

S -Fuzzy Left h -Ideal of Hemirings

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Abstract—The notion of S -fuzzy left h -ideals in a hemiring is introduced and its basic properties are investigated. We also study the homomorphic image and preimage of S -fuzzy left h -ideal of hemirings. Using a collection of left h -ideals of a hemiring, S -fuzzy left h -ideal of hemirings are established. The notion of a finite-valued S -fuzzy left h -ideal is introduced, and its characterization is given. S -fuzzy relations on hemirings are discussed. The notion of direct product and S -product are introduced and some properties of the direct product and S -product of S -fuzzy left h -ideal of hemiring are also discussed.

Keywords—hemiring, left h -ideal, anti fuzzy h -ideal, S -fuzzy left h -ideal, t -conorm, homomorphism.

I. INTRODUCTION

THE concept of fuzzy subset was introduced by L.A. Zadeh [8]. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. B. Schweizer and A. Sklar [5,6] introduced the notions of Triangular norm (t -norm) and Triangular conorm (t -conorm). Triangular norm (t -norm) and Triangular conorm (t -conorm or s -norm) are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators respectively. The ideal theory plays an important role in algebraic structure. La Torre [7] studied the notion of h -ideals and k -ideals in hemirings. Then Y.B Jun et. al [4] introduced the notion of fuzzy h -ideal of hemirings and discussed related properties. First, Abu Osman [1] introduced the notion of fuzzy subgroup with respect to t -norm. Following this, J.Zhan [9] introduced the notion of T -fuzzy left h -ideal of hemirings. Then, J.Zhan [10] introduced the notion of fuzzy hyper ideals in hyper near-rings with respect to t -norm. Recently, Y.U Cho et. al [3] introduced the notion of fuzzy subalgebras with respect to t -conorm of BCK -algebras and M.Akram et. al [2] introduced the notion of sensible fuzzy ideal with respect to t -conorm in BCK -algebras. Using the idea of [2] and [3], in this paper we introduce the notion of S -fuzzy left h -ideal of hemirings and investigate its related properties. Also, we review several results described in [9] using t -conorm.

II. PRELIMINARIES

An algebra $(R; +, \cdot)$ is said to be a *semiring* if $(R; +)$ and $(R; \cdot)$ are semigroups satisfying $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in R$. A semiring R is said to be *additively commutative* if $a + b = b + a$ for all $a, b, c \in S$. A semiring R may have an identity 1, defined by

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1. $a \cdot a = a \cdot 1$ and a zero 0, defined by $0 + a = a = a + 0$ and $a \cdot 0 = 0 = 0 \cdot a$ for all $a \in R$. A semiring R is said to be a *hemiring* if it is an additively commutative with zero. A non-empty subset I of R is said to be a *left (resp., right ideal)* if $x, y \in I$ and $r \in R$ imply that $x + y \in I$ and $rx \in I$ (resp., $xr \in I$). If I is both left and right ideal of R , we say I is a *two-sided ideal, or simply ideal*, of R . A left ideal I of a semiring R is said to be a *k-ideal* if $a \in I$ and $x \in R$, and if $x + a \in I$ or $a + x \in I$ then $x \in I$. Right k -ideal is defined dually, and two-sided k -ideal or simply a k -ideal is both a left and a right k -ideal. A left ideal I of a hemiring R is called a *left h-ideal* if $x + a + z = b + z$ implies that $x \in I$ for all $x, y \in R$ and $a, b \in R$. Right h -ideals are defined similarly.

Definition 2.1: Let X be a non-empty set. A fuzzy subset of X is a function $\mu : X \rightarrow [0, 1]$. Let μ be the fuzzy subset of a set X . For a fixed $0 \leq t \leq 1$, the set

$$L(\mu; t) = \{x \in X : \mu(x) \leq t\}$$

is called a *lower level set* or simply *level set* of μ .

Definition 2.2: A fuzzy subset μ of a hemiring R is said to be *fuzzy left (resp., right) ideal* of R if
(FI1) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ and
(FI2) $\mu(xy) \geq \mu(y)$ (resp., $\mu(xy) > \mu(x)$)
for all $x, y \in R$.

If μ is a *fuzzy ideal* of R if it is both fuzzy left and a fuzzy right ideal of R .

Definition 2.3: A fuzzy subset μ of a hemiring R is said to be an *anti fuzzy left (resp., right) ideal* of R if
(AFI1) $\mu(x + y) \leq \max\{\mu(x), \mu(y)\}$ and
(AFI2) $\mu(xy) \leq \mu(y)$ (resp., $\mu(xy) \leq \mu(x)$)
for all $x, y \in R$.

If μ is an *anti fuzzy ideal* of R if it is both an anti fuzzy left and anti fuzzy right ideal of R .

Definition 2.4: Let R and R' be hemirings. A mapping $f : R \rightarrow R'$ is said to be a *homomorphism* if
 $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$
for all $x, y \in R$.

Definition 2.5: A fuzzy subset μ of a hemiring R is said to be a *fuzzy left (resp., right) h-ideal* of R if
(AFI1) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ and
(AFI2) $\mu(xy) \geq \mu(y)$ (resp., $\mu(xy) \geq \mu(x)$)
for all $x, y \in R$.

(AFI3) If $x + a + z = b + z$ implies that
 $\mu(x) \geq \min\{\mu(a), \mu(b)\}$, for all $a, b, x, z \in S$.

If μ is *fuzzy h-ideal* of R if it is both a fuzzy left and fuzzy right h -ideal of R .

Definition 2.6: A fuzzy subset μ of a hemiring R is said to be an *anti fuzzy left (resp., right) h-ideal* of R if
(AFI1) $\mu(x + y) \leq \max\{\mu(x), \mu(y)\}$ and
(AFI2) $\mu(xy) \leq \mu(y)$ (resp., $\mu(xy) \leq \mu(x)$)

for all $x, y \in R$.

(AFI3) If $x + a + z = b + z$ implies that $\mu(x) \leq \max\{\mu(a), \mu(b)\}$, for all $a, b, x, z \in S$.

If μ is an anti fuzzy h -ideal of R it is both an anti fuzzy left h -ideal and anti fuzzy right h -ideal of R .

Definition 2.7: A triangular conorm (t -conorm) is a mapping $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that satisfies the following conditions:

(S1) $S(x, 0) = x$,

(S2) $S(x, y) = S(y, x)$,

(S3) $S(x, S(y, z)) = S(S(x, y), z)$,

(S4) $S(x, y) \leq S(x, z)$ whenever $y \leq z$,

for all $x, y, z \in [0, 1]$.

Replacing 0 by 1 in condition S , we obtain the concept of t -norm T .

Proposition 2.8: For a t -conorm S .Then the following statement holds $S(x, y) \geq \max(x, y)$, for all $x, y \in [0, 1]$.

Definition 2.9: Let S be a t -conorm. A fuzzy subset μ in a hemiring R is called sensible with respect to S if $Im\mu \subseteq \Delta_S$, where $\Delta_S = \{t \in [0, 1] | S(t, t) = t\}$.

III. S-FUZZY LEFT H-IDEALS IN HEMIRINGS

In what follows, R and S denotes a hemiring and t -conorm respectively, unless otherwise specified.

Definition 3.1: A fuzzy subset μ of R is called a S -fuzzy left ideal of a hemiring R (briefly, fuzzy left ideal with respect to t -conorm) if it satisfies the following conditions:

(SFI1) $\mu(x + y) \leq S(\mu(x), \mu(y))$,

(SFI2) $\mu(xy) \leq \mu(y)$, for all $x, y \in S$.

S -fuzzy right ideals are defined similarly.

Definition 3.2: A S -fuzzy ideal μ of R is said to be a S -fuzzy left h -ideal if it satisfies the following condition:

(SFI3) $x + a + z = b + z$ implies that $\mu(x) \leq S(\mu(a), \mu(b))$, for all $a, b, x, z \in S$.

S -fuzzy right h -ideals are defined similarly.

Definition 3.3: A S -fuzzy left h -ideal μ of R is said to be a sensible if it satisfies the sensible property.

Example 3.4: Let R be the set of natural numbers including 0, and R is a hemiring with usual addition and multiplication. Define a fuzzy subset $\mu : R \rightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} 0 & \text{if } x \text{ is even or } 0, \\ 1 & \text{otherwise.} \end{cases}$$

and let $S_m : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a function defined by $S_m(\alpha, \beta) = \min\{x + y, 1\}$ for all $x, y \in [0, 1]$. Then, S_m is a t -conorm. By routine calculation, we know that μ is a sensible S -fuzzy left h -ideal of R .

Proposition 3.5: Let S be a t -conorm. Then, every sensible S -fuzzy left h -ideal μ of a hemiring R is a anti fuzzy left h -ideal of R .

Proof: The proof is obtained dually by using the notion of t -conorm S instead of t -norm T in [9]. ■

Corollary 3.6: If μ is a sensible S -fuzzy left h -ideal of R , then each non-empty level subset $L(\mu; t)$ of μ is a left h -ideal of R .

Proof: Assume that μ is a sensible S -fuzzy left h -ideal of R and $L(\mu; t)$ is a non-empty level subset of μ in R .

(i) Since $L(\mu; t)$ is a non-empty level subset of μ , there exists

$x, y \in L(\mu; t)$, $\mu(x + y) \leq S(\mu(x), \mu(y)) = t$.

Thus $x + y \in L(\mu; t)$.

(ii) Let $x, y \in L(\mu; t)$, such that $\mu(xy) \leq \mu(y) \leq t$.

Thus $xy \in L(\mu; t)$.

(iii) Let $a, b, x, z \in L(\mu; t)$, If $x + a + z = b + z$ implies that $\mu(x) \leq S(\mu(a), \mu(b)) = t$. Thus $x \in L(\mu; t)$

Hence, $L(\mu; t)$ is a left h -ideal of R . ■

The following example shows that there exists a t -conorm S such that an anti fuzzy h -ideal of R may not be an sensible S -fuzzy left h -ideal of R .

Example 3.7: Let R be a hemiring in Example[3.4]. Define a fuzzy subset $\mu : R \rightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} \frac{1}{5} & \text{if } x \text{ is even or } 0, \\ \frac{1}{3} & \text{otherwise.} \end{cases}$$

is an anti fuzzy h -ideal of R .

Let $\nu = (0, 1)$ and define the binary operation S_ν on $(0, 1)$ as follows

$$S_\nu(\alpha, \beta) = \begin{cases} \max\{\alpha, \beta\} & \text{if } \min\{\alpha, \beta\} = 0, \\ 0 & \text{max}\{\alpha, \beta\} > 0, \alpha + \beta \geq 1 + \nu \\ \nu & \text{otherwise.} \end{cases}$$

Then, S_ν is a t -conorm. It is easy to check that μ is a S -fuzzy left h -ideal of R , but

$$S_\nu(\mu(0), \mu(0)) = S_\nu\left(\frac{1}{5}, \frac{1}{5}\right) = \nu \neq \mu(0).$$

Hence, μ is not a sensible S -fuzzy left h -ideal R .

Now, we consider the following theorem.

Theorem 3.8: Let S be a t -conorm and let μ be a sensible fuzzy subset in a hemiring R , then μ is a sensible S -fuzzy left h -ideal of R if and only if each non-empty level subset $L(\mu; t)$ of μ is a left h -ideal of R .

Proof: The necessary condition can be given by corollary[3.6]. Conversely, assume that each non-empty level subset $L(\mu; t)$ is a left h -ideal of R .

(i) Let $x, y \in R$. Let if possible, $\mu(x + y) > S(\mu(x), \mu(y))$. Set $t_0 := \frac{1}{2}\{\mu(x + y) + S(\mu(x), \mu(y))\}$, we have $x \in L(\mu; t_0)$ and $y \in L(\mu; t_0)$, since $L(\mu; t)$ is a left h -ideal of R . Then $x + y \in L(\mu; t_0)$ and $\mu(x + y) \leq t_0$, a contradiction. Thus $\mu(x + y) \leq S(\mu(x), \mu(y))$.

(ii) If $x, y \in L(\mu; t)$ then $xy \in L(\mu; t)$. Then

$\mu(xy) \leq \mu(y) \leq t$. Thus $\mu(xy) \leq \mu(y)$.

(iii) Let $a, b, x, z \in R$. If $x + a + z = b + z$ implies that $x \in L(\mu; t)$. Define $t = \min\{\mu(a), \mu(b)\}$. Then $\mu(x) \leq t = \min\{\mu(a), \mu(b)\}$. Thus $\mu(x) \leq \max\{\mu(a), \mu(b)\}$.

Hence, μ is a sensible S -fuzzy left h -ideal of R . ■

Definition 3.9: Let R be a hemiring and a family of fuzzy sets $\{\mu_i | i \in I\}$ in R . Then the union $\left(\bigvee_{i \in I} \mu_i\right)$ of $\{\mu_i | i \in I\}$ is defined by

$$\left(\bigvee_{i \in I} \mu_i\right)(x) = \sup\{\mu_i(x) | i \in I\}$$

Theorem 3.10: If $\{\mu_i | i \in I\}$ is a family of S -fuzzy left h -ideal of R , then $\left(\bigvee_{i \in I} \mu_i\right)(x)$ is a S -fuzzy left h -ideal of R .

Proof: Let $\{\mu_i | i \in I\}$ be a family of S -fuzzy left h -ideal of R . ■

(i) For all $x, y \in R$, we have

$$\begin{aligned} \left(\bigvee_{i \in I} \mu_i\right)(x+y) &= \sup \{\mu_i(x+y) | i \in I\} \\ &\leq \sup \{S(\mu_i(x), \mu_i(y)) | i \in I\} \\ &= S(\sup \{\mu_i(x) | i \in I\}, \sup \{\mu_i(y) | i \in I\}) \\ &= S\left(\left(\bigvee_{i \in I} \mu_i\right)(x), \left(\bigvee_{i \in I} \mu_i\right)(y)\right) \end{aligned}$$

(ii) For all $x, y \in R$, we have

$$\begin{aligned} \left(\bigvee_{i \in I} \mu_i\right)(xy) &= \sup \{\mu_i(xy) | i \in I\} \\ &\leq \sup \{S(\mu_i(x)) | i \in I\} \\ &= S\left(\left(\bigvee_{i \in I} \mu_i\right)(x)\right) \end{aligned}$$

(iii) For all $a, b, x, z \in R$ and if $x+a+z = b+z$ then

$$\begin{aligned} \left(\bigvee_{i \in I} \mu_i\right)(x) &= \sup \{\mu_i(x) | i \in I\} \\ &\leq \sup \{S(\mu_i(a), \mu_i(b)) | i \in I\} \\ &= S(\sup \{\mu_i(a) | i \in I\}, \sup \{\mu_i(b) | i \in I\}) \\ &= S\left(\left(\bigvee_{i \in I} \mu_i\right)(a), \left(\bigvee_{i \in I} \mu_i\right)(b)\right) \end{aligned}$$

Hence $\left(\bigvee_{i \in I} \mu_i\right)$ is a S -fuzzy left h -ideal of R . ■

Definition 3.11: Let $f : R \rightarrow R'$ be a mapping, where R and R' are non-empty sets and μ is a fuzzy subset of R . The *preimage* of μ under f written μ^f , is a fuzzy subset of R defined by $\mu^f = \mu(f(x))$, for all $x \in R$.

Theorem 3.12: Let $f : R \rightarrow R'$ be a homomorphism of hemirings. If μ is a S -fuzzy left h -ideal of R' , then μ^f is S -fuzzy left h -ideal of R .

Proof: Suppose μ is a S -fuzzy left h -ideal of R' , then

(i) For all $x, y \in R$, we have

$$\begin{aligned} \mu^f(x+y) &= \mu(f(x+y)) = \mu(f(x)+f(y)) \\ &\leq S(\mu(f(x)), \mu(f(y))) \\ &= S(\mu^f(x), \mu^f(y)) \end{aligned}$$

(ii) For all $x, y \in R$, we have

$$\begin{aligned} \mu^f(xy) &= \mu(f(xy)) = \mu(f(x)f(y)) \\ &\leq \mu(f(y)) = \mu^f(y) \end{aligned}$$

(iii) For all $a, b, x, z \in R$ and if $x+a+z = b+z$ then

$$\begin{aligned} \mu^f(x) &= \mu(f(x)) \\ &\leq S(\mu(f(a)), \mu(f(b))) \\ &= S(\mu^f(a), \mu^f(b)) \end{aligned}$$

Hence μ^f is a S -fuzzy left h -ideal of R . ■

Theorem 3.13: Let $f : R \rightarrow R'$ be a homomorphism of hemirings. If μ^f is a S -fuzzy left h -ideal of R , then μ is S -fuzzy left h -ideal of R' .

Proof: Suppose μ is a S -fuzzy left h -ideal of R' , then

(i) Let $x', y' \in R'$, there exists $x, y \in R$ such that $f(x) = x'$ and $f(y) = y'$, we have

$$\begin{aligned} \mu(x'+y') &= \mu(f(x)+f(y)) \\ &= \mu(f(x+y)) \\ &= \mu^f(x+y) \\ &\leq S(\mu^f(x), \mu^f(y)) \\ &= S(\mu(f(x)), \mu(f(y))) \\ &= S(\mu(x'), \mu(y')) \end{aligned}$$

(ii) Let $x', y' \in R'$, there exists $x, y \in R$ such that $f(x) = x'$ and $f(y) = y'$, we have

$$\begin{aligned} \mu(x'y') &= \mu(f(x)f(y)) = \mu(f(xy)) \\ &= \mu^f(xy) \\ &\leq \mu^f(y) \\ &= \mu(f(y)) \\ &= \mu(y') \end{aligned}$$

(iii) Let $a', b', x', z' \in R'$, there exists $a, b, x, z \in R$ such that $f(a) = a', f(b) = b', f(x) = x', f(z) = z'$. If $x'+a'+z' = b'+z'$. Then $f(x+a+z) = f(b+z)$ and so $f(x)+f(a)+f(z) = f(b)+f(z)$. It follows that

$$\begin{aligned} \mu(x') &= \mu(f(x)) \\ &= \mu^f(x) \\ &\leq S(\mu^f(a), \mu^f(b)) \\ &= S(\mu(f(a)), \mu(f(b))) \\ &= S(\mu(a'), \mu(b')) \end{aligned}$$

Hence μ is a S -fuzzy left h -ideal of R' . ■

Definition 3.14: Let f be a mapping defined on R . If ν is a fuzzy subset in $f(R)$, then the fuzzy subset $\mu = \nu \circ f$ in R (i.e., the fuzzy subset defined by $\mu(x) = \nu(f(x))$ for all $x \in R$) is called the *preimage* of ν under f .

Proposition 3.15: An onto homomorphic preimage of a S -fuzzy left h -ideal R is S -fuzzy left h -ideal.

Proof: The proof is obtained dually by using the notion of t -conorm S instead of t -norm T in [9, Proposition 3.10]. ■

Let μ be a fuzzy subset in a hemiring R and f be a mapping defined on R . Then the fuzzy subset μ^f in $f(R)$ defined by $\mu^f(y) = \inf_{x \in f^{-1}(y)} \mu(x)$ for all $y \in f(R)$ is called the *image* of μ under f . A fuzzy subset μ in R is said to have an *inf property* if for every subset $H \subseteq R$, there exists $h_0 \in H$ such that $\mu(h_0) = \inf_{h \in H} \mu(h)$.

Proposition 3.16: An onto homomorphic image of S -fuzzy left h -ideal with inf property is S -fuzzy left h -ideal.

Proof: Let $f : R \rightarrow R'$ be an onto homomorphism of semirings and let μ be a S -fuzzy left h -ideal of R with the inf property.

(i) Given $x', y' \in R'$, we let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ be such that

$$\mu(x_0) = \inf_{h \in f^{-1}(x')} \mu(h), \quad \mu(y_0) = \inf_{h \in f^{-1}(y')} \mu(h)$$

respectively. Then, we have

$$\begin{aligned} \mu^f(x' + y') &= \inf_{z \in f^{-1}(x'+y')} \mu(z) \leq \max\{\mu(x_0), \mu(y_0)\} \\ &\leq S(\mu(x_0), \mu(y_0)) \\ &= S\left(\inf_{h \in f^{-1}(x')} \mu(h), \inf_{h \in f^{-1}(y')} \mu(h)\right) \\ &= S(\mu^f(x'), \mu^f(y')) \end{aligned}$$

(ii) Given $x', y' \in R'$, we let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ be such that

$$\mu(x_0) = \inf_{h \in f^{-1}(x')} \mu(h), \quad \mu(y_0) = \inf_{h \in f^{-1}(y')} \mu(h)$$

respectively. Then, we have

$$\begin{aligned} \mu^f(x'y') &= \inf_{z \in f^{-1}(x'y')} \mu(z) \leq \mu(y_0) \\ &= \inf_{h \in f^{-1}(y')} \mu(h) = \mu^f(y') \end{aligned}$$

(ii) Given $a', b', x', y' \in R'$, we let $a_0 \in f^{-1}(a')$, $b_0 \in f^{-1}(b')$, $x_0 \in f^{-1}(x')$, $z_0 \in f^{-1}(z')$ be such that

$$\begin{aligned} \mu(a_0) &= \inf_{h \in f^{-1}(a')} \mu(h), \quad \mu(b_0) = \inf_{h \in f^{-1}(b')} \mu(h) \\ \mu(x_0) &= \inf_{h \in f^{-1}(x')} \mu(h), \quad \mu(z_0) = \inf_{h \in f^{-1}(z')} \mu(h) \end{aligned}$$

respectively. If $x' + a' + z' = b' + z'$ then $x_0 + a_0 + z_0 = b_0 + z_0$, where $(x_0 + a_0 + z_0) \in f^{-1}(x' + a' + z')$ and $(b_0 + z_0) \in f^{-1}(b' + z')$, we have

$$\begin{aligned} \mu^f(x') &= \inf_{z \in f^{-1}(x')} \mu(z) \leq \max\{\mu(a_0), \mu(b_0)\} \\ &= S\left(\inf_{h \in f^{-1}(a')} \mu(h), \inf_{h \in f^{-1}(b')} \mu(h)\right) \\ &= S(\mu^f(a'), \mu^f(b')) \end{aligned}$$

Hence, μ^f is a S -fuzzy left h -ideal of R' .

Definition 3.17: A t -conorm S on $[0, 1]$ is called a continuous t -conorm if S is a continuous function from $[0, 1] \times [0, 1] \rightarrow [0, 1]$ with respect to usual topology.

We observe that the function "max" is always a continuous t -conorm

Proposition 3.18: Let S be a continuous t -conorm and let f be a homomorphism on a hemiring R . If μ is a S -fuzzy left h -ideal of R , then μ^f is a S -fuzzy left h -ideal of $f(R)$.

Proof: Let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1 + y_2)$, where $y_1 + y_2 \in f(R)$. Consider the set $A_1 + A_2 = \{x \in R \mid x = a_1 + a_2 \text{ for some } a_1 \in A_1, a_2 \in A_2\}$

If $x \in A_1 + A_2$, then $x = x_1 + x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ so that we have $f(x) = f(x_1 + x_2) = f(x_1) + f(x_2) = y_1 + y_2$, that is, $x \in f^{-1}(y_1 + y_2) = A_{12}$. Thus,

$A_1 + A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned} \mu^f(y_1 + y_2) &= \inf\{\mu(x) \mid x \in f^{-1}(x_1 + x_2)\} \\ &= \inf\{\mu(x) \mid x \in A_{12}\} \\ &\leq \inf\{\mu(x) \mid x \in A_1 + A_2\} \\ &\leq \inf\{\mu(x_1 + x_2) \mid x_1 \in A_1, x_2 \in A_2\} \\ &\leq \inf\{S(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1, x_2 \in A_2\} \end{aligned}$$

Since S is continuous for every $\epsilon > 0$, we see that if

$$\begin{aligned} \inf\{\mu(x_1) \mid x_1 \in A_1\} - x_1^* &\leq \delta \text{ and} \\ \inf\{\mu(x_2) \mid x_2 \in A_2\} - x_2^* &\leq \delta, \text{ then} \end{aligned}$$

$$S(\inf\{\mu(x_1) \mid x_1 \in A_1\}, \inf\{\mu(x_2) \mid x_2 \in A_2\}) - S(x_1^*, x_2^*) \leq \epsilon$$

Choose $a_1 \in A_1$ and $a_2 \in A_2$, such that

$$\begin{aligned} \inf\{\mu(x_1) \mid x_1 \in A_1\} - \mu(a_1) &\leq \delta \text{ and} \\ \inf\{\mu(x_2) \mid x_2 \in A_2\} - \mu(a_2) &\leq \delta, \text{ then} \end{aligned}$$

$$\begin{aligned} S(\inf\{\mu(x_1) \mid x_1 \in A_1\}, \inf\{\mu(x_2) \mid x_2 \in A_2\}) \\ - S(\mu(a_1), \mu(a_2)) \leq \epsilon \end{aligned}$$

Thus, we have

$$\begin{aligned} (i) \mu^f(y_1 + y_2) &\leq \inf\{S(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1, x_2 \in A_2\} \\ &= S(\inf\{\mu(x_1) \mid x_1 \in A_1\}, \inf\{\mu(x_2) \mid x_2 \in A_2\}) \\ &= S(\mu^f(y_1), \mu^f(y_2)) \end{aligned}$$

(ii) Similarly, we can prove that

$$\mu^f(y_1 y_2) \leq \mu^f(y_2)$$

(iii) Now, let $a_1, b_1, x_1, z_1 \in f(R)$ be such that $x_1 + a_1 + z_1 = b_1 + z_1$. We can prove that

$$\mu^f(x_1) \leq S(\mu^f(a_1), \mu^f(b_1))$$

Hence, μ^f is a S -fuzzy left h -ideal of $f(R)$. ■

Lemma 3.19: Let T be a t -norm. Then t -conorm S can be defined as

$$S(x, y) = 1 - T(1 - x, 1 - y).$$

Proof: Straightforward. ■

Theorem 3.20: A fuzzy subset μ of R is a T -fuzzy left h -ideal if and only if its complement μ^c is a S -fuzzy left h -ideal of R .

Proof: Let μ be a T -fuzzy left h -ideal of R .

(i) For all $x, y \in R$, we have

$$\begin{aligned} \mu^c(x + y) &= 1 - \mu(x + y) \\ &\leq 1 - T(\mu(x), \mu(y)) \\ &= 1 - T(1 - \mu^c(x), 1 - \mu^c(y)) \\ &= S(\mu^c(x), \mu^c(y)) \end{aligned}$$

(ii) For all $x, y \in R$, we have

$$\mu^c(xy) = 1 - \mu(xy) \leq 1 - \mu(y) = \mu^c(y)$$

(iii) For all $a, b, x, z \in R$, if $x + a + z = b + z$ such that

$$\begin{aligned} \mu^c(x) &= 1 - \mu(x) \\ &\leq 1 - T(\mu(a), \mu(b)) \\ &= 1 - T(1 - \mu^c(a), 1 - \mu^c(b)) \\ &= S(\mu^c(a), \mu^c(b)) \end{aligned}$$

Hence $\mu^c(x)$ is a S -fuzzy left h -ideal of R .
The converse is proved similarly.

IV. CHAIN CONDITIONS

Let μ and ν be a fuzzy subset in a hemiring R .Then the S - h -product of μ and ν is defined by

$$\mu \circ_h \nu(x) = \begin{cases} \inf (S(\mu(a_i), \mu(b_i)) \mid i = 1, 2) & \text{if } x \text{ can be expressed as} \\ & x + a_1b_1 + z = a_2b_2 + z, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.1: Let μ and ν be a fuzzy subset of R .If they are S -fuzzy left h -ideal of R ,then so $\mu \cup \nu$,where $\mu \cup \nu$ is defined by $(\mu \cup \nu)(x) = S(\mu(x), \nu(x))$ for all $x \in R$.Moreover,If μ and ν are a S -fuzzy right h -ideal and a S -fuzzy left h -ideal respectively,then $\mu \circ_h \nu \subseteq \mu \cup \nu$

Proof: The proof is obtained dually by using the notion of t -conorm S instead of t -norm T in [9,proposition 4.2]. ■

Theorem 4.2: Let μ be a fuzzy subset in R and $Im(\mu) = \{\alpha_0, \alpha_1, \dots, \alpha_k\}$,where $\alpha_i < \alpha_j$ whenever $i > j$.Let $\{A_n \mid n = 0, 1, \dots, k\}$ be a family of ideals of R such that

- (i) $A_0 \subset A_1 \subset \dots \subset A_k = R$,
- (ii) $\mu(A_n^*) = \alpha_n$,where $A_n^* = A_n \setminus A_{n-1}, A_{-1} = \phi$ for $n = 0, 1, \dots, k$.

Then μ is a S -fuzzy left h -ideal of R .

Proof: Suppose $\{A_n \mid n = 0, 1, \dots, k\}$ be a family of ideals of R .

(i) For all $x, y \in R$,Then we discuss the following cases:If $x + y \in A_n$ and $y \in A_n$ such that $x \in A_n$,since A_n is an ideal of R .thus

$$\mu(x + y) \leq \alpha_n = S(\mu(x), \mu(y)).$$

If $x + y \notin A_n^*$ and $y \notin A_n^*$,then the following four cases arise:

- 1) $x + y \in R \setminus A_n$ and $y \in R \setminus A_n$
- 2) $x + y \in A_{n-1}$ and $y \in A_{n-1}$
- 3) $x + y \in R \setminus A_n$ and $y \in A_{n-1}$
- 4) $x + y \in A_{n-1}$ and $y \in R \setminus A_n$

But,in either cases,we know that

$$\mu(x + y) \leq S(\mu(x), \mu(y)).$$

If $x + y \in R \setminus A_n^*$ and $y \notin A_n^*$ then either $y \in A_{n-1}$ or $x \in R \setminus A_n$. It follows that either $x \in A_n$ or $x \in R \setminus A_n$.Thus

$$\mu(x + y) \leq S(\mu(x), \mu(y)).$$

If $x + y \notin R \setminus A_n^*$ and $y \in A_n^*$ then by similar process we have

$$\mu(x + y) \leq S(\mu(x), \mu(y)).$$

(ii) Similarly, for $x, y \in R$,we have

$$\mu(xy) \leq \mu(y).$$

(iii)For all $a, b, x, z \in R$,If $x + a + z = b + z$ such that $a \in A_n$ and $b \in A_n$ then $x \in A_n$.By the above process it is easy to show that

$$\mu(x) \leq S(\mu(a), \mu(b)).$$

Hence μ is a S -fuzzy left h -ideal of R . ■

Theorem 4.3: Let $\{A_n \mid n \in N\}$ be a family of h -ideals of a hemiring R which is nested,that is, $R = A_1 \supset A_2 \supset \dots$.Let μ be a fuzzy subset in R defined by

$$\mu(x) = \begin{cases} \frac{1}{n+1} & \text{if } x \in A_n \setminus A_{n+1}, n = 1, 2, 3, \dots, \\ 0 & \text{if } x \in \bigcap_{n=1}^{\infty} A_n. \end{cases}$$

for all $x \in R$.Then μ is a S -fuzzy left h -ideal of R .

Proof: Let $x, y \in R$.

(i)Suppose that $x \in A_k \setminus A_{k+1}$ and $y \in A_r \setminus A_{r+1}$ for $k = 1, 2, \dots; r = 1, 2, \dots$.Without loss of generality,we may assume that $k \leq r$.Then $x + y \in A_k$ and so

$$\mu(x + y) \leq \frac{1}{k+1} = \max\{\mu(x), \mu(y)\} \leq S(\mu(x), \mu(y))$$

If $x, y \in \bigcap_{n=1}^{\infty} A_n$ then $x + y \in \bigcap_{n=1}^{\infty} A_n$ and thus

$$\mu(x + y) = 0 = S(\mu(x), \mu(y))$$

If $x \in \bigcap_{n=1}^{\infty} A_n$ then $y \notin \bigcap_{n=1}^{\infty} A_n$,then there exists $i \in N$ such that $y \in A_i \setminus A_{i+1}$.It follows that $x + y \in A_i$ so that

$$\mu(x + y) \leq \frac{1}{i+1} = \max\{\mu(x), \mu(y)\} \leq S(\mu(x), \mu(y))$$

Similarly,we know that

$$\mu(x + y) \leq S(\mu(x), \mu(y))$$

for all $x \notin \bigcap_{n=1}^{\infty} A_n$ then $y \in \bigcap_{n=1}^{\infty} A_n$.

(ii)Now,if $y \in A_r \setminus A_{r+1}$ for some $k = 1, 2, \dots$,then $xy \in A_k$ for all $x \in R$.Thus

$$\mu(x + y) \leq \frac{1}{k+1} = \mu(y)$$

If $y \in \bigcap_{n=1}^{\infty} A_n$ then $xy \in \bigcap_{n=1}^{\infty} A_n$ for all $x \in R$.Thus

$$\mu(xy) = 0 = \mu(y)$$

(iii) Now,let $a, b, x, z \in R$ be such that $x + a + z = b + z$.If $a, b \in A_r \setminus A_{r+1}$ for some $r = 1, 2, 3, \dots$, then $x \in A_r$ as A_r is a left h -ideal of R .Thus

$$\mu(x) \leq \frac{1}{k+1} = \max\{\mu(a), \mu(b)\} \leq S(\mu(a), \mu(b))$$

If $a, b \in \bigcap_{n=1}^{\infty} A_n$ then $x \in \bigcap_{n=1}^{\infty} A_n$ and so

$$\mu(x) = 0 = S(\mu(a), \mu(b))$$

Assume that $a \in A_r \setminus A_{r+1}$ for some $r = 1, 2, 3, \dots$,and $b \in \bigcap_{n=1}^{\infty} A_n$ (or, $a \in \bigcap_{n=1}^{\infty} A_n$ and $b \in A_r \setminus A_{r+1}$ for some $r = 1, 2, 3, \dots$).Then $x \in A_r$ and so

$$\mu(x) \leq \frac{1}{r+1} = \max\{\mu(a), \mu(b)\} \leq S(\mu(a), \mu(b))$$

Hence, μ is a S -fuzzy left h -ideal of R . ■

Let $\mu : R \rightarrow [0, 1]$ be a fuzzy subset of R . The smallest S -fuzzy left h -ideal containing μ is called the S -fuzzy left h -ideal generated by μ , and μ is said to be n -valued if $\mu(R)$ is a finite set of n elements. When no specific n is intended, we call μ a finite-valued fuzzy subset.

Theorem 4.4: A S -fuzzy left h -ideal ν of R is finite valued if and only if a finite-valued fuzzy subset μ of R is generated by ν .

Proof: If $\nu : R \rightarrow [0, 1]$ is a finite-valued S -fuzzy left h -ideal of R , then one may choose $\mu = \nu$. Consequently, assume that $\mu : R \rightarrow [0, 1]$ is a n -valued fuzzy subset with n distinct values t_1, t_2, \dots, t_n , where $t_1 < t_2 < \dots < t_n$. Let G^i be the inverse image of t_i under μ , that is, $G^i = \mu^{-1}(t_i)$. Obviously, $\bigcup_{i=1}^j G^i \subseteq \bigcup_{i=1}^r G^i$ when $j < r$. We denote by A^j the left h -ideal of R generated by the set $\bigcup_{i=1}^j G^i$. Then we have the following chain of left h -ideals:

$$A^1 \supseteq A^2 \supseteq \dots \supseteq A^n = R$$

Define a fuzzy $\nu : R \rightarrow [0, 1]$ by

$$\nu(x) = \begin{cases} t_n & \text{if } x \in A^n, \\ t_j & \text{if } x \in A^j \setminus A^{j-1}; j = 1, 2, \dots, n-1 \end{cases}$$

We claim that ν is a S -fuzzy left h -ideal of R and μ is generated by ν . Let $x, y \in R$ and let i and j be the largest integer such that $x \in A^i$ and $y \in A^j$. We may assume that $i < j$ without loss of generality. Then $x + y \in A^i$ and $xy \in A^i$ and so

$$\begin{aligned} \nu(x + y) &\leq t_j = \max\{t_i, t_j\} = \max\{\nu(x), \nu(y)\} \\ &\leq S(\nu(x), \nu(y)) \end{aligned}$$

and

$$\nu(xy) \leq t_j = \nu(y)$$

Now, let $a, b, x, z \in R$ be such that $x + a + z = b + z$. If $a \in A^i$ and $b \in A^j$ for some $i < j$, then $a, b \in A^i$ and so $x \in A^i$ as A^i is a h -ideal of R . Thus

$$\begin{aligned} \nu(x) &\leq t_j = \max\{t_i, t_j\} = \max\{\nu(a), \nu(b)\} \\ &\leq S(\nu(a), \nu(b)) \end{aligned}$$

Hence, μ is a S -fuzzy left h -ideal of R .

If $x \in R$ and $\mu(x) = t_j$, then $x \in G^j$ and so $x \in A^j$. But we get $\nu(x) \leq t_j = \mu(x)$. Consequently, $\nu \subseteq \mu$. Let γ be any S -fuzzy left h -ideal of R which is a subset of μ . Then, $\bigcup_{i=1}^j G^i = L(\gamma; t_j) \subseteq L(\mu; t_j)$, and thus $A^j \subseteq L(\gamma; t_j)$. Hence, $\gamma \subseteq \mu$ and μ is generated by ν . Note that $|Im\mu| = n = |Im\nu|$. Thus completing the proof. ■

A semiring R is said to be *left h -artinian* (see [9]) if it satisfies the descending chain condition on left h -ideals of R .

Theorem 4.5: If R is a h -artinian hemiring, then every S -fuzzy left h -ideal of R is finite valued.

Proof: Let $\mu : R \rightarrow [0, 1]$ be a S -fuzzy left h -ideal of R which is not finite valued. Then, there exists sequence of distinct numbers $\mu(0) = t_1 > t_2 > \dots > t_n$, where $t_1 = \mu(x_i)$ for some $x_i \in R$. This sequence induces an infinite sequence of distinct left h -ideals of R :

$$L(\mu; t_1) \supset L(\mu; t_2) \supset \dots \supset L(\mu; t_n) \supset \dots$$

This is a contradiction. ■

Combining Theorem 8 and Theorem 9, we have the following corollary.

Corollary 4.6: If R is a h -artinian hemiring, then every S -fuzzy left h -ideal of R is generated by a finite fuzzy subset in R .

V. S-PRODUCT OF S-FUZZY LEFT h -IDEALS

Definition 5.1: (see [2]) A fuzzy relation on any set R is a fuzzy subset $\mu : R \times R$.

Definition 5.2: Let S be a t -conorm. If μ is a fuzzy relation on a set R and ν is a fuzzy set in R , then μ is a S -fuzzy relation on ν if $\mu_\nu(x, y) \geq S(\nu(x), \nu(y))$, for all $x, y \in R$.

Definition 5.3: Let S be a t -conorm. Let μ and ν be a fuzzy subset of R . Then direct S -product of μ and ν is defined by $(\mu \times \nu)(x, y) = S(\mu(x), \nu(y))$, for all $x, y \in R$.

Lemma 5.4: Let S be a t -conorm. Let μ and ν be a fuzzy subset of R . Then,

- (i) $\mu \times \nu$ is a S -fuzzy relation on S .
- (ii) $L(\mu \times \nu; t) = L(\mu; t) \times L(\nu; t)$, for all $t \in [0, 1]$

Proof: The proof is obvious. ■

Definition 5.5: Let S be a t -conorm. Let μ be a fuzzy subset of R , then μ is said to be the strongest S -fuzzy relation on R if $\mu_\nu(x, y) \geq S(\nu(x), \nu(y))$, for all $x, y \in R$.

Lemma 5.6: For given fuzzy subset ν in a set R , let μ_ν be the strongest S -fuzzy relation on R . Then

$$L(\mu_\nu; t) = L(\mu; t) \times L(\nu; t), \text{ for all } t \in [0, 1].$$

Proof: The proof is obvious. ■

Proposition 5.7: For given fuzzy subset ν in a set R , let μ_ν be the strongest S -fuzzy relation on R . If μ_ν is a sensible S -fuzzy left h -ideal of $R \times R$, then $\nu(a) \geq \nu(0)$ for $a \in R$.

Proof: If μ_ν is a sensible S -fuzzy left h -ideal of $R \times R$, then $\mu_\nu(a, a) \geq \mu_\nu(0, 0)$ for $a \in R$. This means $S(\nu(a), \nu(a)) \geq S(\nu(0), \nu(0))$ for $a \in R$. Since μ is sensible, then $\nu(a) \geq \nu(0)$ for $a \in R$. ■

The following proposition is an immediate consequence of lemma 5.6.

Proposition 5.8: Let μ and ν be S -fuzzy left h -ideal of R , then the level left h -ideals of μ_ν are given by $L(\mu_\nu; t) = L(\mu; t) \times L(\nu; t)$, for all $t \in R$.

Theorem 5.9: Let S be a t -conorm. Let μ and ν be S -fuzzy left h -ideal of R , then $\mu \times \nu$ is a S -fuzzy left h -ideal of $R \times R$.

Proof: Suppose μ and ν be S -fuzzy left h -ideal of R . Let $\mu \times \nu$ is a S -fuzzy left h -ideal of $R \times R$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any element of $R \times R$. Then,

(i)

$$\begin{aligned} (\mu \times \nu)(x + y) &= (\mu \times \nu)((x_1, x_2) + (y_1, y_2)) \\ &= (\mu \times \nu)((x_1 + y_1, x_2 + y_2)) \\ &= S(\mu(x_1 + y_1), \nu(x_2 + y_2)) \\ &\leq S(S(\mu(x_1), \mu(y_1)), S(\nu(x_2), \nu(y_2))) \\ &= S(S(\mu(x_1), \nu(x_2)), S(\mu(y_1), \nu(y_2))) \\ &= S((\mu \times \nu)(x_1, x_2), (\mu \times \nu)(y_1, y_2)) \\ &= S((\mu \times \nu)(x), (\mu \times \nu)(y)) \end{aligned}$$

(ii)

$$\begin{aligned} (\mu \times \nu)(xy) &= (\mu \times \nu)((x_1, x_2)(y_1, y_2)) \\ &= (\mu \times \nu)((x_1y_1, x_2y_2)) \\ &= S(\mu(y_1), \nu(y_2)) \\ &= (\mu \times \nu)(y_1, y_2) \\ &= (\mu \times \nu)(y) \end{aligned}$$

(iii) Let $x = (x_1, x_2)$, $z = (z_1, z_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be such that $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$. Then,

$$\begin{aligned} (\mu \times \nu)(x) &= (\mu \times \nu)((x_1, x_2)) \\ &= S(\mu(x_1), \nu(x_2)) \\ &\leq S(S(\mu(a_1), \mu(b_1))), S(\nu(a_2), \nu(b_2))) \\ &= S(S(\mu(a_1), \nu(a_2))), S(\mu(b_1), \nu(b_2))) \\ &= S((\mu \times \nu)(a_1, a_2), (\mu \times \nu)(b_1, b_2)) \\ &= S((\mu \times \nu)(a), (\mu \times \nu)(b)) \end{aligned}$$

Thus, $\mu \times \nu$ is a S -fuzzy left h -ideal of $R \times R$. ■

Corollary 5.10: Let S be a t -conorm. Let μ and ν be a sensible S -fuzzy left h -ideal of R , then $\mu \times \nu$ is a sensible S -fuzzy left h -ideal of $R \times R$.

Proof: By Theorem 5.9, we have $\mu \times \nu$ is a S -fuzzy left h -ideal of $R \times R$. Let $x = (x_1, x_2)$ be any element in $R \times R$, then

$$\begin{aligned} (\mu \times \nu)(x) &= (\mu \times \nu)((x_1, x_2)) \\ &= S(\mu(x_1), \nu(x_2)) \\ &= S(S(\mu(x_1), \mu(x_1))), S(\nu(x_2), \nu(x_2))) \\ &= S(S(\mu(x_1), \nu(x_2))), S(\mu(x_1), \nu(x_2))) \\ &= S((\mu \times \nu)(x_1, x_2), (\mu \times \nu)(x_1, x_2)) \\ &= S((\mu \times \nu)(x), (\mu \times \nu)(x)) \end{aligned}$$

Hence, $\mu \times \nu$ is a sensible S -fuzzy left h -ideal of $R \times R$. ■
As the converse of Corollary 5.10, we have a following question: If $\mu \times \nu$ is a sensible S -fuzzy left h -ideal of $R \times R$, then are both μ and ν sensible S -fuzzy left h -ideal of R ? The following example gives a negative answer.

Example 5.11: Let R be a hemiring with $|R| \geq 2$ and let $t \in [0, 1]$. Define a sensible fuzzy subset μ and ν in R by $\mu(x) = 1$ and

$$\nu(x) = \begin{cases} 1 & \text{if } x = 0, \\ t & \text{otherwise.} \end{cases}$$

for all $x \in R$, respectively.

If $x = 0$, then $\nu(x) = 1$, and thus

$$(\mu \times \nu)(x, x) = S(\mu(x), \nu(x)) = S(1, 1) = 1$$

If $x \neq 0$, then $\nu(x) = t$, and thus

$$(\mu \times \nu)(x, x) = S(\mu(x), \nu(x)) = S(1, t) = 1$$

That is, $\mu \times \nu$ is a constant function, and so $\mu \times \nu$ is a sensible S -fuzzy left h -ideal of $R \times R$. Now, μ is a sensible S -fuzzy left h -ideal of R , but ν is not a sensible S -fuzzy left h -ideal of R , since for $x \neq 0$, we have $\nu(0) = 1 > t = \nu(x)$.

Now, we generalize the product of two S -fuzzy left h -ideal of R to the product of n S -fuzzy left h -ideal. We first need to generalize the domain of t -conorm R to $\prod_{i=1}^n [0, 1]$ as follows.

Definition 5.12: The function $S_n : \prod_{i=1}^n [0, 1] \rightarrow [0, 1]$ is defined by

$$S_n(\alpha_1, \alpha_2, \dots, \alpha_n) =$$

$$S(\alpha_i, S_{n-1}(\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n))$$

for all $1 \leq i \leq n$, where $n \geq 2$, $S_2 = S$ and $S_1 = \text{identity}$.

Lemma 5.13: For a t -conorm S and every $\alpha_i, \beta_i \in [0, 1]$, where $1 \leq i \leq n, n \geq 2$, we have

$$\begin{aligned} S_n(S(\alpha_1, \beta_1), S(\alpha_2, \beta_2), \dots, S(\alpha_n, \beta_n)) \\ = S(S_n(\alpha_1, \alpha_2, \dots, \alpha_n), S_n(\beta_1, \beta_2, \dots, \beta_n)). \end{aligned}$$

Proposition 5.14: Let S be a t -conorm. Let $\{R_i\}_{i=1}^n$ be the finite collection of hemirings and $R = \prod_{i=1}^n R_i$ the S -product of S_i . Let μ_i be a S -fuzzy left h -ideal of S_i , where $1 \leq i \leq n$. Then, $\mu = \prod_{i=1}^n \mu_i$ defined by

$$\begin{aligned} \mu(x_1, x_2, \dots, x_n) &= \prod_{i=1}^n \mu_i(x_1, x_2, \dots, x_n) \\ &= S_n(\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)) \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in R$ is a S -fuzzy left h -ideal of R .

Proof: The proof is similar to the proof of Theorem 10. ■

Definition 5.15: Let μ and ν be fuzzy subset in R . Then, the S -product of μ and ν , written as

$$[\mu \cdot \nu]_S(x) = S(\mu(x), \nu(x))$$

for all $x \in R$.

Theorem 5.16: Let μ and ν be S -fuzzy left h -ideal of R . If S^* is a t -conorm which dominates S , that is,

$$S^*(S(\alpha, \beta), S(\gamma, \delta)) \geq S^*(S^*(\alpha, \beta), S^*(\gamma, \delta))$$

for all $\alpha, \beta, \gamma, \delta \in R$. Then S^* -product of μ and ν , $[\mu \cdot \nu]_{S^*}$ is a S^* -fuzzy left h -ideal of S .

Proof: Let $x, y \in R$, then we have

(i)

$$\begin{aligned} [\mu \cdot \nu]_{S^*}(x + y) &= S^*(\mu(x + y), \nu(x + y)) \\ &\leq S^*(S(\mu(x), \mu(y)), S(\nu(x), \nu(y))) \\ &\leq S^*(S(\mu(x), \nu(x)), S(\mu(y), \nu(y))) \\ &= S([\mu \cdot \nu]_{S^*}(x), [\mu \cdot \nu]_{S^*}(y)) \end{aligned}$$

(ii)

$$\begin{aligned} [\mu \cdot \nu]_{S^*}(xy) &= S^*(\mu(xy), \nu(xy)) \\ &\leq S^*(\mu(y), \nu(y)) \\ &= [\mu \cdot \nu]_{S^*}(y) \end{aligned}$$

(iii) Now, let $a, b, x, z \in R$ be such that $x + a + z = b + z$. Then

$$\begin{aligned} [\mu \cdot \nu]_{S^*}(x) &= S^*(\mu(x), \nu(x)) \\ &\leq S^*(S(\mu(a), \mu(b)), S(\nu(a), \nu(b))) \\ &\leq S^*(S(\mu(a), \nu(a)), S(\mu(b), \nu(b))) \\ &= S([\mu \cdot \nu]_{S^*}(a), [\mu \cdot \nu]_{S^*}(b)) \end{aligned}$$

Hence, $[\mu \cdot \nu]_{S^*}$ is a S^* -fuzzy left h -ideal of R . ■

Theorem 5.17: Let $R \rightarrow R'$ be an onto homomorphism of hemirings. Let S^* be a t -conorm such that S^* dominates S . Let μ and ν be S -fuzzy left h -ideal of S' . If $[\mu \cdot \nu]_{S^*}$ is the S^* -product of μ and ν , and $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{S^*}$ is the S^* -product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$, then

$$f^{-1}([\mu \cdot \nu]_{S^*}) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{S^*}$$

Proof: Let $x \in R$, then we have

$$\begin{aligned} f^{-1}([\mu \cdot \nu]_{S^*})(x) &= [\mu \cdot \nu]_{S^*}(f(x)) \\ &= S^*(\mu(f(x)), \nu(f(x))) \\ &= S^*(f^{-1}(\mu(x)), f^{-1}(\nu(x))) \\ &= [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{S^*}(x) \end{aligned}$$

Theorem 5.18: Let ν be a sensible fuzzy subset of R . Let μ_ν be the strongest S -fuzzy relation on R . Then ν is a sensible S -fuzzy left h -ideal of R if and only if μ_ν is a sensible S -fuzzy left h -ideal of $R \times R$.

Proof: Suppose that ν is a sensible S -fuzzy left h -ideal of R . Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of $R \times R$. Then,

$$\begin{aligned} \mu_\nu(x+y) &= \mu_\nu((x_1, x_2) + (y_1, y_2)) \\ &= \mu_\nu((x_1+y_1, x_2+y_2)) \\ &= S(\nu(x_1+y_1), \nu(x_2+y_2)) \\ &\leq S(S(\nu(x_1), \nu(y_1)), S(\nu(x_2), \nu(y_2))) \\ &= S(S(\nu(x_1), \nu(x_2)), S(\nu(y_1), \nu(y_2))) \\ &= S(\mu_\nu(x_1, x_2), S(\mu_\nu(y_1, y_2))) \\ &= S(\mu_\nu(x), S(\mu_\nu(y))) \end{aligned}$$

$$\begin{aligned} \mu_\nu(xy) &= \mu_\nu((x_1, x_2)(y_1, y_2)) \\ &= \mu_\nu((x_1y_1, x_2y_2)) \\ &= S(\nu(x_1y_1), \nu(x_2y_2)) \\ &\leq \mu_\nu((x_1, x_2)(y_1, y_2)) \\ &= \mu_\nu(y_1, y_2) \\ &= \mu_\nu(y) \end{aligned}$$

(iii) Let $x = (x_1, x_2)$, $z = (z_1, z_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be such that $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$. Then,

$$\begin{aligned} \mu_\nu(x) &= \mu_\nu((x_1, x_2)) \\ &= S(\nu(x_1), \nu(x_2)) \\ &\leq S(S(\nu(a_1), \nu(b_1)), S(\nu(a_2), \nu(b_2))) \\ &= S(S(\nu(a_1), \nu(a_2)), S(\nu(b_1), \nu(b_2))) \\ &= S(\mu_\nu(a_1, a_2), S(\mu_\nu(b_1, b_2))) \\ &= S(\mu_\nu(a), S(\mu_\nu(b))) \end{aligned}$$

Thus, μ_ν is a S -fuzzy left h -ideal of $R \times R$.

(iv) For any $x = (x_1, x_2) \in R \times R$, then

$$\begin{aligned} S(\mu_\nu(x), \mu_\nu(x)) &= S(\mu_\nu(x_1, x_2), \mu_\nu(x_1, x_2)) \\ &= S(S(\nu(x_1), \nu(x_2)), S(\nu(x_1), \nu(x_2))) \\ &= S(S(\nu(x_1), \nu(x_1)), S(\nu(x_2), \nu(x_2))) \\ &= S(\nu(x_1), \nu(x_2)) \\ &= \mu_\nu(x_1, x_2) \\ &= \mu_\nu(x) \end{aligned}$$

Hence, μ_ν is a sensible S -fuzzy left h -ideal of R .

Conversely, suppose that μ_ν is a sensible S -fuzzy left h -ideal of $R \times R$. Let $x, y \in R$, we have

$$\begin{aligned} \nu(x+y) &= S(\nu(x+y), \nu(x+y)) \\ &= \mu_\nu(x+y, x+y) \\ &= \mu_\nu((x, x) + (y, y)) \\ &\leq S(\mu_\nu(x, x), \mu_\nu(y, y)) \\ &= S(S(\nu(x), \nu(x)), S(\nu(y), \nu(y))) \\ &= S(\nu(x), \nu(y)) \end{aligned}$$

$$\begin{aligned} \nu(xy) &= S(\nu(xy), \nu(xy)) \\ &= \mu_\nu(xy, xy) \\ &\leq \mu_\nu(y, y) \\ &= S(\nu(y), \nu(y)) \\ &= \nu(y) \end{aligned}$$

(iii) Let $a, b, x, z \in R$ be such that $(x, x) + (a, a) + (z, z) = (b, b) + (z, z)$. Since μ_ν is a sensible S -fuzzy left h -ideal of $R \times R$. Then

$$\begin{aligned} \nu(x) &= S(\nu(x), \nu(x)) \\ &= \mu_\nu(x, x) \\ &= \mu_\nu((x, x) + (y, y)) \\ &\leq S(\mu_\nu(a, a), \mu_\nu(b, b)) \\ &= S(S(\nu(a), \nu(a)), S(\nu(b), \nu(b))) \\ &= S(\nu(a), \nu(b)) \end{aligned}$$

Consequently, ν is a sensible S -fuzzy left h -ideal of R . ■

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