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Seven step Adams Type Block Method With Continuous Coefficient For Periodic Ordinary Differential Equation

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Abstract—We consider the development of an eight order Adam's type method, with A-stability property discussed by expressing them as a one-step method in higher dimension. This makes it suitable for solving variety of initial-value problems. The main method and additional methods are obtained from the same continuous scheme derived via interpolation and collocation procedures. The methods are then applied in block form as simultaneous numerical integrators over non-overlapping intervals. Numerical results obtained using the proposed block form reveals that it is highly competitive with existing methods in the literature.

Keywords—Block Adam's type Method; Periodic Ordinary Differential Equation; Stability.

I. Introduction

We propose seven-step eighth order LMM for first order IVPs of the form

$$y' = f(t,y),\ y(t_0) = y_0\ ,\ x \ \epsilon\ [t_0, T_n] \tag{1}$$

where f satisfies the Lipschitz condition as given in Henrici [1]). The main method is conventionally written as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2}$$

Which has $2k+1$ unknown parameters α's and β's and therefore can be of order $2k$, where k is the step number, however, according to Dahlquist[2], the order of (2) cannot exceed $k+1$ (k is odd) or $k+2$ (k is even) for the method to be stable.

Several types of block methods for the solution of (1) have been proposed in literature for the numerical solution of systems of ordinary differential equations (ODEs). A list of related references may be found in [[3],[4],[5] ,[6],[7],[8],[9]] and references therein. Our algorithm is based on collocation and interpolation methods designed with periodic stiff systems of ODEs in mind. Most of the block methods in literature use a predictor- corrector approach requiring starting value through the use of special methods for their implementation for instance (see[[5] ,[8],[9]]), while our methods as in [[**?**],[10],[11],] preserve the Runge Kuta traditional advantage of being self starting . we propose a class of continuous Adam's type methods that are assembled into block matrix equations for solving first order ODEs. The continuous representation of the algorithm generates main discrete methods to provide the approximate solution y_{n+j} at the points $t = t_{n+j}, j = 1, \ldots, 7$.

The paper is presented as follows: In section 2, we discuss the basic idea behind the algorithm and obtain a continuous representation $Y(t)$ for the exact solution $y(t)$ which is used to generate members of the block method for solving (1). In section 3, we present the order of accuracy of the method. section 4 we present the analysis of our block algorithm. In section 5, we show the accuracy of our methods. Finally, in section 6 we present some concluding remarks.

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II. Formulation of the method

Our objective is to derive a continuous method and use it to generate the standard method (2) and additional methods which are combined to form our continuous block Adams type methods (CABM) of order 8. We proceed by seeking an approximation to the exact solution by assuming a continuous solution of the form

$$Y(t) = \sum_{j=0}^{8} b_i \phi_j(t) \tag{3}$$

where b_j are unknown coefficients to be determined and $\phi_j t$ are polynomial basis function of degree 8. We thus construct the seven step continuous Adam's method with $\phi_j t = t^j, j = 0, \ldots, 8$ by imposing that the interpolating function (3) coincides with the analytical solution at the point $t_{n+i}, i = 6$ to obtain the equation

$$\sum_{j=0}^{8} b_j t_{n+i}^j = y_{n+6} \tag{4}$$

If the function (3) satisfies the differential equation (1) at the points $t_{n+i}, i = 0, 1, \ldots, 7$ we obtain the following set of eight equations

$$\sum_{j=0}^{8} j b_j t_{n+i}^{j-1} = f_{n+i}, \qquad i = 0, 1, \ldots, 7 \tag{5}$$

where y_{n+i} is the approximation for the exact solution $y(t_{n+i}), f_{n+i} = f(t_{n+i}, y_{n+i})$ and n is the grid index. It should be noted that equation (4) and (5) lead to a system of equations which must be solved to obtain the coefficients

$b_j, j = 0, \ldots, 8$ which are substituted into (3) and after some algebraic computation, our continuous representation yields the form

$$Y(t) = y_{n+6} + h \sum_{i=0}^{7} \beta_i(t) f_{n+i} \tag{6}$$

where $\beta_i(t)$ are continuous coefficients. The method (6) is then used to generate the $7-step$ standard Adams method (7) at point $t = t_{n+7}$.

The additional methods are the obtained by evaluating at the points $t = t_{n+i}, i = 0, \ldots, 5$. Thus we have the additional methods as (8).

The methods (7) and (8) are then combined and implemented as a one block, self starting methods to simultaneously generate the solution $\{y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+6}, y_{n+7}\}$ for equation (1) at points $\{t_{n+1}, t_{n+2}, t_{n+3}, t_{n+4}, t_{n+5}, t_{n+6}, t_{n+7}\}$

III. Order of accuracy and Local truncation error

Following Fatunla [12] and Lambert [13] we define the local truncation error associated with (3) to be the linear difference operator Assuming that $y(t)$ is sufficiently differentiable, we can expand the terms in (10) as a Taylor series about the point t to obtain the expression

$$L[y(t), h] = C_0 y(t) + C_1 y'(t) + \ldots + C_s h^s y^{(s)}(t) + \ldots,$$

where the constant coefficients $C_s, s = 0, 1, \ldots$ are given as follows:

$C_0 = \sum_{j=0}^{k} \alpha_j,$

$C_1 = \sum_{j=0}^{k} j\alpha_j,$

$\vdots$

$C_s = \frac{1}{s!}[\sum_{j=0}^{k} j^s \alpha_j - s(\sum_{j=0}^{k} j^{s-1} \beta_j)]$

According to [1], we say that the method (2) has order m if

$C_0 = C_1 = \ldots = C_m, \;\; C_{m+1} \neq 0$

therefore, C_m is the error constant and $C_{m+1} h^{m+1} y^{(m+1)}(t_n)$ the principal local truncation error at the point t_n. Thus, we can write the local truncation error (LTE) of the method of order m as

$LTE = C_{m+1} h^{m+1} y^{(m+2)}(t_n) + \bigcirc(h^{m+2}).$

It is established from our calculations that the block Adam's methods (7) together with (8) have order $m = (8, 8, 8, 8, 8, 8, 8)$ and relatively small error constants $(-\frac{33953}{3628800}, \frac{9}{1400}, -\frac{425}{145152}, -\frac{13}{14175}, -\frac{81}{44800}, -\frac{127}{113400}, -\frac{7297}{3628800})^T$ respectively.

IV. Analysis of the method

In what follows, (7) and (8) can be rearranged and rewritten as a matrix finite difference equation of the form

where

$Y_{\omega+1} = (y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+6}, y_{n+7})^T$

$Y_{\omega} = (y_{n-6}, y_{n-5}, y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}, y_n)^T$

$F_{\omega+1} = (f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}, f_{n+7})^T$

$F_{\omega} = (f_{n-6}, f_{n-5}, f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n)^T$

for $\omega = 0, \ldots$ and $n = 0, 7, \ldots, N-7$, and the matrices $A_{(1)}$, $A_{(0)}$, $B_{(1)}$ and $B_{(0)}$ are 7 by 7 matrices whose entries are given by the coefficients of (7) and (8). In particular, the matrices are defined as equation (11).

A. Zero-stability

It is worth noting that zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (10) tends to the difference system

$$A_1 Y_{\omega+1} = A_0 Y_{\omega}$$

whose first characteristic polynomial $\rho(R)$ is given by

$$\rho(R) = \det(RA_{(1)} - A_{(0)}) = R^6(1 - R) \tag{12}$$

Following Fatunla[12], the block method (10) is zero-stable, since from (**??**), $\rho(R) = 0$ satisfies $|R_j| \leq 1$, $j = 1, \ldots, \nu$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1.

1) Consistency: The block method (10) is consistent as it has order $m > 1$. According to Henrici[1],convergent, since $convergence = zerostability + consistency.$

B. Linear stability

The linear stability properties of the CABM are determined by expressing them in the form (10) and applying them to the test equation

$$y' = \lambda y \quad , \quad \lambda < 0$$

which is applied to (10) to yield

$$Y_{\omega+1} = D(z) Y_{\omega} \;,\; z = \lambda h, \tag{13}$$

where the matrix $D(z)$ is given by

$$D(z) = (A_{(1)} - zB_{(1)})^{-1}(A_0 + zB_0)$$

From (13) we obtain the stability function $R(z) : \mathbb{C} \to \mathbb{C}$ which is a rational function with real coefficients given by (14).

The stability domain of the method (or region of absolute stability), S, is defined as

$$S = [z \in \mathbb{C} : R(z) \leq 1] \tag{15}$$

Specifically, when the left-half complex plane is contained in S, the method is said to be A-stable. Below in Fig. 1, we show the plot with rectangle representing the zeros and plus sign representing the poles of (14). The plot in white represents the stability region which corresponds to the stability function (14). Clearly, from the figure, it is obvious that our method is A- stable since according to Hairer and Wanner [14] it has no pole of the stability function (14) in the left half complex plane.

Implementation The implementation of the above block methods is summarized as follows:

Summary

$$y_{n+7} = y_{n+6} + h[\frac{275}{24192}f_n - \frac{11351}{120960}f_{n+1} + \frac{1537}{4480}f_{n+2} - \frac{88547}{120960}f_{n+3} + \frac{123135}{120960}f_{n+4} - \frac{4511}{4480}f_{n+5} + \frac{139849}{120960}f_{n+6} + \frac{5257}{17280}f_{n+7}] \tag{7}$$

$$\left.\begin{aligned}
y_n &= y_{n+6} + h[\tfrac{-41}{140}f_n - \tfrac{54}{35}f_{n+1} - \tfrac{27}{140}f_{n+2} - \tfrac{68}{35}f_{n+3} - \tfrac{27}{140}f_{n+4} - \tfrac{54}{35}f_{n+5} - \tfrac{41}{140}f_{n+6} + 0f_{n+7}]\\
y_{n+1} &= y_{n+6} + h[\tfrac{275}{24129}f_n - \tfrac{9355}{24192}f_{n+1} - \tfrac{1075}{896}f_{n+2} - \tfrac{22375}{24192}f_{n+3} - \tfrac{22375}{24192}f_{n+4} - \tfrac{1075}{896}f_{n+5} - \tfrac{9355}{24192}f_{n+6} + \tfrac{275}{24192}f_{n+7}]\\
y_{n+2} &= y_{n+6} + h[0f_n + \tfrac{8}{945}f_{n+1} - \tfrac{38}{105}f_{n+2} - \tfrac{136}{105}f_{n+3} - \tfrac{664}{945}f_{n+4} - \tfrac{136}{105}f_{n+5} - \tfrac{38}{105}f_{n+6} + \tfrac{8}{945}f_{n+7}]\\
y_{n+3} &= y_{n+6} + h[\tfrac{13}{4480}f_n - \tfrac{117}{4480}f_{n+1} + \tfrac{513}{4480}f_{n+2} - \tfrac{2777}{4480}f_{n+3} - \tfrac{3897}{4480}f_{n+4} - \tfrac{1107}{896}f_{n+5} - \tfrac{337}{896}f_{n+6} + \tfrac{9}{896}f_{n+7}]\\
y_{n+4} &= y_{n+6} + h[\tfrac{1}{756}f_n - \tfrac{2}{189}f_{n+1} + \tfrac{1}{28}f_{n+2} - \tfrac{52}{945}f_{n+3} - \tfrac{1153}{3780}f_{n+4} - \tfrac{46}{35}f_{n+5} - \tfrac{1363}{3780}f_{n+6} + \tfrac{8}{945}f_{n+7}]\\
y_{n+5} &= y_{n+6} + h[\tfrac{13}{4480}f_n - \tfrac{2999}{120960}f_{n+1} + \tfrac{1283}{13440}f_{n+2} - \tfrac{2987}{13440}f_{n+3} + \tfrac{44797}{120960}f_{n+4} - \tfrac{11261}{13440}f_{n+5} - \tfrac{5311}{13440}f_{n+6} + \tfrac{275}{24192}f_{n+7}]
\end{aligned}\right\} \tag{8}$$

$$L[y(t), h] = \sum_{j=0}^{k}\{\alpha_j y(t+jh) - h\beta_j y'(t+jh)\} \tag{9}$$

$$A_{(1)}Y_{\omega+1} = A_{(0)}Y_\omega + h[B^{(1)}F_{\omega+1} + B^{(0)}F_\omega] \tag{10}$$

$$\left.\begin{aligned}
A_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}\\
A_0 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\\
B_{(1)} &= \begin{pmatrix} -\frac{9355}{24192} & -\frac{1075}{896} & -\frac{22375}{24129} & -\frac{22375}{24129} & -\frac{1075}{896} & -\frac{9355}{24192} & \frac{275}{24192}\\ -\frac{8}{945} & -\frac{38}{105} & -\frac{136}{105} & -\frac{664}{945} & -\frac{136}{105} & -\frac{38}{105} & \frac{8}{945}\\ -\frac{117}{4480} & \frac{513}{4480} & -\frac{2777}{4480} & -\frac{3897}{4480} & -\frac{1107}{896} & -\frac{337}{896} & \frac{9}{896}\\ -\frac{2}{189} & \frac{1}{28} & -\frac{52}{945} & -\frac{1153}{3780} & -\frac{46}{35} & -\frac{1363}{3780} & \frac{8}{945}\\ -\frac{2999}{120960} & \frac{1283}{13440} & -\frac{2987}{13440} & \frac{44797}{120960} & -\frac{11261}{13440} & -\frac{5311}{13440} & \frac{275}{24129}\\ -\frac{54}{35} & -\frac{27}{140} & -\frac{68}{35} & -\frac{27}{140} & -\frac{54}{35} & -\frac{41}{140} & 0\\ -\frac{11351}{120960} & \frac{1537}{4480} & -\frac{88547}{120960} & \frac{123133}{120960} & -\frac{4511}{4480} & \frac{139849}{120960} & \frac{5257}{17280} \end{pmatrix}\\
B_{(0)} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{275}{24192}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{13}{4480}\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{756}\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{13}{4480}\\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{41}{140}\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{275}{24192} \end{pmatrix}
\end{aligned}\right\} \tag{11}$$

$$R(z) = \frac{1680 + 5880z + 9660z^2 + 9800z^3 + 6769z^4 + 3283z^5 + 1089z^6 + 210z^7}{1680 - 5880z + 9660z^2 - 9800z^3 + 6769z^4 - 3283z^5 + 1089z^6 - 210z^7} \tag{14}$$

On the partition $I_N : \{a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b$,$n = 0, 1, 2, \ldots, N-1$.

Step 1. Choose N for $k = 7$, $h = \frac{b-a}{N}$ the number of blocks $\pi = \frac{N}{7}$ using (10) $n = 0, \omega = 0$ the values $(y_1, y_2, \ldots, y_7)^T$ are generated simultaneously over the subinterval $[t_0, t_7]$ as y_0 are known from the IVP (1).

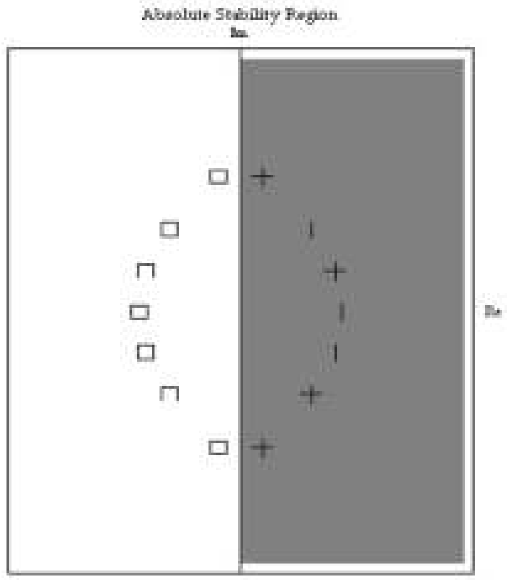

Fig. 1. Stability Region

Step 2. for $n = 8, \omega = 1$, $(y_8, y_9, \ldots, y_{14})^T$ are obtained over the subinterval $[t_7, t_{14}]$ since y_7 is known from the first block.

Step 3. The process is continued for $n = 2k, \ldots, N-k$ and $\omega = 2, \ldots, \pi$ to obtain approximate solutions to (1) on sub-intervals $[t_0, t_k], \ldots, [t_{N-k}, t_N]$ N is a positive integer and n the grid index.

V. Numerical examples

We give three numerical examples to illustrate the accuracy of the method. We find absolute of the approximate solution on the partition π_N as $|y - y(x)|$. The rate of convergence is calculated using the formula $Rate = \log_2(E^{2h}/E^h)$, E^h is the maximum absolute error obtained using the step size h. All computations were carried out using our written Matlab code.

A. Example 1

Our first example is the given linear system on the range on the range $0 \leq t \leq 1$

Example 1:

$$\begin{aligned} y_1' &= -21y_1 + 19y_2 - 20y_3, \ y_1(0) = 1 \\ y_2' &= 19y_1 - 21y_2 + 20y_3, \ y_2 = 0 \\ y_3' &= 40y_1 - 40y_2 + 40y_3, \ y_3 = -1 \end{aligned}$$

The exact solution of the system is given by

$$\begin{aligned} y_1(x) &= \tfrac{1}{2}(e^{-2x} + e^{-40x}(\cos(40x) + \sin(40x))) \\ y_2(x) &= \tfrac{1}{2}(e^{-2x} - e^{-40x}(\cos(40x) + \sin(40x))) \\ y_3(x) &= \tfrac{1}{2}(2e^{-40x}(\sin(40x) - \cos(40x))) \end{aligned}$$

This problem was also solved by Brugnano and Trigiante [6] using the Generalized Backward Backward Differentiation Formulas (GBDF) of order seven. The results for the GBDF are reproduced in table 1 and compared with the results given by the CABM of order eight. It is seen from table 1 that the CABM performs better than the GBDF by gaining three digits in accuracy. In all cases the rate of convergence is consistent with the order of the methods as the step-size is decreased. Thus, for this example, our method is superior in terms of accuracy.

TABLE I
A comparison of methods for Example 1

h	GBDF ($p = 7$)	Rate	CABM ($p = 8$)	Rate
1.00×10^{-2}	1.187×10^{-3}	-	3.953×10^{-6}	-
5.00×10^{-3}	1.389×10^{-5}	6.42	2.913×10^{-8}	7.08
2.50×10^{-3}	1.079×10^{-7}	7.00	2.206×10^{-10}	7.06
1.25×10^{-3}	1.079×10^{-9}	6.64	6.650×10^{-13}	8.36
6.25×10^{-4}	9.409×10^{-12}	6.84	2.689×10^{-15}	7.95

B. Example 2

Example 2: Next, we consider the Bessel ODE (see Vigo-Aguiar and Ramos [15]) given by

$$t^2y'' + ty' + (t^2 - 0.25)y = 0, \ y(1) = \sqrt{\frac{2}{\pi}}\sin 1 \simeq 0.6713967071418031$$

$$y'(1) = (2\cos 1 - \sin 1)/\sqrt{2\pi} \simeq 0.0954005144474746$$

$$\text{Exact} : y(t) = J_{1/2}(t) = \sqrt{\frac{2}{\pi t}}\sin t$$

We reduced the above second order equation to the form (1) as

$$\begin{aligned} &y_1' = y_2 \qquad y_1(1) = \sqrt{\tfrac{2}{\pi}}\sin 1 \simeq 0.6713967071418031 \\ &t^2y_2' + ty_2 + (t^2 - 0.25)y_1 = 0 \\ &\qquad y_2(1) = \tfrac{(2\cos 1 - \sin 1)}{\sqrt{2\pi}} \simeq 0.0954005144474746 \end{aligned}$$

$$\begin{aligned} &\text{Exact} : y_1(t) = J_{1/2}(t) = \sqrt{\tfrac{2}{\pi t}}\sin \\ &\text{Exact} : y_2(t) = J_{1/2}'(t) = \sqrt{\tfrac{2}{\pi t}}\cos - \sqrt{\tfrac{1}{2\pi}}\sqrt{(\tfrac{1}{t})^3}\sin \end{aligned}$$

The theoretical solution at $t = 8$ is $y(8) = \sqrt{\frac{2}{8\pi}}\sin(8) \simeq 0.279092789108058969$. The absolute errors for the y_1-component were obtained at $t = 8$ using our method for fixed step-sizes $h = 7/67, 7/82, 7/97, 7/112, 7/125$ corresponding to the number of steps $N = 67, 82, 97, 112, 125$ as shown in Table 2. Similar results were obtained for the same problem in [15] using the variable-step Falker method of order eight (m = 8) implemented in the predictor corrector mode (PC). It is seen that although we used fixed step-sizes, our method is more efficient in terms the number of function evaluation than the method in [15], which was implemented in a predictor-predictor mode with special techniques for supplying the starting values and for varying the step-size. Our method is self-starting and implemented without predictors.

TABLE II
ABSOLUTE ERRORS, $\|y-\overline{y}\|$, FOR EXAMPLE 2 WHERE
$y(x)=J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$

steps	VAR (m=8)	NFEs	CABM (m=8) 8	NFEs
67	7.1122×10^{-7}	134	2.978×10^{-9}	71
82	9.2632×10^{-8}	164	9.3971×10^{-10}	85
97	8.7834×10^{-9}	194	1.2447×10^{-10}	99
112	1.2108×10^{-10}	224	3.2552×10^{-11}	113
125	2.7068×10^{-11}	250	5.8148×10^{-11}	126

TABLE III
RESULT FOR $CABM_8$ $Error = Max_i|y_i-y(t_i)|$ FOR EXAMPLE 3

h	Steps	Error $CABM_8$
1.00×10^{-1}	200	7.14060×10^{-10}
5.00×10^{-2}	400	1.89718×10^{-12}
2.50×10^{-2}	800	7.08808×10^{-14}
1.25×10^{-2}	1600	1.04916×10^{-14}
6.25×10^{-3}	3200	4.29379×10^{-14}

C. Example 3

Example 3: Finally we solve without comparison our last example the two body problem in the range $0\le t\le 20$

$$\begin{array}{ll} y_1'=y_3, & y_1(0)=1\\ y_2'=-\frac{y_1}{\sqrt{(y_1^2-y_2^2)^3}}, & y_2(0)=0\\ y_3'=y_4, & y_3(0)=0\\ y_4'=-\frac{y_2}{\sqrt{(y_1^2-y_2^2)^3}}, & y_4(0)=1 \end{array}$$

With exact solution given as
$y_1(t)=cos(t), \qquad y_2(t)=sin(t)$

$y_3(t)=-sin(t), \qquad y_4(t)=cos(t)$

VI. CONCLUSION

A seven step continuous block Adams type block method CABM of order eight has been proposed and implemented as self starting methods for solution of ordinary differential equations,. Details of our numerical results indicate that the method is promising and they may be competitive with other methods that are frequently used for periodic ordinary differential equations.

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