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Significance of Splitting Method in Non-linear Grid system for the Solution of Navier-Stokes Equation

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Abstract**—**Solution to unsteady Navier-Stokes equation by Splitting method in physical orthogonal algebraic curvilinear coordinate system, also termed 'Non-linear grid system' is presented. The linear terms in Navier-Stokes equation are solved by Crank-Nicholson method while the non-linear term is solved by the second order Adams-Bashforth method. This work is meant to bring together the advantage of Splitting method as pressure-velocity solver of higher efficiency with the advantage of consuming Non-linear grid system which produce more accurate results in relatively equal number of grid points as compared to Cartesian grid. The validation of Splitting method as a solution of Navier-Stokes equation in Non-linear grid system is done by comparison with the benchmark results for lid driven cavity flow by Ghia and some case studies including Backward Facing Step Flow Problem.

Keywords**—**Navier-Stokes, 'Non-linear grid system', Splitting method.

I. Introduction

THE application of Navier-Stokes equation in solving fluid flow has evolved in the past few decades with numerical method as one of the most inspiring technique that been explored. Amongst the efforts, lid driven cavity flow data by Ghia [1] has been widely used as the benchmark results. In traditional two dimensional solution of viscous incompressible flow, one of the most popular velocity-pressure coupling methods is SIMPLE (Semi-Implicit Method for Pressure-Linked Equation).

SIMPLE technique is found to be low in efficiency since it involves major convergence iteration in determining the pressure values for every main velocity-time iteration. As an alternative, Karniadakis [2] had introduced a new formulation for high-order time-accurate splitting scheme for the solution of the incompressible Navier-Stokes equations.

Principally, flow problems where large gradients are concentrated in a specific region require refinement of resolutions on those regions. Instead of using uniform, high resolution grid distribution in the physical domain, grid points may be clustered in the regions of high flow gradients and broaden at other regions. Non-linear grid system could demonstrate these advantages with direct usage of mathematical models of Navier-Stokes solution derived in Cartesian coordinate with minimum verifications of the discretization methods.

This work is meant to bring together the advantage of Splitting method as pressure-velocity solver of higher efficiency with the advantage of consuming Non-linear grid system which produce more accurate results in relatively equal number of grid points as compared to Cartesian grid.

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II. Mathematical preliminaries for Splitting method

The temporal integration of the Navier-Stokes system is achieved using a semi-implicit splitting method, similar to the method of Karniadakis et. al [2] and others. Consider the Navier-Stokes expression below

$$\frac{\partial \vec{v}}{\partial t} + \vec{N}(\vec{v}) = -\nabla p + \frac{1}{R_e}\vec{L}(\vec{v}), \tag{1}$$

where $\vec{L}$ is the linear viscous term and $\vec{N}$ is the non-linear advective term,

$$\begin{aligned}\vec{L}(\vec{v}) &= \nabla^2 \vec{v},\\ \vec{N}(\vec{v}) &= \vec{v}\cdot\nabla\vec{v}.\end{aligned} \tag{2}$$

Integrate the above equation over one time step, Δt,

$$\int_{t_k}^{t_{k+1}} \frac{\partial \vec{v}}{\partial t}dt + \int_{t_k}^{t_{k+1}} \vec{N}(\vec{v})dt = -\int_{t_k}^{t_{k+1}} \nabla p dt + \int_{t_k}^{t_{k+1}} \frac{1}{R_e}\vec{L}(\vec{v})dt, \tag{3}$$

where k is the time step.

The first term is easily evaluated without approximation,

$$\int_{t_k}^{t_{k+1}} \frac{\partial \vec{v}}{\partial t}dt = \vec{v}^{\,k+1} - \vec{v}^{\,k}. \tag{4}$$

A semi-implicit method treats linear terms implicitly for stability, and non-linear term is achieved with the second-order Adams-Bashforth method,

$$\int_{t_k}^{t_{k+1}} \vec{N}(\vec{v})dt = \left[\frac{3}{2}\vec{N}(\vec{v}^k) - \frac{1}{2}\vec{N}(\vec{v}^{k-1})\right]\Delta t \quad (5)$$

The explicit treatment of the nonlinear term avoids sampling $\vec{N}$ at the leading time step, which would result in nonlinear algebraic equations, requiring further iteration. The pressure term is treated by reversing the order of integration and differentiation, then introducing time-averaged pressure,

$$\int_{t_k}^{t_{k+1}} \nabla p dt = \nabla\left[\int_{t_k}^{t_{k+1}} p dt\right] = \nabla \overline{p}^{k+1}\Delta t. \quad (6)$$

Implicit treatment of the linear viscous term is achieved with the second-order Crank-Nicholson method,

$$\int_{t_k}^{t_{k+1}} \vec{L}(\vec{v})dt = \frac{1}{2}\left[\nabla^2 \vec{v}^{k+1} + \nabla^2 \vec{v}^k\right]\Delta t. \quad (7)$$

The combined difference equation is now,

$$\vec{v}^{k+1} - \vec{v}^k + \left[\frac{3}{2}\vec{N}(\vec{v}^k) - \frac{1}{2}\vec{N}(\vec{v}^{k-1})\right]\Delta t = -\nabla \overline{p}^{k+1}\Delta t + \frac{1}{2R_e}\left[\nabla^2 \vec{v}^{k+1} + \nabla^2 \vec{v}^k\right]\Delta t \quad (7)$$

The continuity equation is imposed at the leading time step,

$$\nabla \cdot \vec{v}^{k+1} = 0. \quad (8)$$

In splitting method, (7) is integrated numerically in three for each time step, each stage addressing the three terms independently. Two intermediate velocity fields, $\hat{\vec{v}}$ and $\hat{\hat{\vec{v}}}$, are introduced in order to achieve this. The three stages are,

$$\begin{aligned} \hat{\vec{v}} - \vec{v}^k &= \left[\frac{3}{2}\vec{N}(\vec{v}^k) - \frac{1}{2}\vec{N}(\vec{v}^{k-1})\right]\Delta t, \\ \hat{\hat{\vec{v}}} - \hat{\vec{v}} &= -\nabla \overline{p}^{k+1}\Delta t, \\ \vec{v}^{k+1} - \hat{\hat{\vec{v}}} &= \frac{1}{2R_e}\left[\nabla^2 \vec{v}^{k+1} + \nabla^2 \vec{v}^k\right]\Delta t. \end{aligned} \quad (9)$$

In order to process the second step, the average pressure, $\overline{p}$, must be determined. The pressure is not needed for the first step, and therefore can be determined after $\hat{\vec{v}}$ .take divergence of (7) and use the continuity equation to obtain the Poisson's equation for pressure,

$$\nabla^2 \overline{p}^{k+1} = \nabla \cdot \left(\frac{\hat{\vec{v}}}{\Delta t}\right), \quad (10)$$

where the nonlinear term is neglected.

All variables require boundary conditions, including $\vec{v}^{k+1}, \hat{\vec{v}}, \hat{\hat{\vec{v}}}$ and $\overline{p}$. The boundary conditions on $\vec{v}^{k+1}$ are the natural boundary conditions, which must be enforced at the final stage if the splitting method. Boundary conditions on $\hat{\vec{v}}$ and $\hat{\hat{\vec{v}}}$ can be chosen to enhance the numerical aspect of the method. Hence,

$$\hat{\vec{v}} \cdot \vec{k} = \hat{\hat{\vec{v}}} \cdot \vec{k} = 0 \quad (11)$$

on all boundaries.

Finally, there are no natural boundary conditions on the pressure since the value of pressure at the boundary depends on the velocity field in the neighborhood of the boundary. Pressure boundary conditions must be approximated from the governing equations. Take the normal component of (7) to get,

$$\vec{k} \cdot \nabla \overline{p}^{k+1} = \vec{k} \cdot \vec{v}^k - \vec{k} \cdot \vec{v}^{k+1} - \vec{k} \cdot \left[\frac{3}{2}\vec{N}(\vec{v}^k) - \frac{1}{2}\vec{N}(\vec{v}^{k-1})\right]\Delta t + \frac{1}{2R_e}\vec{k} \cdot \left[\nabla^2 \vec{v}^{k+1} + \nabla^2 \vec{v}^k\right]\Delta t \quad (12)$$

Karniadakis [2] has shown that all the right hand side terms of above equation can be neglected for large Reynolds number, leaving,

$$\vec{n} \cdot \nabla \overline{p}^{k+1} = 0. \quad (13)$$

For that reason, Karniadakis [2] recommends higher order boundary conditions for a better approximation, especially for low Reynolds number flow.

III. Non-linear grid system Formulation

Regular Cartesian coordinate can be transformed to 'Non-linear grid system' according to the specific requirement by the use of algebraic transformation technique. In generating grid coordinate for flow in a duct, Anderson, Tannehill and Pletcher [3] derived a set of algebraic expressions to transform points in computational Cartesian coordinate to physical Non-linear grid system and vise versa.

For the case of square cavity flow, algebraic expressions are used to cluster grid points near solid boundaries and critical locations such as the corners of a cavity to provide adequate resolutions for the viscous boundary layer and secondary vortices. Since the transformation for flow in a duct was found to be in a single horizontal direction, modification is done for the cavity flow grid by first transforming the horizontal, x direction and then followed by transforming the vertical, y direction.

For a cavity of width L and height H, where β is the clustering parameter, and α defines where the clustering takes place. When $\alpha = 0$ the clustering is at $x=L$ and $y=H$; whereas when $\alpha = 1/2$ clustering is distributed equally at the four sides of the cavity. Fig. 1 (a) shows Non-linear grid system for square cavity (at $\alpha=1/2, \beta=1.25$) and for further analysis, these relationship can also be modified for specific cases like turbulent channel flow (at $\alpha=0, \beta=1.001$) as shown in Fig. 1 (b). Fig. 1 (c) shows the computational domain, (η_1, η_2).

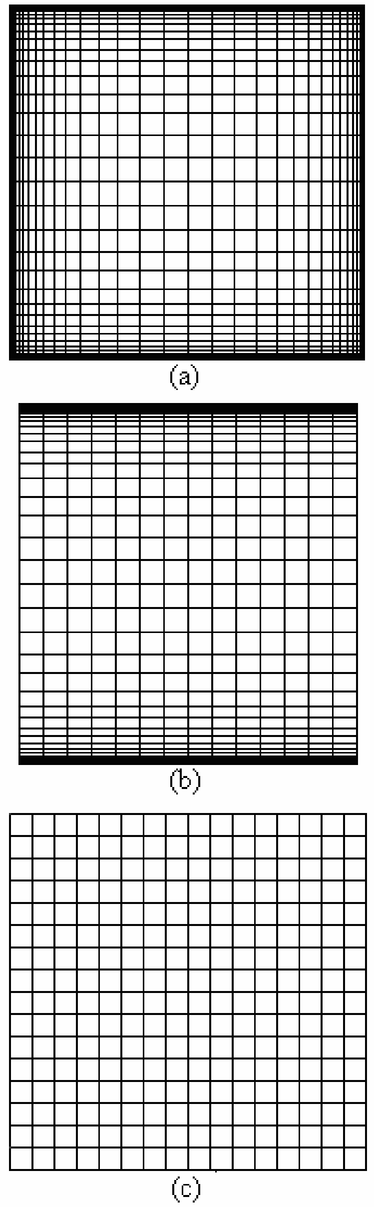

Fig. 1: Non-linear grid system for (a) square cavity flow, (b) channel flow and (c) computational domain.

The algebraic formulation for transformation between physical and computational domain is shown below:

$$\eta_1 = \alpha + (1-\alpha) \frac{ln\{\{\beta + [(2\alpha+1)x/L] - 2\alpha\}/\{\beta - [(2\alpha+1)x/L] + 2\alpha\}\}}{ln[(\beta+1)/(\beta-1)]} \quad (14)$$

$$\eta_2 = \alpha + (1-\alpha) \frac{ln\{\{\beta + [(2\alpha+1)y/H] - 2\alpha\}/\{\beta - [(2\alpha+1)y/H] + 2\alpha\}\}}{ln[(\beta+1)/(\beta-1)]} \quad (15)$$

or inversely,

$$x = L\frac{(2\alpha+\beta)[(\beta+1)/(\beta-1)]^{(\eta_1-\alpha)/(1-\alpha)} + 2\alpha - \beta}{(2\alpha+1)\{1+[(\beta+1)/(\beta-1)]^{(\eta_1-\alpha)/(1-\alpha)}\}} \quad (16)$$

$$y = H\frac{(2\alpha+\beta)[(\beta+1)/(\beta-1)]^{(\eta_1-\alpha)/(1-\alpha)} + 2\alpha - \beta}{(2\alpha+1)\{1+[(\beta+1)/(\beta-1)]^{(\eta_1-\alpha)/(1-\alpha)}\}} \quad (17)$$

IV. Numerical Results and Discussions

The introduction of Splitting method in Non-linear grid system produces more efficient scheme as compared to the scheme using Splitting method in Cartesian coordinate. Since the code of Splitting method in Non-linear grid system involves more tedious equations than the one in Cartesian coordinate, it is also observed that the efficiency of a single iteration for the code in Non-linear grid system is less than in Cartesian coordinate.

Fig. 2 show the efficiency comparison to reach steady state for Splitting methods in both coordinate system. The resolutions of both coordinate systems are set at 33 X 33 and the time increment of 0.0005. The Reynolds number is varied from 100 to 1000 and the efficiencies observed. The data shows that for low Reynolds number of 100, Cartesian coordinate gives a better efficiency than Non-linear grid system. As Reynolds number increases, the efficiency of Cartesian coordinate is relatively decreases as compared to Non-linear grid system.

The Non-linear grid system gives more efficient result for Reynolds number as low as 400 with resolution of 33 X 33. It clearly shows that even though the Splitting method in Non-linear grid system deals with more tedious equations, it converges to steady state faster as compared to Splitting method in Cartesian coordinate.

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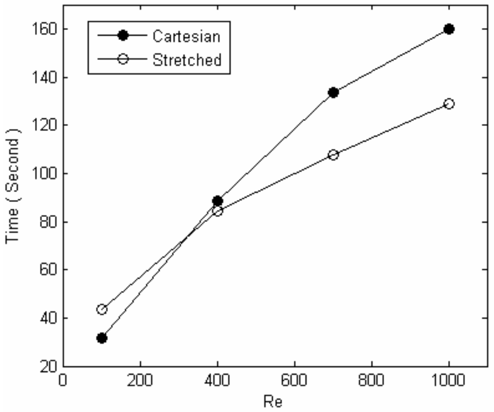
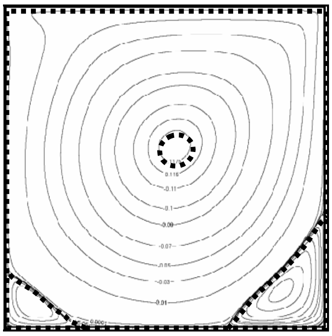
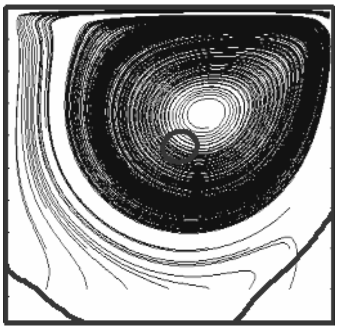


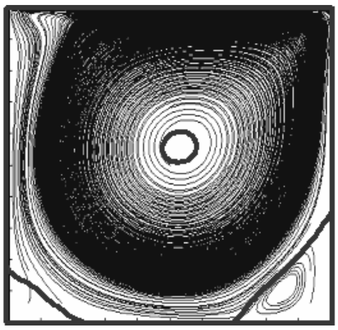
Fig. 2: Efficiency comparison to reach steady state (Resolution 33 X33).



(a) Ghia 129X129 resolution



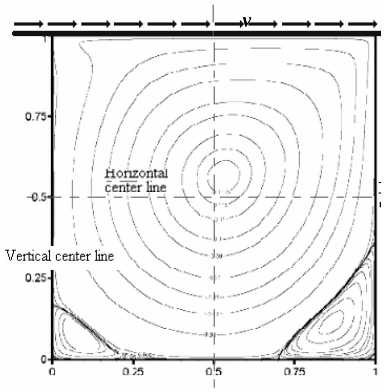
(b) Cartesian 33X33 resolution



(c) Non-linear grid 33X33 resolution

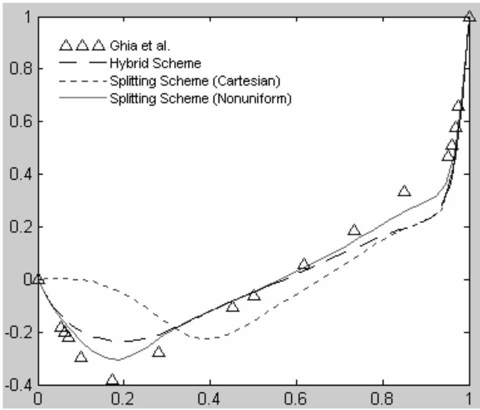
Fig. 3: Streamline comparison of Ghia, Splitting in Cartesian and Splitting in Non-linear grid system.

For a better visual comparison, Fig. 3 shows the streamline comparison between the benchmark result of steady Ghia solution in 129X129 resolution with Splitting unsteady solution in both Cartesian and Non-linear grid system in only 33X33 resolutions. Reynolds number of 1000 and time increment of 0.001 was chosen for the comparison.

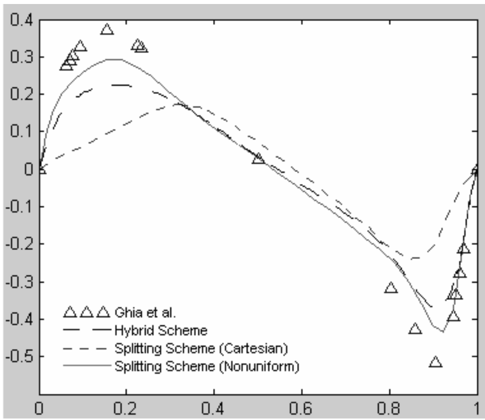


x

(a) Center lines of cavity.



(b) Horizontal velocity



(c) Vertical velocity

Fig. 4: Accuracy comparison for Re 1000 between benchmark Ghia steady solution with steady Hybrid, unsteady Splitting in 33X33 cartesian and in Non-linear grid system.

Fig. 4 shows that the results gained by Splitting method in Non-linear grid system for resolution as low as 33X33 is more comparable to the benchmark Ghia result as compared to the Hybrid steady solution and Splitting unsteady solution in Cartesian coordinate. This proves that the introduction of Splitting method in Non-linear grid system promotes a higher level of efficient and accuracy for the solution of Navier-Stokes equation.

V. Conclusions

Solution to Navier-Stokes equation by Splitting method in physical orthogonal algebraic curvilinear coordinate system, also termed *'Non-linear grid system'* was successfully presented. The combination of highly efficient velocity-pressure coupling, Splitting method with the advantage of accuracy enhancing factor in utilizing Non-linear grid system has proven to produce more accurate and efficient Navier-Stokes equation solver compared to solutions by Splitting method developed in Cartesian coordinate. The developed Non-linear grid system has the advantages of low resolution at less critical location - middle of the cavity for more efficient scheme, high resolution on the cavity boundaries for higher accuracy and divergence rate, and higher resolution at the cavity corners for faster establishment of secondary vortices.

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