Simplex Method for Solving Linear Programming Problems with Fuzzy Numbers

S. H. Nasseri, E. Ardil, A. Yazdani, and R. Zaefarian

***Abstract*—**The fuzzy set theory has been applied in many fields, such as operations research, control theory, and management sciences, etc. In particular, an application of this theory in decision making problems is linear programming problems with fuzzy numbers. In this study, we present a new method for solving fuzzy number linear programming problems, by use of linear ranking function. In fact, our method is similar to simplex method that was used for solving linear programming problems in crisp environment before.

***Keywords*—**Fuzzy number linear programming, ranking function, simplex method.

I. Introduction

FUZZY linear programming first formulated by Zimmermann [10]. Recently, these problems are considered in several kinds, that is, it is possible that some coefficients of the problem in the objective function, technical coefficients, the right-hand side coefficients or decision making variables be fuzzy number [3], [4], [5], [6], [7], [8], [9]. In this work, we focus on the linear programming problems with fuzzy numbers in the objective function. Verdegay and et al [4], [9] proposed the equivalent parametric linear programming problems for these problems by use of a certain membership function and proposed a dual method for fuzzy number linear programming problems. Here, we first explain the concept of the comparison of fuzzy numbers by introducing a linear ranking function. Moreover, we describe basic feasible solution for the FNLP problems and state optimality conditions for these problems. Finally, we provide some important results for FNLP problems and we propose simplex algorithm for solving these problems.

II. Definitions and Notations

We first review necessary backgrounds of fuzzy sets theory.

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A. Fuzzy Sets

Let X be a classical set of objects, called the universe, whose generic elements are denoted by x . The membership in a crisp subset of X is often viewed as characteristic function $\mu_A(x)$ from X to $\{0,1\}$ such that:

$$\mu_A(x)=1 \quad \text{, if } \quad x \in A$$
$$\quad\quad =0 \quad \text{, otherwise}$$

where $\{0,1\}$ is called a valuation set.

If the valuation set is allowed to be the real interval $[0,1]$, A is called a *fuzzy set* proposed by Zadeh [2]. $\mu_A(x)$ is the degree of membership of x in A . The closer the value of $\mu_A(x)$ is to 1, the more x belong to A . Therefore, A is completely characterized by the set of ordered pairs:

$$A=\{(x,\mu_A(x))\,|\,x\in X\}.$$

The *support* of a fuzzy set A is the crisp subset of X and is presented as:

$$SuppA=\{x\in X \mid \mu_A(x)>0\}.$$

The $\alpha-level$ ($\alpha-cut$) set of a fuzzy set A is a crisp subset of X and is denoted by

$$A_\alpha=\{x\in X \mid \mu_A(x)\geq\alpha\}.$$

A fuzzy set A in X is *convex* if $\mu_A(\lambda x+(1-\lambda)y)\geq\min\{\mu_A(x),\mu_A(y)\}$ $x,y\in X$, and $\lambda\in[0,1]$. Alternatively, a fuzzy set is convex if all $\alpha-level$ sets are convex. Note that in this paper we suppose that $X=R$.

A *fuzzy number* A is a convex normalized fuzzy set on the real line R such that

1) It exists at least one $x_0\in R$ with $\mu_A(x_0)=1$.

2) $\mu_A(x)$ is piecewise continuous.

Among the various types of fuzzy numbers, triangular and trapezoidal fuzzy numbers are of the most important. Note that, in this study we only consider trapezoidal fuzzy numbers.

A fuzzy number is a *trapezoidal fuzzy number* if the membership function of it be in the following form:

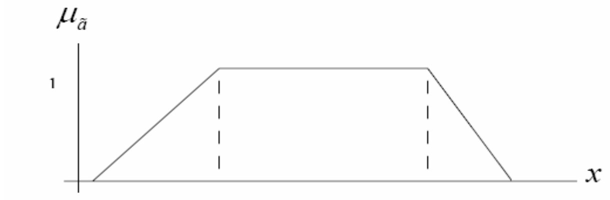

Fig. 1 Trapezoidal Fuzzy Number

We show any trapezoidal fuzzy number by $\tilde{a}=(a^L,a^U,\alpha,\beta)$, where the support of $\tilde{a}$ is $(a^L-\alpha,a^U+\beta)$, and the modal set of $\tilde{a}$ is $[a^L,a^U]$. Let $F(R)$ be the set of trapezoidal fuzzy numbers. In the next subsection we describe arithmetic on $F(R)$.

B. Arithmetic on Fuzzy Numbers

Let $\tilde{a}=(a^L,a^U,\alpha,\beta)$ and $\tilde{b}=(b^L,b^U,\gamma,\theta)$ be two trapezoidal fuzzy numbers and $x\in R$. Then, the results of applying fuzzy arithmetic on the trapezoidal fuzzy numbers as shown in the following:

Image of $\tilde{a}$: $-\tilde{a}=(-a^U,-a^L,\beta,\alpha)$

Addition: $\tilde{a}+\tilde{b}=(a^L+b^L,a^U+b^U,\alpha+\gamma,\beta+\theta)$

Scalar Multiplication:

$$x>0,\ x\tilde{a}=(xa^L,xa^U,x\alpha,x\beta)$$
$$x<0,\ x\tilde{a}=(xa^U,xa^L,-x\beta,-x\alpha)$$

III. Ranking Functions

A convenient method for comparing of the fuzzy numbers is by use of ranking functions. A ranking function is a map from $F(R)$ into the real line. Now, we define orders on $F(R)$ as following:

$$\tilde{a}\underset{\Re}{\geq}\tilde{b} \text{ if and only if } \Re(\tilde{a})\geq\Re(\tilde{b}) \tag{1}$$

$$\tilde{a}\underset{\Re}{>}\tilde{b} \text{ if and only if } \Re(\tilde{a})>\Re(\tilde{b}) \tag{2}$$

$$\tilde{a}\underset{\Re}{=}\tilde{b} \text{ if and only if } \Re(\tilde{a})=\Re(\tilde{b}) \tag{3}$$

where $\tilde{a}$ and $\tilde{b}$ are in $F(R)$. It is obvious that we may write $\tilde{a}\underset{\Re}{\leq}\tilde{b}$ if and only if $\tilde{b}\underset{\Re}{\geq}\tilde{a}$. Since there are many ranking function for comparing fuzzy numbers we only apply linear ranking functions. So, it is obvious that if we suppose that $\Re$ be any linear ranking function, then

i) $\tilde{a}\underset{\Re}{\geq}\tilde{b}$ if and only if $\tilde{a}-\tilde{b}\underset{\Re}{\geq}0$ if and only if $-\tilde{b}\underset{\Re}{\geq}-\tilde{a}$

ii) If $\tilde{a}\underset{\Re}{\geq}\tilde{b}$ and $\tilde{c}\underset{\Re}{\geq}\tilde{d}$, then $\tilde{a}+\tilde{c}\underset{\Re}{\geq}\tilde{b}+\tilde{d}$.

One suggestion for a linear ranking function as following:

$$\Re(\tilde{a})=a^L+a^U+\frac{1}{2}(\beta-\alpha) \tag{4}$$

where $\tilde{a}=(a^L,a^U,\alpha,\beta)\in F(R)$.

IV. Fuzzy Linear Programming

In this section, we introduce fuzzy linear programming (FLP) problems. So, we first define linear programming problems.

A. Linear Programming

A linear programming (LP) problem is defined as:

$$\begin{aligned} \text{Max}\quad & z=cx \\ \text{s.t.}\quad & Ax=b \\ & x\geq 0 \end{aligned} \tag{5}$$

where $c=(c_1,...,c_n), b=(b_1,...,b_m)^T$, and $A=[a_{ij}]_{m\times n}$.

In the above problem, all of the parameters are crisp [1]. Now, if some of the parameters be fuzzy numbers we obtain a fuzzy linear programming which is defined in the next subsection.

B. Fuzzy Linear Programming

Suppose that in the linear programming problem some parameters be fuzzy numbers. Hence, it is possible that some coefficients of the problem in the objective function, technical coefficients, the right-hand side coefficients or decision making variables be fuzzy number [3], [4], [5], [6], [7], [8], [9]. Here, we consider the linear programming problems with fuzzy numbers in the objective function.

V. Fuzzy Number Linear Programming

A fuzzy number linear programming (FNLP) problem is defined as follows:

$$\begin{aligned} \text{Max}\quad & \tilde{z}\underset{\Re}{=}\tilde{c}x \\ \text{s.t.}\quad & Ax=b \\ & x\geq 0 \end{aligned} \tag{6}$$

where $b\in R^m, x\in R^n, A\in R^{m\times n}, \tilde{c}^T\in(F(R))^n$, and $\Re$ is a linear ranking function.

Definition 5.1. We say that vector $x\in R^n$ is a feasible solution to (6) if and only if x satisfies the constraints of the problem.

Definition 5.2. A feasible solution x_* is an optimal solution for (6), if for all feasible solution x for (6), we have $\tilde{c}x_*\underset{\Re}{\geq}\tilde{c}x$.

A. Fuzzy Basic Feasible Solution

Here, we introduce basic feasible solutions for FNLP problems. Consider the system $Ax=b$ and $x\geq 0$, where A is an $m\times n$ matrix and b is an m vector. Now, suppose that $rank(A,b)=rank(A)=m$. Partition after possibly rearranging the columns of A as $[\ B\ ,\ N\ ]$ where B,

$m \times m$, is nonsingular. It is obvious that $rank(B) = m$. The point $x = (x_B{}^T, x_N{}^T)^T$ where $x_B = B^{-1}b$, $x_N = 0$ is called a *basic solution* of the system. If $x_B \geq 0$, then x is called a *basic feasible solution* (BFS) of the system. Here B *is* called the *basic matrix* and N is called the non*basic matrix*. The components of x_B are called a *basic variables,* and the components of x_N are called *nonbasic variables.* If $x_B > 0$, then is x called a *nondegenerate basic feasible solution*, and if at least one component of x_B is zero, then x is called a *degenerate basic feasible solution.*

The following theorem characterizes optimal solutions. The result corresponds to the so-called nondegenerate problems, where all fuzzy basic variables corresponding to every basis B are nonzero (and hence positive) [5].

Theorem 5.1. Assume the FNLP problem is nondegenerate. A basic feasible solution $x_B = B^{-1}b, x_N = 0$ is optimal to (6) if and only if $\tilde{z}_j \underset{\Re}{\geq} \tilde{c}_j$ for all $1 \leq j \leq n$.

Proof. Suppose that $x_* = (x_B^T \ x_N^T)^T$ is a basic feasible solution to (6), where $x_B = B^{-1}b, x_N = 0$. Then $\tilde{z}_* \underset{\Re}{=} \tilde{c}_B x_B \underset{\Re}{=} \tilde{c}_B B^{-1}b.$ On the other hand, for all feasible solution x, we have $b = Ax = Bx_B + Nx_N$. Hence, we may obtain

$$\tilde{z} \underset{\Re}{=} \tilde{c}x \underset{\Re}{=} \tilde{c}_B x_B + \tilde{c}_N x_N \underset{\Re}{=} \tilde{c}_B B^{-1}b - \sum_{j \neq B_i} (\tilde{c}_B B^{-1} a_j - \tilde{c}_j) x_j$$

Then,
$$\tilde{z} \underset{\Re}{=} \tilde{z}_* - \sum_{j \neq B_i} (\tilde{z}_j - \tilde{c}_j) x_j \tag{7}$$

Therefore, the results follow immediately from (7) and the assumptions of theorem.

In the next section, we propose simplex method for solving FNLP problems.

VI. Simplex Method for the FNLP Problems

Consider the FNLP problem as is defined in (6).

$$\max \ \tilde{z} \underset{\Re}{=} \tilde{c}_B x_B + \tilde{c}_N x_N$$
$$\text{s.t.} \quad Bx_B + Nx_N = b$$
$$x_B, x_N \geq 0$$

Hence, we may write $x_B + B^{-1}Nx_N = B^{-1}b$. Therefore, $\tilde{z} + (\tilde{c}_B B^{-1}N - \tilde{c}_N)x_N \underset{\Re}{=} \tilde{c}_B B^{-1}b$. Currently $x_N = 0$, and then $x_B = B^{-1}b$, and $\tilde{z} \underset{\Re}{=} \tilde{c}_B B^{-1}b$. Then we rewrite the above FNLP problem in the following tableau format**:**

	$\tilde{z}$	x_B	x_N	R.H.S.
$\tilde{z}$	1	$\tilde{0}$	$\tilde{c}_B B^{-1}N - \tilde{c}_N$	$\tilde{c}_B B^{-1}b$
x_B	0	I	$B^{-1}N$	$B^{-1}b$

The above tableau gives us all the information we need to proceed with the simplex method. The fuzzy cost row in the above tableau is $\tilde{\gamma} \underset{\Re}{=} (\tilde{c}_B B^{-1}a_j - \tilde{c}_j)_{j \neq B_i}$, which consists of the $\tilde{\gamma}_j \underset{\Re}{=} \tilde{z}_j - \tilde{c}_j$'s for the nonbasic variables. According to the optimality condition for these problems we are at the optimal solution if $\tilde{\gamma}_j \underset{\Re}{\geq} 0$ for all $j \neq B_i$. On the other hand, if $\tilde{\gamma}_k \underset{\Re}{<} 0$, for a $k \neq B_i$ then we may exchange x_{B_r} with x_k. Then we compute the vector $y_k = B^{-1}a_k$. If $y_k \leq 0$, then x_k can be increase indefinitely, and then the optimal objective is unbounded. On the other hand, if y_k has at least one positive component, then the increase in will be blocked by one of the current basic variables, which drops to zero.

Theorem 6.1. If in a simplex tableau, an l exists such that $\tilde{z}_l - \tilde{c}_l \underset{\Re}{<} 0$ and there exists a basic index i such that $y_{il} > 0$, then a pivoting row r can be found so that pivoting on y_{rl} will yield a feasible tableau with a corresponding nondecreasing fuzzy objective value.

Proof. We need a criterion for choosing a basic variable to leave the basis so that the new simplex tableau will remain feasible and the new objective value is nondecreasing. Assume column l is the pivot column. Also, suppose that is $x = (x_B{}^T, x_N{}^T)^T$ a basic feasible solution to the FNLP problem, where $x_B = B^{-1}b$, and $x_N = 0$. Then, the corresponding fuzzy objective value is $\tilde{z} \underset{\Re}{=} \tilde{c}_B B^{-1}b \underset{\Re}{=} \tilde{c}_B y_0$.

On the other hand, for any basic feasible solution to the FNLP problem, we have

$$x_B + \sum_{j \neq B_i} y_j x_j = y_0 \tag{8}$$

where $y_j = B^{-1}a_j$.

So, if x_l enters into the basis we may write

$$x_B = y_0 - y_l x_l \tag{9}$$

Since, we want x_B be feasible, hence $x_{B_i} \geq 0$, or

$$y_{i0} - y_{il} x_l \geq 0, \text{ for all } i = 1, \ldots, m.$$

If $y_{il} \leq 0$, then it is obvious that the above condition is hold. Hence, for all $y_{il} > 0$, we need to have

$$x_l \leq \frac{y_{i0}}{y_{il}} \tag{10}$$

To satisfy (10) it is sufficient to let

$$\frac{y_{r0}}{y_{rl}} = \min\{\frac{y_{i0}}{y_{il}} \mid y_{il} > 0\} \tag{11}$$

Also, for any basic feasible solution to the FNLP problem, we have

$$\tilde{z} \underset{\Re}{=} \tilde{c}_B y_0 - \sum_{j \neq B_i} (\tilde{z}_j - \tilde{c}_j) x_j \tag{12}$$

So, if we enter x_l into the basis we have

$$\tilde{z} \underset{\Re}{=} \tilde{c}_B y_0 - (\tilde{z}_l - \tilde{c}_l) x_l \,. \tag{13}$$

We note that the new objective value is nondecreasing, since

$$\tilde{z} \underset{\Re}{=} \tilde{c}_B y_0 - (\tilde{z}_l - \tilde{c}_l) x_l \underset{\Re}{\geq} \tilde{c}_B y_0 \,, \tag{14}$$

Using the fact that $(\tilde{z}_l - \tilde{c}_l) x_l \underset{\Re}{\leq} 0$.

Theorem 6.2. If for any basic feasible solution to the FNLP problem there is some column not in basis for which $\tilde{z}_l - \tilde{c}_l \underset{\Re}{<} 0$ and $y_{il} \leq 0, i = 1,...,m$, then the FNLP problem has an unbounded solution.

Proof. Suppose that x_B is a basic solution to the FNLP problem, so

$$x_{B_i} + \sum_{j \neq B_i} y_{ij} x_j = y_{i0},\ i = 1,...,m,\ j = 1,...,n, \tag{15}$$

or

$$x_{B_i} = y_{i0} - \sum_{j \neq B_i} y_{ij} x_j,\ i = 1,...,m,\ j = 1,...,n. \tag{16}$$

Now, if we enter x_l into the basis, then we have $x_l > 0$, and $x_j = 0$, for all $j \neq B_i \cup l$. Since $y_{il} \leq 0, i = 1,...,m$, hence

$$y_{i0} - y_{il} x_l \geq 0 \tag{17}$$

Therefore, the current basic solution will remain feasible. Now, the value of $\hat{z}$ for the above feasible solution as following:

$$\hat{z} \underset{\Re}{=} \tilde{c}_B x_B + \tilde{c}_N x_N \underset{\Re}{=} \sum_{i=1}^{m} \tilde{c}_{B_i} (y_{i0} - y_{il} x_l) + \tilde{c}_l x_l$$

$$\underset{\Re}{=} \sum_{i=1}^{m} \tilde{c}_{B_i} y_{i0} - (\sum_{i=1}^{m} \tilde{c}_{B_i} y_{il} - \tilde{c}_l) x_l$$

$$\underset{\Re}{=} \tilde{c}_B y_0 - (\tilde{c}_B y_l - \tilde{c}_l) x_l \underset{\Re}{=} \tilde{z} - (\tilde{z}_l - \tilde{c}_l) x_l$$

So,

$$\hat{z} \underset{\Re}{=} \tilde{z} - (\tilde{z}_l - \tilde{c}_l) x_l \,. \tag{18}$$

Hence, we can enter x_l into the basis with arbitrarily large value. Then, from (18) we have unbounded solution.

VII. A Numerical Example

For an illustration of the above method we solve a FNLP problem by use of simplex method.

Example 8.1.

$$\text{Max } \tilde{z} \underset{\Re}{=} (5,8,2,5) x_1 + (6,10,2,6) x_2$$

$$\text{s.t. } 2x_1 + 3x_2 \leq 6$$

$$5x_1 + 4x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

We may rewrite

$$2x_1 + 3x_2 + x_3 = 6$$

$$5x_1 + 4x_2 + x_4 = 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

We may write the first feasible simplex tableau as follows:

basis	x_1	x_2	x_3	x_4	*R.H.S.*
$\tilde{z}$	$(-8,-5,5,2)$	$(-10,-6,6,2)$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
x_3	2	3	1	0	6
x_4	5	4	0	1	10

Since $(\tilde{z}_1 - \tilde{c}_1, \tilde{z}_2 - \tilde{c}_2) \underset{\Re}{=} ((-8,-5,5,2),(-10,-6,6,2))$, and $(\gamma_1, \gamma_2) = (\Re(\tilde{\gamma}_1), \Re(\tilde{\gamma}_2)) = (-14.5, -18)$, then x_2 enters the basis and the leaving variable is x_3. The new tableau is:

basis	x_1	x_2	x_3	x_4	*R.H.S.*
$\tilde{z}$	$(-4,\frac{5}{3},\frac{19}{3},6)$	$\tilde{0}$	$(2,\frac{10}{3},\frac{2}{3},2)$	$\tilde{0}$	$(12,20,4,12)$
x_2	$\frac{2}{3}$	1	$\frac{1}{3}$	0	2
x_4	$\frac{7}{3}$	0	$\frac{-4}{3}$	1	2

Now, from $(\tilde{\gamma}_1, \tilde{\gamma}_3) \underset{\Re}{=} ((-4,\frac{5}{3},\frac{19}{3},6),(2,\frac{10}{3},\frac{2}{3},2))$ and $(\Re(\tilde{\gamma}_1), \Re(\tilde{\gamma}_3)) = (\gamma_1, \gamma_3) = (\frac{-5}{2}, 6)$, it follows that x_1 is an entering variable and x_4 is a leaving variable.

The last tableau is shown in the below.

basis	x_1	x_2	x_3	x_4	$R.H.S.$
$\tilde{z}$	$\tilde{0}$	$\tilde{0}$	$(\frac{-2}{7},\frac{30}{7},\frac{30}{7},\frac{38}{7})$	$(\frac{-5}{7},\frac{12}{7},\frac{18}{7},\frac{19}{7})$	$(\frac{90}{7},\frac{148}{7},\frac{32}{7},\frac{90}{7})$
x_2	0	1	$\frac{5}{7}$	$\frac{-2}{7}$	$\frac{10}{7}$
x_1	1	0	$\frac{-4}{7}$	$\frac{3}{7}$	$\frac{6}{7}$

$$\tilde{w} \underset{\Re}{=} \tilde{c}_B B^{-1} \underset{\Re}{=} (\tilde{c}_2,\tilde{c}_1)B^{-1} \underset{\Re}{=} ((\tfrac{-2}{7},\tfrac{30}{7},\tfrac{30}{7},\tfrac{38}{7}),(\tfrac{-5}{7},\tfrac{12}{7},\tfrac{18}{7},\tfrac{19}{7}))$$

$$\tilde{w}b \underset{\Re}{=} \tilde{c}_B B^{-1}b \underset{\Re}{=} (\tfrac{90}{7},\tfrac{148}{7},\tfrac{32}{7},\tfrac{90}{7}),\ \Re(\tilde{c}_B B^{-1}b) = \frac{267}{7}$$

$$(\tilde{\gamma}_3,\tilde{\gamma}_4) \underset{\Re}{=} (\tilde{w}N - \tilde{c}_N) \underset{\Re}{=} ((\tfrac{-2}{7},\tfrac{30}{7},\tfrac{30}{7},\tfrac{38}{7}),(\tfrac{-5}{7},\tfrac{12}{7},\tfrac{18}{7},\tfrac{19}{7}))$$

$$(\gamma_3,\gamma_4) = (\Re(\tilde{\gamma}_3),\Re(\tilde{\gamma}_4)) = (\tfrac{32}{7},\tfrac{15}{14}) > 0,\ \tilde{\gamma}_2 \underset{\Re}{=} \tilde{\gamma}_1 \underset{\Re}{=} \tilde{0}.$$

Now, using optimality condition there is not any variable that enters the basis. Therefore, this basis is optimal.

VIII. Conclusion

We considered fuzzy number linear programming problems and introduced the basic feasible solution for these problems. Finally, we obtained some important results and we proposed a new algorithm for solving these problems directly, by use of linear ranking function.

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