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Second Sub-Harmonic Resonance in Vortex-Induced Vibrations of a Marine Pipeline Close to the Seabed

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***Abstract*—**In this paper, using the method of multiple scales, the second sub-harmonic resonance in vortex-induced vibrations (VIV) of a marine pipeline close to the seabed is investigated based on a developed wake oscillator model. The amplitude-frequency equations are also derived. It is found that the oscillation will increase all the time when both discriminants of the amplitude-frequency equations are positive while the oscillation will decay when the discriminants are negative.

***Keywords*—**Vortex-induced vibrations, marine pipeline, seabed, sub-harmonic resonance.

I. INTRODUCTION

PIPELINES play an important role in the offshore and subsea engineering to transport oil and gas. To understand the VIV characteristics of a pipeline close to a plane boundary, a large number of experimental tests were performed and the results show that the gap-to-diameter ratio plays an important effect in both amplitudes and resonance ranges of the vibrating pipeline [1]-[5]. Moreover, using various Computational Fluid Dynamics (CFD) methods, the vortex has been studied [6]-[9], and the VIV is also undertaken numerically [10]-[12]. However, such methods are limited by their extensive computational requirements for simulations at realistic Reynolds numbers [13].

To investigate the vortex shedding past a circular cylinder near a wall, various models and methods were used, such as a new wake oscillator model proposed by [14], two-dimensional standard high Reynolds number k–ε turbulence model [15], Arbitrary Lagrangian Eulerian (ALE) scheme [16] and so on. Especially, the model by Jin and Dong that extends the classic Van der Pol equation in the wake oscillator model is capable of capturing the different vortex shedding modes and predicting variations of vibration amplitudes with the reduced velocity at a much lower cost than that with other models. Vibration characteristics of the cylinder for a range of gap ratios were systematically studied by solving the model equations numerically.

In this paper, the focus is on the second sub-harmonic resonance in VIV of a pipeline close to the Seabed based on the wake oscillator model of Jin and Dong [14].

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II. THE ANALYTICAL SOLUTION

A. Equations of Motion

To study the VIV of a marine pipeline close to the seabed, the model of an elastically supported rigid circular cylinder of diameter D, subjected to a stationary and uniform flow with the free stream velocity U, is used as depicted in Fig. 1.

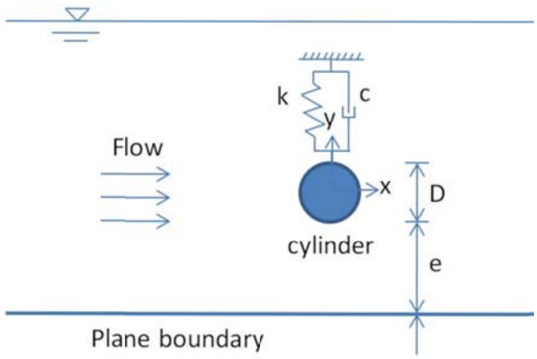

Fig. 1 Spring supported cylinder close to a plane boundary subjected to current [14]

The general structure response equation in a plane cross-flow is usually formulated in terms of a oscillator variable q as [17]:

$$m_e \frac{d^2y}{dt^2} + \left(c + \gamma\rho D^2 \Omega_f\right)\frac{dy}{dt} + ky = \frac{1}{4}\rho U^2 D C_{L0} q. \tag{1}$$

The fluid wake effect around a cylinder with the influence of a boundary can be given as [14]:

$$\frac{d^2q}{dt^2} + \lambda\Omega_f(q^2-1)\frac{dq}{dt} + {\Omega_f}^2 q + \alpha f_n \Omega_f q^2 = \beta\, A/D \frac{d^2y}{dt^2} \tag{2}$$

where y is the displacement of the cylinder in the cross-flow direction with respect to time t, Ω_f is the vortex-shedding angular frequency, $\Omega_f = 2\pi S_t U/D$, the mass for unit length of the cylinder, m_e is taken into account both the mass of structure m and the fluid-added mass $m_f = mc_a$, in which C_a is the added mass coefficient, ρ is the density of the fluid, U is the flow velocity in the free stream, D is the diameter of the cylinder, k is a spring constant and c is the structure damping $c = 2m_e\xi\Omega_s$ and ξ is the reduced damping, Ω_s is the structural angular frequency $\Omega_s = (k/m_e)^{0.5}$. The damping of flow $\gamma = C_D/4\pi S_t$ and the drag coefficient $C_D = C_{D0}(1+2y/D)S_t U_r$, in which C_{D0} is reference drag coefficient, and the reduced flow velocity $U_r = U/f_n D$. α and β are the coefficients and the functions of mass ratio and gap ratio [14], $q = 2\,C_L/C_{L0}$, is a flow variable that is commonly referred to as a reduced vortex lift coefficient in which C_L is the lift coefficient and C_{L0} is the reference lift coefficient, usually taken as a constant of 0.3. ε and A are parameters which need to be determined empirically and have typical values $\lambda = 0.3$ and $A = 12$ [17].

For the ease of computation, the equations of motion can be non-dimensionalised with the following scaled quantities:

$$y^* = \mathrm{y/D};\ m^* = 4m/\pi\rho D^2;\ w^* = \mathrm{w/w_c};\ {\Omega_f}^* = \Omega_f/\mathrm{w_c};\ c^* = \mathrm{c/(mw_c)};\ \tau = \mathrm{w_c t}$$

where $\mathrm{w_c}$ is a chosen arbitrary frequency. Introducing these quantities into (1) and (2) gives the following on-dimensional equations :

$$\frac{d^2y}{d\tau^2} + \mathrm{w}^2\mathrm{y} = \Gamma_1 q{\Omega_f}^2 - \Gamma_2\frac{{\Omega_f}^2}{w}\frac{dy}{d\tau} - 2\Gamma_2\frac{{\Omega_f}^2}{w}\frac{ydy}{d\tau} - c\frac{dy}{d\tau} \quad (3)$$

$$\frac{d^2q}{d\tau^2} + {\Omega_f}^2 q = -\lambda\Omega_f(q^2-1)\frac{dq}{d\tau} - \alpha f_n\Omega_f q^2 + \beta A\frac{d^2y}{d\tau^2} \quad (4)$$

where $\Gamma_1 = \mathrm{C_{L0}}/\left(4\pi^3{S_t}^2 m\right)$; $\Gamma_2 = \mathrm{C_{D0}}/(\pi^2 S_t m)$, and all the asterisks in (3)and (4) are omitted for simplicity.

B. Second Sub-Harmonic Solutions

Clearly, due to the presence of the quadratic and cubic nonlinearity terms, it is not possible to obtain the exact analytical solutions of (3) and (4). However, for primary resonance approximate solutions may be obtained using the Method of Multiple Scales. Distinguishing the quadratic items from the other ones in (3) and (4) with a parameter ε ($\varepsilon \ll 1$), (3) and (4) can be written as [18]:

$$\frac{d^2y}{d\tau^2} + \mathrm{w}^2\mathrm{y} = \Gamma_1 q{\Omega_f}^2 - \varepsilon\Gamma_2\frac{{\Omega_f}^2}{w}\frac{dy}{d\tau} - 2\varepsilon\Gamma_2\frac{{\Omega_f}^2}{w}\frac{ydy}{d\tau} - \varepsilon c\frac{dy}{d\tau} \quad (5)$$

$$\frac{d^2q}{d\tau^2} + {\Omega_f}^2 q = -\varepsilon^2\lambda\Omega_f q^2\frac{dq}{d\tau} + \varepsilon\lambda\Omega_f\frac{dq}{d\tau} - \varepsilon\alpha f_n\Omega_f q^2 + \beta A\frac{d^2y}{d\tau^2} \quad (6)$$

Let $\mathrm{T}_n = \varepsilon^n\tau$, then the approximate solutions to (5) and (6) can be expressed as:

$$y(\tau,\varepsilon) = \sum_{i=1}^{n}\varepsilon^i\, y_i(\mathrm{T_0},\mathrm{T_1},\mathrm{T_2},\ldots,\mathrm{T}_n) \quad (7)$$

$$q(\tau,\varepsilon) = \sum_{i=1}^{n}\varepsilon^i\, q_i(\mathrm{T_0},\mathrm{T_1},\mathrm{T_2},\ldots,\mathrm{T}_n) \quad (8)$$

To study the Second Sub-harmonic Resonance, the non-dimensional shedding frequency Ω_f (representing the frequency of external forcing) is expressed with the non-dimensional structural frequency w and a detuning parameter σ as:

$$\Omega_f = 2w + \varepsilon\sigma \quad (9)$$

After substituting (7)-(9) into (5) and (6) and equating the coefficients of $\varepsilon^i(i = 0{,}1)$ on the both sides of the equations, the system is expanded into six coupled equations, which can be solved sequentially:

ε^0:

$$ {D_0}^2 y_0 + w^2 y_0 = \mathrm{A}{D_0}^2 y_0 \quad (10)$$

$$ {D_0}^2 q_0 + w^2 q_0 = 4\Gamma_1 q_0 w^2 \quad (11)$$

ε^1:

$$ {D_0}^2 y_1 + w^2 y_1 = -2D_0D_1y_0 + 4\Gamma_1 q_1 w^2 + 4w\sigma q_0 - 4wD_0y_0 - 8\Gamma_2 w y_0 D_0 y_0 - cD_0y_0 \quad (12)$$

$$ {D_0}^2 q_1 + w^2 q_1 = -2D_0D_1q_0 - 4w\sigma q_0 + 2\lambda w D_0 q_0 - 2\Gamma_3 w^2 {q_0}^2 + \mathrm{A}{D_0}^2 y_1 + 2\mathrm{A}D_0D_1y_0 \quad (13)$$

where $D_0, D_1, D_2, {D_0}^2, {D_1}^2$ represent the first and second order derivatives, respectively, and $D_0 = \frac{\partial}{\partial T_0}$; $D_1 = \frac{\partial}{\partial T_1}$; ${D_0}^2 = \frac{\partial^2}{\partial {T_0}^2}$; ${D_1}^2 = \frac{\partial^2}{\partial {T_1}^2}$.

The general solutions of (10) and (11) can be written in the form:

$$y_0 = Y_a(\mathrm{T_1})e^{(g+ih)\mathrm{T_0}} + \overline{Y_a}(\mathrm{T_1})e^{(g-ih)\mathrm{T_0}} + Y_b(\mathrm{T_1})e^{(-g+ih)\mathrm{T_0}} + \overline{Y_b}(\mathrm{T_1})e^{(-g-ih)\mathrm{T_0}} \quad (14)$$

$$q_0 = \frac{1}{4\Gamma_1 q_1 w^2}\left\{[w^2 + (g+ih)^2]Y_a(\mathrm{T_1})e^{(g+ih)\mathrm{T_0}} + [w^2 + (g-ih)^2]\overline{Y_a}(\mathrm{T_1})e^{(g-h)\mathrm{T_0}} + [w^2 + (-g+ih)^2]Y_b(\mathrm{T_1})e^{(-g+ih)\mathrm{T_0}} + [w^2 + (-g-ih)^2]\overline{Y_b}(\mathrm{T_1})e^{(-g-h)\mathrm{T_0}}\right\} \quad (15)$$

where $\overline{Y_a}$ and $\overline{Y_b}$ are the complex conjugates of Y_a and Y_b, respectively. Then, substituting (14) and (15) into (12) and (13) gives:

$$\begin{aligned} {D_0}^4 y_1 + (-3w^2 - 4\Gamma_1 w^2 A){D_0}^2 y_1 + 4w^4 y_1 = &\Big\{-2(g+ih)^3 Y_a' + \frac{\sigma}{w}Y_a[w^2 + (g+ih)^2](g+ih)^2 - 4\Gamma_2 w(g+ih)^3 Y_a - c(g+ih)^3 Y_a - 2(g+ih)Y_a' + 4w\sigma Y_a[w^2 + (g+ih)^2] - 16w^3\Gamma_2(g+ih)Y_a - 2[w^2 + (g+ih)^2](g+ih){Y_a}' \\ &- 16\Gamma_1 w^3\sigma Y_a[w^2 + (g+ih)^2] + 8\lambda\Gamma_1 w^3[w^2 + (g+ih)^2](g+ih)Y_a - 8\Gamma_1 w^2 A(g+ih){Y_a}'\Big\}e^{(g+ih)\mathrm{T_0}} + \Big\{-2(-g+ih)^3 Y_b' + \frac{\sigma}{w}Y_b[w^2 + (-g+ih)^2](-g+ih)^2 \\ &- 4\Gamma_2 w(-g+ih)^3 Y_b - c(-g+ih)^3 Y_a - 2(-g+ih)Y_b' + 4w\sigma Y_a[w^2 + (-g+ih)^2] - 16w^3\Gamma_2(-g+ih)Y_b - 2[w^2 + (-g+ih)^2](-g+ih){Y_b}' \\ &- 16\Gamma_1 w^3\sigma Y_b[w^2 + (-g+ih)^2] + 8\lambda\Gamma_1 w^3[w^2 + (-g+ih)^2](-g+ih)Y_b - 8\Gamma_1 w^2 A(-g+ih){Y_b}'\Big\}e^{(-g+ih)\mathrm{T_0}} + NST \end{aligned} \quad (16)$$

where NST represents other terms, and ($'$) means derivative with respect to time $\mathrm{T_1}$.

It is recognized that the terms related to $e^{(g+ih)\mathrm{T_0}}$ and $e^{(-g+ih)\mathrm{T_0}}$ have the secular properties, which can cause a disproportionate increase in the relative magnitude of the additional correction generated at this order of perturbation [18]. In order to eliminate the secular properties of final solution, the terms related to$e^{(g+ih)\mathrm{T_0}}$ and $e^{(-g+ih)\mathrm{T_0}}$ in (16) have to be set equal to zero, then the solutions of (17) and (18) can be obtained:

$$-2(g+ih)^3Y_a' + \frac{\sigma}{w}Y_a[w^2+(g+ih)^2](g+ih)^2 - 4\Gamma_2 w(g+ih)^3Y_a - c(g+ih)^3Y_a - 2(g+ih)Y_a' + 4w\sigma Y_a[w^2+(g+ih)^2] - 16w^3\Gamma_2(g+ih)Y_a - 2[w^2+(g+ih)^2](g+ih)Y_a' - 16\Gamma_1 w^3\sigma Y_a[w^2+(g+ih)^2] + 8\lambda\Gamma_1 w^3[w^2+(g+ih)^2](g+ih)Y_a - 8\Gamma_1 w^2 A(g+ih)Y_a' = 0 \quad (17)$$

$$-2(-g+ih)^3Y_b' + \frac{\sigma}{w}Y_b[w^2+(-g+ih)^2](-g+ih)^2 - 4\Gamma_2 w(-g+ih)^3Y_b - c(-g+ih)^3Y_a - 2(-g+ih)Y_b' + 4w\sigma Y_a[w^2+(-g+ih)^2] - 16w^3\Gamma_2(-g+ih)Y_b - 2[w^2+(-g+ih)^2](-g+ih)Y_b' - 16\Gamma_1 w^3\sigma Y_b[w^2+(-g+ih)^2] + 8\lambda\Gamma_1 w^3[w^2+(-g+ih)^2](-g+ih)Y_b - 8\Gamma_1 w^2 A(-g+ih)Y_b' = 0 \quad (18)$$

It is convenient to express Y_a and $Y_b$as:

$$Y_a = A_r + iA_i \quad (19)$$

$$Y_b = B_r + iB_i \quad (20)$$

Then substituting (19) and (20) into (17) and (18) and separating real and imaginary parts leads to:

$$M_{11}A_r' + M_{12}A_i' + M_{13}A_r + M_{14}A_i = 0 \quad (21)$$

$$M_{21}A_r' + M_{22}A_i' + M_{23}A_i + M_{24}A_r = 0 \quad (22)$$

$$M_{31}B_r' + M_{32}B_i' + M_{33}B_r + M_{34}B_i = 0 \quad (23)$$

$$M_{41}B_r' + M_{42}B_i' + M_{43}B_i + M_{44}B_r = 0 \quad (24)$$

The solutions of (21)-(24) can be expressed as:

$$A_r = a_r e^{\lambda_1 T_1} \quad (25)$$

$$A_i = a_i e^{\lambda_1 T_1} \quad (26)$$

$$B_r = b_r e^{\lambda_2 T_1} \quad (27)$$

$$B_i = b_i e^{\lambda_2 T_1} \quad (28)$$

Substituting (25)-(28) into (21)-(24), it can be obtained:

$$\begin{bmatrix} M_{11}\lambda_1 + M_{13} & M_{12}\lambda_1 + M_{14} \\ M_{21}\lambda_1 + M_{24} & M_{22}\lambda_1 + M_{23} \end{bmatrix} \begin{bmatrix} a_r \\ a_i \end{bmatrix} = 0 \quad (29)$$

$$\begin{bmatrix} M_{11}\lambda_2 + M_{13} & M_{12}\lambda_2 + M_{14} \\ M_{21}\lambda_2 + M_{24} & M_{22}\lambda_2 + M_{23} \end{bmatrix} \begin{bmatrix} b_r \\ b_i \end{bmatrix} = 0 \quad (30)$$

To get the trivial solutions,

$$\begin{vmatrix} M_{11}\lambda_1 + M_{13} & M_{12}\lambda_1 + M_{14} \\ M_{21}\lambda_1 + M_{24} & M_{22}\lambda_1 + M_{23} \end{vmatrix} = 0 \quad (31)$$

and

$$\begin{vmatrix} M_{11}\lambda_2 + M_{13} & M_{12}\lambda_2 + M_{14} \\ M_{21}\lambda_2 + M_{24} & M_{22}\lambda_2 + M_{23} \end{vmatrix} = 0 \quad (32)$$

Then

$$(M_{11}M_{12} - M_{12}M_{21})\lambda_1{}^2 + (M_{11}M_{23} + M_{13}M_{22} - M_{12}M_{24} - M_{14}M_{21})\lambda_1 + (M_{13}M_{23} - M_{14}M_{24}) = 0 \quad (33)$$

$$(M_{31}M_{32} - M_{32}M_{41})\lambda_2{}^2 + (M_{31}M_{43} + M_{33}M_{42} - M_{32}M_{44} - M_{34}M_{41})\lambda_2 + (M_{33}M_{43} - M_{34}M_{44}) = 0 \quad (34)$$

The discriminants of the quadratic equations (33) and (34) can be obtained:

$$\Delta_1 = (M_{11}M_{23} + M_{13}M_{22} - M_{12}M_{24} - M_{14}M_{21})^2 - 4(M_{11}M_{12} - M_{12}M_{21})(M_{13}M_{23} - M_{14}M_{24}) \quad (35)$$

$$\Delta_2 = (M_{31}M_{43} + M_{33}M_{42} - M_{32}M_{44} - M_{34}M_{41})^2 - 4(M_{31}M_{32} - M_{32}M_{41})(M_{33}M_{43} - M_{34}M_{44}) \quad (36)$$

If both of the discriminants are negative ($\Delta_1 < 0$ and $\Delta_2 < 0$), then there are no real roots of (33) and (34). In the second sub-harmonic resonance in VIV of a marine pipeline close to the seabed, it means that the oscillation will decay with the increase of time. However, If both discriminants are positive ($\Delta_1 > 0$ and $\Delta_2 > 0$.), then there are two distinct roots of (33) and (34). In the second sub-harmonic resonance in VIV of a marine pipeline close to the seabed, it means that the oscillation will increase all the time. In this case, the attention has to be paid to the max allowable amplitude of the pipeline. When the amplitude of the pipeline exceeds the allowable one, the damage of pipeline will take place, which may lead to a big loss.

III. Conclusion

The second sub-harmonic resonance in VIV of a pipeline close to the seabed is studied using the wake oscillator method. The main purpose of this work was to derive the amplitude-frequency equation with regard to second sub-harmonic resonance of the cylinder from a recently derived wake oscillator model and solve the equation analytically using the multiple scales method. Moreover, a method of predicting the trend of oscillation is proposed, which may be helpful in the engineering.

Appendix

$$Z_1 = 5w^2 - 4\Gamma_1 w^2 A$$

$$Z_2 = \sqrt{40w^4\Gamma_1 A - 9w^4 - 16\Gamma_1{}^2 w^4 A^2}$$

$$c_\lambda = \frac{-Z_1}{2}$$

$$d_\lambda = \frac{Z_2}{2}$$

$$g = -\sqrt{\left(\sqrt{c_\lambda{}^2 + d_\lambda{}^2} + c_\lambda\right)/2}$$

$$h = \sqrt{\left(\sqrt{{c_\lambda}^2 + {d_\lambda}^2} - c_\lambda\right)/2}$$

$$M_{11} = -2(g^3 - 3gh^2) - 8w^2 g - 2[(w^2 + g^2 - h^2)g - 2gh^2] + 8Aw^2\Gamma_1 g$$

$$M_{12} = 2(3g^2 h - h^3) + 8w^2 h + 2[(w^2 + g^2 - h^2)h + 2g^2 h] - 8Aw^2\Gamma_1 h$$

$$M_{13} = -\frac{\sigma}{w}[(w^2 + g^2 - h^2)(g^2 - h^2) - 4g^2h^2] - 4\Gamma_2 w(g^3 - 3gh^2) - c(g^3 - 3gh^2) + 4w\sigma(w^2 + g^2 - h^2) - 16w^3\Gamma_2 g - 4w^2cg - 16\sigma w^3\Gamma_1(w^2 + g^2 - h^2) - 8\lambda w^3\Gamma_1[(w^2 + g^2 - h^2)g - 2gh^2]$$

$$M_{14} = \frac{2\sigma gh}{w}(w^2 + 2g^2 - 2h^2) + 4\Gamma_2 w(3g^2h - h^3) + c(3g^2h - h^3) - 8ghw\sigma + 16w^3\Gamma_2 h + 4w^2ch + 32\sigma w^3\Gamma_1 gh + 8\lambda w^3\Gamma_1[(w^2 + g^2 - h^2)h + 2g^2h]$$

$$M_{21} = -M_{12}, M_{22} = M_{11}, M_{23} = -M_{14}, M_{24} = M_{13}$$

$$M_{31} = -2(-g^3 + 3gh^2) + 8w^2 g + 2[(w^2 + g^2 - h^2)g + 2gh^2] - 8Aw^2\Gamma_1 g$$

$$M_{32} = 2(3g^2 h - h^3) + 8w^2 h + 2[(w^2 + g^2 - h^2)h + 2g^2 h] - 8Aw^2\Gamma_1 h$$

$$M_{33} = -\frac{\sigma}{w}[(w^2 + g^2 - h^2)(g^2 - h^2) - 4g^2h^2] + 4\Gamma_2 w(g^3 - 3gh^2) + c(g^3 - 3gh^2) + 4w\sigma(w^2 + g^2 - h^2) + 16w^3\Gamma_2 g + 4w^2cg - 16\sigma w^3\Gamma_1(w^2 + g^2 - h^2) - 8\lambda w^3\Gamma_1[(w^2 + g^2 - h^2)g + 2gh^2]$$

$$M_{34} = \frac{-2\sigma gh}{w}(w^2 + 2g^2 - 2h^2) + 4\Gamma_2 w(3g^2h - h^3) + c(3g^2h - h^3) + 8ghw\sigma + 16w^3\Gamma_2 h + 4w^2ch - 32\sigma w^3\Gamma_1 gh + 8\lambda w^3\Gamma_1[(w^2 + g^2 - h^2)h + 2g^2h]$$

$$M_{41} = -M_{32}, M_{42} = M_{31}, M_{43} = -M_{34}, M_{44} = M_{33}$$

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