i=0}^{k-2} y_{n,i}\right). \end{aligned}$$

Thus $N_i(\bar{y}) = N_i\left(\sum_{j=0}^{\infty} y_{1,j}, \cdots, \sum_{j=0}^{\infty} y_{n,j}\right) = \sum_{j=1}^{\infty} y_{i,j}$. Hence $y_i = \sum_{j=0}^{\infty} y_{i,j}$.

V. Numerical Examples

Ex.1: Consider the system of linear differential equations [10]:

$$\begin{aligned} y_1' &= y_3 - \cos(t), \quad y_1(0) = 1, \\ y_2' &= y_3 - e^t, \quad y_2(0) = 0, \\ y_3' &= y_1 - y_2, \quad y_3(0) = 2. \qquad (9) \end{aligned}$$

Equivalent system of integral equations is

$$\begin{aligned} y_1 &= (1 - \sin(t)) + \int_0^t y_3 dt = f_1(t) + N_1(y_1, y_2, y_3); \\ y_2 &= (1 - e^t) + \int_0^t y_3 dt = f_2(t) + N_2(y_1, y_2, y_3); \\ y_3 &= 2 + \int_0^t (y_1 - y_2)\, dt = f_3(t) + N_3(y_1, y_2, y_3). \end{aligned}$$

Using revised NIM we get an iterative scheme:

$$\begin{aligned} & y_{1,0} = 1 - \sin(t),\ y_{2,0} = 1 - e^t,\ y_{3,0} = 2; \\ & y_{1,1} = 2t,\ y_{2,1} = 2t,\ y_{3,1} = -2 + e^t + \cos(t); \\ & y_{1,2} = -1 + e^t - 2t + \sin(t), \\ & y_{2,2} = -1 + e^t - 2t + \sin(t),\ y_{3,2} = 0; \\ & y_{1,3} = 0,\ y_{2,3} = 0,\ y_{3,3} = 0. \end{aligned}$$

Thus, the solution of system (9) is $y_1 = e^t$, $y_2 = \sin(t)$, $y_3 = e^t + \cos(t)$.

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Ex.2: Consider the system of nonlinear differential equations:

$$\begin{aligned} y_1^{'} &= 2y_2^2, \quad y_1(0)=1, \\ y_2^{'} &= e^{-t}y_1, \quad y_2(0)=1, \\ y_3^{'} &= y_2+y_3, \quad y_3(0)=0. \end{aligned} \tag{10}$$

Integrating we get

$$\begin{aligned} y_1 &= 1+2\int_0^t y_2^2 dt = f_1(t)+N_1(y_1,y_2,y_3), \\ y_2 &= 1+\int_0^t e^{-t}y_1 dt = f_2(t)+N_2(y_1,y_2,y_3), \\ y_3 &= \int_0^t (y_2-y_3)\,dt = f_3(t)+N_3(y_1,y_2,y_3). \end{aligned}$$

The revised NIM leads to

$$\begin{aligned} &y_{1,0}=1,\ y_{2,0}=1,\ y_{3,0}=0; \\ &y_{1,1}=2t,\ y_{2,1}=3-3e^{-t}-2te^{-t}, \\ &y_{3,1}=-5+4t+5e^{-t}+2te^{-t}; \\ &y_{1,2}=-63+30t+80e^{-t}-4t^2e^{-2t}-16te^{-2t}-17e^{-2t}, \\ &y_{2,2}=\frac{196}{27}+\frac{209}{27}e^{-3t}-48e^{-2t}+33e^{-t}+\frac{56}{9}te^{-3t} \\ &-16te^{-2t}-30te^{-t}+\frac{4}{3}t^2e^{-3t}, \\ &y_{3,2}=\frac{-395}{27}-\frac{91}{27}e^{-3t}+28e^{-2t}-10e^{-t}+\frac{61}{27}t \\ &-\frac{64}{27}te^{-3t}+8te^{-2t}+28te^{-t}+2t^2-\frac{4}{9}t^2e^{-3t}, \end{aligned}$$

and so on. In Fig.1, Fig.2 and Fig.3 we compare the solutions of (10) with the solutions by standard NIM and by revised ADM [10]. Solid line shows exact solution, dotted line shows solution by revised NIM, dashed line shows standard NIM solution and long dashed line shows revised ADM solution.

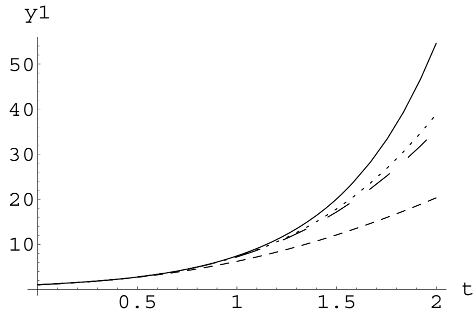

Fig. 1: (Ex.2, y_1)

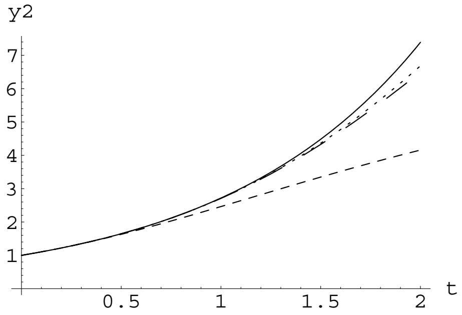

Fig. 2: (Ex.2,y_2)

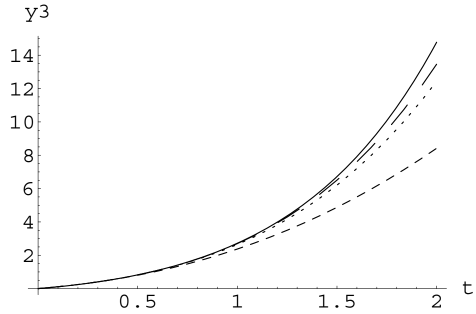

Fig. 3: (Ex.2, y_3)

Solid line = exact solution, dashed line = standard NIM, long dashed line = revised ADM, dotted line = revised NIM

Ex.3: Consider system of nonlinear partial differential equations [11]:

$$\begin{aligned} &\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}+u=1, \quad u(x,0)=e^x, \\ &\frac{\partial v}{\partial t}-u\frac{\partial v}{\partial x}-v=1, \quad v(x,0)=e^{-x}. \end{aligned} \tag{11}$$

The system (11) is equivalent to

$$\begin{aligned} u &= (e^x+t)-\int_0^t \left(u+v\frac{\partial u}{\partial x}\right)dt, \\ v &= (e^{-x}+t)+\int_0^t \left(v+u\frac{\partial v}{\partial x}\right)dt. \end{aligned} \tag{12}$$

The exact solution of (15) is $u(x,t)=e^{x-t}$, $v(x,t)=e^{t-x}$.

Employing revised NIM to (15), we get

$$\begin{aligned} &u_0=e^x+t,\ v_0=e^{-x}+t; \\ &u_1=\frac{-t}{2}(2+t)(1+e^x), \\ &v_1=\frac{t}{6}e^{-x}\left(6+t^2+e^x(-6+6t+t^2)\right); \end{aligned}$$

and so on. In Fig.4, Fig.5, Fig.6 and Fig.7 we draw 3-term solutions and exact solutions of (11), it is clear from figures that the 3-term solutions are in agreement with the exact solutions.

Ex.4: Consider system representing nonlinear chemical reaction [12]

$$\begin{aligned} y_1^{'} &= -y_1, \quad y_1(0)=1, \\ y_2^{'} &= y_1-y_2^2, \quad y_2(0)=0, \\ y_3^{'} &= y_2^2, \quad y_3(0)=0. \end{aligned} \tag{13}$$

The system (13) is equivalent to

$$\begin{aligned} y_1 &= 1-\int_0^t y_1 dt; \\ y_2 &= \int_0^t y_1 dt-\int_0^t y_2^2 dt; \\ y_3 &= \int_0^t y_2^2 dt. \end{aligned}$$

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Applying revised NIM, we get

$$
\begin{aligned}
&y_{1,0} = 1,\ y_{2,0} = 0,\ y_{3,0} = 0;\\
&y_{1,1} = -t,\ y_{2,1} = \frac{-t}{2}(t-2),\ y_{3,1} = \frac{t^3}{60}(20 - 15t + 3t^2);\\
&y_{1,2} = \frac{t^2}{2},\ y_{2,2} = \frac{t^3}{60}(10 - 15t + 3t^2),\\
&y_{3,2} = \frac{t^5}{831600}(-55440 + 92400t - 38280t^2 - 3465t^3\\
&\quad +7315t^4 - 2079t^5 + 189t^6),
\end{aligned}
$$

$\ldots$.

Fig.8, Fig.9 and Fig.10 represents the 3-term solutions of (13). Note that these graphs are in agreement with the graphs given in [12].

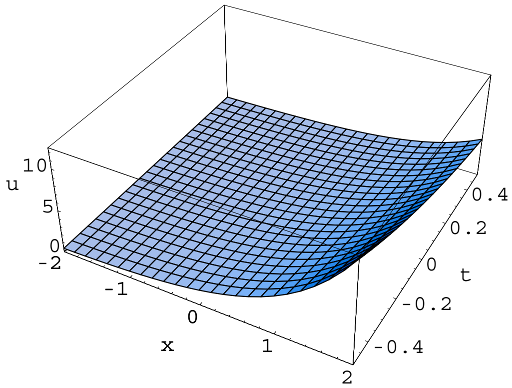

Fig. 4: (Ex.3, 3-term solution u)

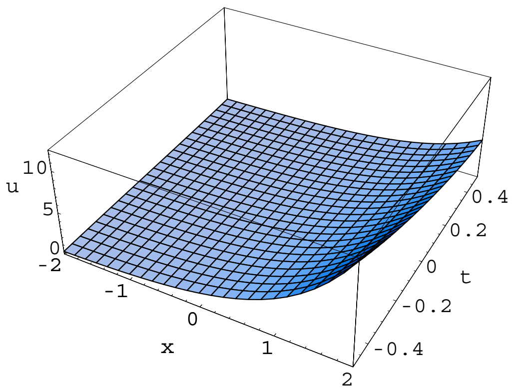

Fig. 5: (Ex.3, Exact solution $u = e^{x-t}$)

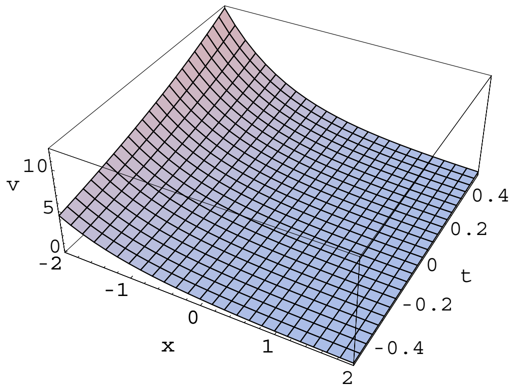

Fig. 6: (Ex.3, 3-term solution v)

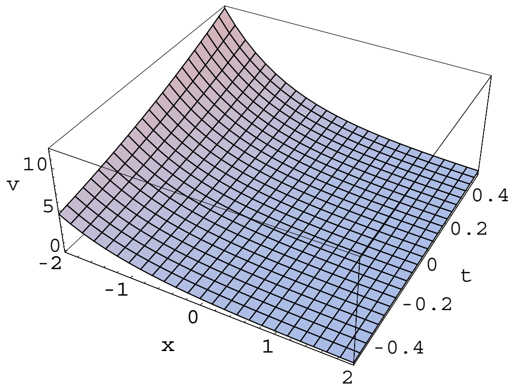

Fig. 7: (Ex.3, Exact solution $v = e^{t-x}$)

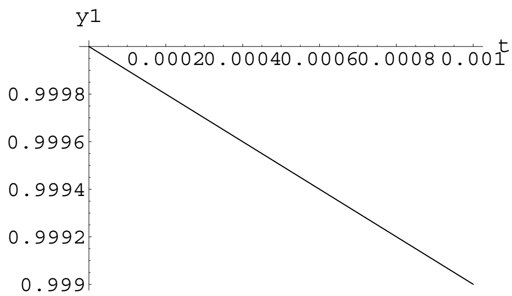

Fig. 8: (Ex.4, 3-term solution y_1)

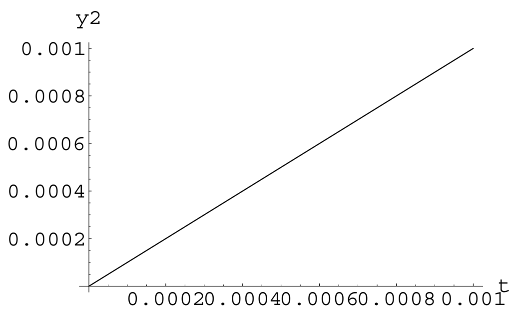

Fig. 9: (Ex.4, 3-term solution y_2)

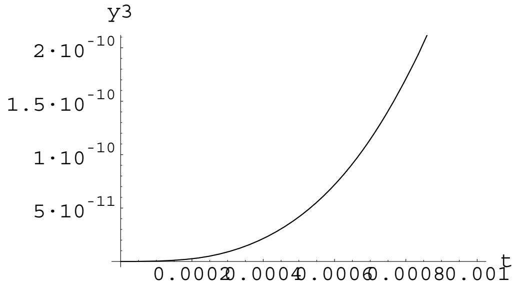

Fig. 10: (Ex.4, 3-term solution y_3)

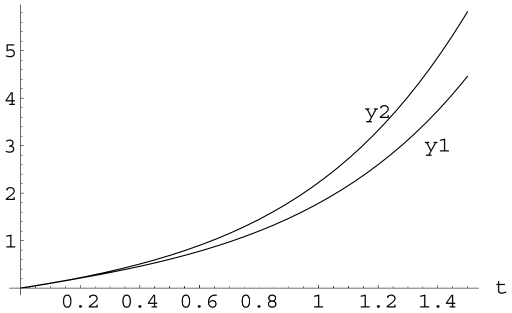

Fig. 11: (Ex.5, 4-term solutions)

Ex.5: Consider the system of nonlinear fractional differential equations

$$\begin{aligned} D_t^{1.3} y_1 &= y_1 + y_2^2, \quad y_1(0) = 0,\, y_1'(0) = 1, \\ D_t^{2.4} y_2 &= y_1 + 5y_2, \quad y_2(0) = 0,\, y_2'(0) = y_2''(0) = 1. \end{aligned} \tag{14}$$

In view of (3), this system is equivalent to the following system of equations

$$\begin{aligned} y_1 &= t + I_t^{1.3}\left(y_1 + y_2^2\right); \\ y_2 &= t + \frac{t^2}{2} + I_t^{2.4}\left(y_1 + 5y_2\right). \end{aligned} \tag{15}$$

Applying revised NIM to (15) we get

$$\begin{aligned} & y_{1,0} = t,\, y_{2,0} = t + \frac{t^2}{2}; \\ & y_{1,1} = 0.372656t^{2.3} + 0.225852t^{3.3} \\ & \quad +0.157571t^{4.3} + 0.0297305t^{5.3}, \\ & y_{2,1} = 0.591944t^{3.4} + 0.1121t^{4.4} + 0.01379t^{4.7} \\ & \quad +0.004838t^{5.7} + 0.002166t^{6.7} + 0.000281t^{7.7}; \\ & y_{1,2} = 0.0747312t^{3.6} + 0.0324918t^{4.6} \\ & \quad + \cdots + 3.44154 \times 10^{-8}t^{15.7} + 2.0608 \times 10^{-9}t^{16.7}, \\ & y_{2,2} = 0.0604101t^{5.8} + 0.00138889t^{6} + \cdots \\ & \quad +3.63177 \times 10^{-11}t^{18.1} + 1.90145 \times 10^{-12}t^{19.1}, \end{aligned}$$

and so on. Fig.11 represents the 4-term approximate solutions of (14).

Ex.6: Consider the system of nonlinear fractional differential equations

$$\begin{aligned} D_t^{\alpha} y_1 &= -y_1 + y_2 y_3, \quad y_1(0) = 1, \\ D_t^{\alpha} y_2 &= -y_2 y_3 - 2y_2^2, \quad y_2(0) = 2, \\ D_t^{\alpha} y_3 &= y_2^2, \quad y_3(0) = 0, \quad 0 < \alpha \le 1. \end{aligned} \tag{16}$$

Applying (3), we get equivalent system of integral equations

$$\begin{aligned} y_1 &= 1 + I_t^{\alpha}\left(-y_1 + y_2 y_3\right); \\ y_2 &= 2 + I_t^{\alpha}\left(-y_2 y_3 - 2y_2^2\right); \\ y_3 &= I_t^{\alpha}\left(y_2^2\right). \end{aligned}$$

In view of revised NIM,

$$\begin{aligned} & y_{1,0} = 1,\, y_{2,0} = 0,\, y_{3,0} = 0; \\ & y_{1,1} = -\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ & y_{2,1} = -\frac{8t^{\alpha}}{\Gamma(\alpha+1)}, \\ & y_{3,1} = \frac{4t^{\alpha}}{\Gamma(\alpha+1)}\left(1 - \frac{2^{3-2\alpha}\sqrt{\pi}t^{\alpha}}{\Gamma(\alpha+0.5)} + \frac{4^{2+\alpha}t^{2\alpha}\Gamma(\alpha+0.5)}{\sqrt{\pi}\Gamma(1+3\alpha)}\right), \end{aligned}$$

and so on.

VI. Conclusions

In this article a modification of NIM, termed as 'revised NIM' has been presented. It has been applied successfully to solve a variety of problems formulated in terms of systems of functional equations. Revised NIM gives series solution which converges faster relative to the series obtained by NIM. The solutions obtained are highly in agreement with the exact solutions.

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