Robust $\mathcal{H}_\infty$ Filter Design for Uncertain Fuzzy Descriptor Systems: LMI-Based Design

Wudhichai Assawinchaichote* and Sing Kiong Nguang

Abstract— **This paper examines the problem of designing a robust $\mathcal{H}_\infty$ filter for a class of uncertain fuzzy descriptor systems described by a Takagi-Sugeno (TS) fuzzy model. Based on a linear matrix inequality (LMI) approach, LMI-based sufficient conditions for the uncertain nonlinear descriptor systems to have an $\mathcal{H}_\infty$ performance are derived. To alleviate the ill-conditioning resulting from the interaction of slow and fast dynamic modes, solutions to the problem are given in terms of linear matrix inequalities which are independent of the singular perturbation ε, when ε is sufficiently small. The proposed approach does not involve the separation of states into slow and fast ones and it can be applied not only to standard, but also to nonstandard uncertain nonlinear descriptor systems. A numerical example is provided to illustrate the design developed in this paper.**

Index Terms— $\mathcal{H}_\infty$ **control; Takagi-Sugeno (TS) fuzzy model; Linear Matrix Inequalities (LMIs); Descriptor systems**

I. INTRODUCTION

The problem of control design for descriptor systems system has been intensively studied by a number of researchers for the past three decades; see [1]-[6]. This is due not only to theoretical interest but also to the relevance of this topic in control engineering applications. Descriptor systems or so called singularly perturbed systems are dynamical systems with multiple time-scales. Descriptor systems often occur naturally due to the presence of small "parasitic" parameter, typically small time constants, masses, etc.

The main purpose of the singular perturbation approach to analysis and design is the alleviation of high dimensionality and ill-conditioning resulting from the interaction of slow and fast dynamics modes. The separation of states into slow and fast ones is a nontrivial modelling task demanding insight and ingenuity on the part of the analyst. In state space, such systems are commonly modelled using the mathematical framework of singular perturbations, with a small parameter, say ε, determining the degree of separation between the "slow" and "fast" modes of the system.

In the last few years, many researchers have studied the $\mathcal{H}_\infty$ filter design for a general class of linear descriptor systems. In [3], the authors have investigated the decomposition solution of $\mathcal{H}_\infty$ filter gain for singularly perturbed systems.

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The reduced-order $\mathcal{H}_\infty$ optimal filtering for system with slow and fast modes has been considered in [4]. Although many researchers have studied linear descriptor systems for many years, the $\mathcal{H}_\infty$ filtering design for nonlinear descriptor systems remains as an open research area. This is because, in general, nonlinear singularly perturbed systems can not be easily separated into slow and fast subsystems.

Fuzzy system theory enables us to utilize qualitative, linguistic information about a highly complex nonlinear system to construct a mathematical model for it. Recent studies show that a fuzzy linear model can be used to approximate global behaviors of a highly complex nonlinear system; see for example, [7]-[19]. In this fuzzy linear model, local dynamics in different state space regions are represented by local linear systems. The overall model of the system is obtained by "blending" these linear models through nonlinear fuzzy membership functions. Unlike conventional modelling where a single model is used to describe the global behaviour of a system, the fuzzy modelling is essentially a multi-model approach in which simple sub-models (linear models) are combined to describe the global behaviour of the system.

What we intend to do in this paper is to design a robust $\mathcal{H}_\infty$ filter for a class of nonlinear descriptor systems with nonlinear on both fast and slow variables. First, we approximate this class of nonlinear descriptor systems by a Takagi-Sugeno fuzzy model. Then based on an LMI approach, we develop an $\mathcal{H}_\infty$ filter such that the $\mathcal{L}_2$-gain from an exogenous input to an estimate error is less or equal to a prescribed value. To alleviate the ill-conditioning resulting from the interaction of slow and fast dynamic modes, solutions to the problem are given in terms of linear matrix inequalities which are independent of the singular perturbation ε, when ε is sufficiently small. The proposed approach does not involve the separation of states into slow and fast ones and it can be applied not only to standard, but also to nonstandard nonlinear descriptor systems.

This paper is organized as follows. In Section II, system descriptions and definitions are presented. In Section III, based on an LMI approach, we develop a technique for designing a robust $\mathcal{H}_\infty$ filter for the system described in section II. The validity of this approach is demonstrated by an example from a literature in Section IV. Finally in Section V, conclusions are given.

II. SYSTEM DESCRIPTIONS

In this section, we generalize the TS fuzzy system to represent a TS fuzzy descriptor system with parametric

uncertainties. As in [19], we examine a TS fuzzy descriptor system with parametric uncertainties as follows:

$$
\begin{aligned}
E_\varepsilon \dot{x}(t) &= \sum_{i=1}^r \mu_i(\nu(t)) \Big[[A_i + \Delta A_i] x(t) \\
&+ [B_{1_i} + \Delta B_{1_i}] w(t) + [B_{2_i} + \Delta B_{2_i}] u(t) \Big], \quad x(0) = 0 \\
z(t) &= \sum_{i=1}^r \mu_i(\nu(t)) \Big[[C_{1_i} + \Delta C_{1_i}] x(t) \\
&+ [D_{12_i} + \Delta D_{12_i}] u(t) \Big] \\
y(t) &= \sum_{i=1}^r \mu_i(\nu(t)) \Big[[C_{2_i} + \Delta C_{2_i}] x(t) \\
&+ [D_{21_i} + \Delta D_{21_i}] w(t) \Big]
\end{aligned}
\tag{1}
$$

where $E_\varepsilon = \begin{bmatrix} I & 0 \\ 0 & \varepsilon I \end{bmatrix}$, $\varepsilon > 0$ is the singular perturbation parameter, $\nu(t) = [\nu_1(t) \; \cdots \; \nu_\vartheta(t)]$ is the premise variable vector that may depend on states in many cases, $\mu_i(\nu(t))$ denotes the normalized time-varying fuzzy weighting functions for each rule (i.e., $\mu_i(\nu(t)) \geq 0$ and $\sum_{i=1}^r \mu_i(\nu(t)) = 1$), ϑ is the number of fuzzy sets, $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the input, $w(t) \in \Re^p$ is the disturbance which belongs to $\mathcal{L}_2[0, \infty)$, $y(t) \in \Re^\ell$ is the measurement and $z(t) \in \Re^s$ is the controlled output, the matrices $A_i, B_{1_i}, B_{2_i}, C_{1_i}, C_{2_i}, D_{12_i}$ and D_{21_i} are of appropriate dimensions, and the matrices ΔA_i, ΔB_{1_i}, ΔB_{2_i}, ΔC_{1_i}, ΔC_{2_i}, ΔD_{12_i} and ΔD_{21_i} represent the uncertainties in the system and satisfy the following assumption.

Assumption 1:

$$
\Delta A_i = F(x(t), t) H_{1_i}, \quad \Delta B_{1_i} = F(x(t), t) H_{2_i},
$$

$$
\Delta B_{2_i} = F(x(t), t) H_{3_i}, \quad \Delta C_{1_i} = F(x(t), t) H_{4_i},
$$

$$
\Delta C_{2_i} = F(x(t), t) H_{5_i}, \quad \Delta D_{12_i} = F(x(t), t) H_{6_i}
$$

$$
and \quad \Delta D_{21_i} = F(x(t), t) H_{7_i}
$$

where H_{j_i}, $j = 1, 2, \cdots, 7$ are known matrix functions which characterize the structure of the uncertainties. Furthermore, the following inequality holds:

$$
\| F(x(t), t) \| \leq \rho
\tag{2}
$$

for any known positive constant ρ.

Next, let us recall the following definition.

Definition 1: Suppose γ is a given positive number. A system (1) is said to have an $\mathcal{L}_2$-gain less than or equal to γ if

$$
\int_0^{T_f} \Big(z(t) - \hat{z}(t) \Big)^T \Big(z(t) - \hat{z}(t) \Big) dt
$$

$$
\leq \gamma^2 \left[\int_0^{T_f} w^T(t) w(t) dt \right]
\tag{3}
$$

with $x(0) = 0$, where $(z(t) - \hat{z}(t))$ is the estimated error output, for all $T_f \geq 0$ and $w(t) \in \mathcal{L}_2[0, T_f]$.

III. ROBUST $\mathcal{H}_\infty$ FUZZY FILTER DESIGN

Without loss of generality, in this section, we assume that $u(t) = 0$. Let us recall the system (1) with $u(t) = 0$ as follows:

$$
\begin{aligned}
E_\varepsilon \dot{x}(t) &= \sum_{i=1}^r \mu_i \Big[[A_i + \Delta A_i] x(t) \\
&+ [B_{1_i} + \Delta B_{1_i}] w(t) \Big], \quad x(0) = 0 \\
z(t) &= \sum_{i=1}^r \mu_i \Big[[C_{1_i} + \Delta C_{1_i}] x(t) \Big] \\
y(t) &= \sum_{i=1}^r \mu_i \Big[[C_{2_i} + \Delta C_{2_i}] x(t) \\
&+ [D_{21_i} + \Delta D_{21_i}] w(t) \Big].
\end{aligned}
\tag{4}
$$

We are now aiming to design a full order dynamic $\mathcal{H}_\infty$ fuzzy filter of the form

$$
\begin{aligned}
E_\varepsilon \dot{\hat{x}}(t) &= \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j \Big[\hat{A}_{ij}(\varepsilon) \hat{x}(t) + \hat{B}_i y(t) \Big] \\
\hat{z}(t) &= \sum_{i=1}^r \hat{\mu}_i \hat{C}_i \hat{x}(t)
\end{aligned}
\tag{5}
$$

where $\hat{x}(t) \in \Re^n$ is the filter's state vector, $\hat{z} \in \Re^s$ is the estimate of $z(t)$, $\hat{A}_{ij}(\varepsilon)$, $\hat{B}_i$ and $\hat{C}_i$ are parameters of the filter which are to be determined, and $\hat{\mu}_i$ denotes the normalized time-varying fuzzy weighting functions for each rule (i.e., $\hat{\mu}_i \geq 0$ and $\sum_{i=1}^r \hat{\mu}_i = 1$), such that the inequality (3) holds. Clearly, in real control problems, all of the premise variables are not necessarily measurable. In this section, we then consider the designing of the robust $\mathcal{H}_\infty$ fuzzy filter into two cases as follows.

A. Case I–$\nu(t)$ is available for feedback

The premise variable of the fuzzy model $\nu(t)$ is available for feedback which implies that μ_i is available for feedback. Thus, we can select our filter that depends on μ_i as follows:

$$
\begin{aligned}
E_\varepsilon \dot{\hat{x}}(t) &= \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j \Big[\hat{A}_{ij}(\varepsilon) \hat{x}(t) + \hat{B}_i y(t) \Big] \\
\hat{z}(t) &= \sum_{i=1}^r \mu_i \hat{C}_i \hat{x}(t).
\end{aligned}
\tag{6}
$$

Before presenting our next results, the following lemma is recalled.

Lemma 1: Consider the system (4). Given a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ and a positive constant δ, if there exist matrices $X_\varepsilon = X_\varepsilon^T$, $Y_\varepsilon = Y_\varepsilon^T$, $\mathcal{B}_i(\varepsilon)$ and $\mathcal{C}_i(\varepsilon)$, $i = 1, 2, \cdots, r$, satisfying the following ε-dependent linear matrix inequalities:

$$
\begin{bmatrix} X_\varepsilon & I \\ I & Y_\varepsilon \end{bmatrix} > 0
\tag{7}
$$

$$
X_\varepsilon > 0
\tag{8}
$$

$$
Y_\varepsilon > 0
\tag{9}
$$

$$
\Psi_{11_{ii}}(\varepsilon) < 0, \quad i = 1, 2, \cdots, r
\tag{10}
$$

$$
\Psi_{22_{ii}}(\varepsilon) < 0, \quad i = 1, 2, \cdots, r
\tag{11}
$$

$$
\Psi_{11_{ij}}(\varepsilon) + \Psi_{11_{ji}}(\varepsilon) < 0, \quad i < j \leq r
\tag{12}
$$

$$
\Psi_{22_{ij}}(\varepsilon) + \Psi_{22_{ji}}(\varepsilon) < 0, \quad i < j \leq r
\tag{13}
$$

where

$$\Psi_{11_{ij}}(\varepsilon) = \begin{pmatrix} \begin{pmatrix} E_\varepsilon^{-1} A_i Y_\varepsilon + Y_\varepsilon A_i^T E_\varepsilon^{-1} \\ + \gamma^{-2} E_\varepsilon^{-1} \tilde{B}_{1_i} \tilde{B}_{1_j}^T E_\varepsilon^{-1} \end{pmatrix} & (*)^T \\ \left[Y_\varepsilon \tilde{C}_{1_i}^T + E_\varepsilon^{-1} \mathcal{C}_i^T(\varepsilon) \tilde{D}_{12}^T \right]^T & -I \end{pmatrix}$$

$$\Psi_{22_{ij}}(\varepsilon) = \begin{pmatrix} \begin{pmatrix} A_i^T E_\varepsilon^{-1} X_\varepsilon + X_\varepsilon E_\varepsilon^{-1} A_i \\ + \mathcal{B}_i(\varepsilon) C_{2_i} + C_{2_i}^T \mathcal{B}_j^T(\varepsilon) \\ + \tilde{C}_{1_i}^T \tilde{C}_{1_j} \end{pmatrix} & (*)^T \\ \left[X_\varepsilon E_\varepsilon^{-1} \tilde{B}_{1_i} + \mathcal{B}_i(\varepsilon) \tilde{D}_{21_j} \right]^T & -\gamma^2 I \end{pmatrix}$$

with

$$\tilde{B}_{1_i} = \begin{bmatrix} \delta I & I & 0 & B_{1_i} & 0 \end{bmatrix},$$

$$\tilde{C}_{1_i} = \begin{bmatrix} \frac{\gamma\rho}{\delta} H_{1_i}^T & \frac{\gamma\rho}{\delta} H_{5_i}^T & \sqrt{2}\lambda\rho H_{4_i}^T & \sqrt{2}\lambda C_{1_i}^T \end{bmatrix}^T,$$

$$\tilde{D}_{12} = \begin{bmatrix} 0 & 0 & 0 & -\sqrt{2}\lambda I \end{bmatrix}^T,$$

$$\tilde{D}_{21_i} = \begin{bmatrix} 0 & 0 & \delta I & D_{21_i} & I \end{bmatrix}$$

and $\lambda = \left(1 + \rho^2 \sum_{i=1}^r \sum_{j=1}^r \left[\| H_{2_i}^T H_{2_j} \| + \| H_{7_i}^T H_{7_j} \| \right] \right)^{\frac{1}{2}},$

then the prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ is guaranteed. Furthermore, a suitable filter is of the form (6) with

$$\begin{aligned} \hat{A}_{ij}(\varepsilon) &= E_\varepsilon \left[Y_\varepsilon^{-1} - X_\varepsilon \right]^{-1} \mathcal{M}_{ij}(\varepsilon) Y_\varepsilon^{-1} \\ \hat{B}_i &= E_\varepsilon \left[Y_\varepsilon^{-1} - X_\varepsilon \right]^{-1} \mathcal{B}_i(\varepsilon) \\ \hat{C}_i &= \mathcal{C}_i(\varepsilon) E_\varepsilon^{-1} Y_\varepsilon^{-1} \end{aligned} \qquad (14)$$

where

$$\mathcal{M}_{ij}(\varepsilon) = -A_i^T E_\varepsilon^{-1} - X_\varepsilon E_\varepsilon^{-1} A_i Y_\varepsilon$$
$$- \left[Y_\varepsilon^{-1} - X_\varepsilon \right] E_\varepsilon^{-1} \hat{B}_i C_{2_j} Y_\varepsilon - \tilde{C}_{1_i}^T \left[\tilde{C}_{1_j} Y_\varepsilon + \tilde{D}_{12} \hat{C}_j Y_\varepsilon \right]$$
$$- \gamma^{-2} \left\{ X_\varepsilon E_\varepsilon^{-1} \tilde{B}_{1_i} + \left[Y_\varepsilon^{-1} - X_\varepsilon \right] E_\varepsilon^{-1} \hat{B}_i \tilde{D}_{21_i} \right\} \tilde{B}_{1_i}^T E_\varepsilon^{-1}.$$

Proof: Due to limited pages, the proof has been omitted. ∎

Remark 1: The LMIs given in Lemma 1 may become ill-conditioned when ε is sufficiently small, which is always the case for the descriptor systems. In general, these ill-conditioned LMIs are very difficult to solve. Thus, to alleviate these ill-conditioned LMIs, we have the following ε-independent well-posed LMI-based sufficient conditions for the uncertain fuzzy descriptor systems to obtain the prescribed $\mathcal{H}_\infty$ performance. □

Theorem 1: Consider the system (4). Given a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ and a positive constant δ, if there exist matrices X_0, Y_0, $\mathcal{B}_{0_i}$ and $\mathcal{C}_{0_i}$, $i = 1, 2, \cdots, r$, satisfying the following ε-independent linear matrix inequalities:

$$\begin{bmatrix} X_0 E + D X_0 & I \\ I & Y_0 E + D Y_0 \end{bmatrix} > 0 \quad (15)$$

$$E X_0^T = X_0 E, \quad X_0^T D = D X_0, \quad X_0 E + D X_0 > 0 \quad (16)$$

$$E Y_0^T = Y_0 E, \quad Y_0^T D = D Y_0, \quad Y_0 E + D Y_0 > 0 \quad (17)$$

$$\Psi_{11_{ii}} < 0, \quad i = 1, 2, \cdots, r \quad (18)$$

$$\Psi_{22_{ii}} < 0, \quad i = 1, 2, \cdots, r \quad (19)$$

$$\Psi_{11_{ij}} + \Psi_{11_{ji}} < 0, \quad i < j \le r \quad (20)$$

$$\Psi_{22_{ij}} + \Psi_{22_{ji}} < 0, \quad i < j \le r \quad (21)$$

where $E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$

$$\Psi_{11_{ij}} = \begin{pmatrix} A_i Y_0^T + Y_0 A_i^T + \gamma^{-2} \tilde{B}_{1_i} \tilde{B}_{1_j}^T & (*)^T \\ \left[Y_0 \tilde{C}_{1_i}^T + \mathcal{C}_{0_i}^T \tilde{D}_{12}^T \right]^T & -I \end{pmatrix}$$

$$\Psi_{22_{ij}} = \begin{pmatrix} \begin{pmatrix} A_i^T X_0 + X_0 A_i + \mathcal{B}_{0_i} C_{2_j} \\ + C_{2_i}^T \mathcal{B}_{0_j}^T + \tilde{C}_{1_i}^T \tilde{C}_{1_j} \end{pmatrix} & (*)^T \\ \left[X_0 \tilde{B}_{1_i} + \mathcal{B}_{0_i} \tilde{D}_{21_j} \right]^T & -\gamma^2 I \end{pmatrix}$$

with

$$\tilde{B}_{1_i} = \begin{bmatrix} \delta I & I & 0 & B_{1_i} & 0 \end{bmatrix},$$

$$\tilde{C}_{1_i} = \begin{bmatrix} \frac{\gamma\rho}{\delta} H_{1_i}^T & \frac{\gamma\rho}{\delta} H_{5_i}^T & \sqrt{2}\lambda\rho H_{4_i}^T & \sqrt{2}\lambda C_{1_i}^T \end{bmatrix}^T,$$

$$\tilde{D}_{12} = \begin{bmatrix} 0 & 0 & 0 & -\sqrt{2}\lambda I \end{bmatrix}^T,$$

$$\tilde{D}_{21_i} = \begin{bmatrix} 0 & 0 & \delta I & D_{21_i} & I \end{bmatrix}$$

and $\lambda = \left(1 + \rho^2 \sum_{i=1}^r \sum_{j=1}^r \left[\| H_{2_i}^T H_{2_j} \| + \| H_{7_i}^T H_{7_j} \| \right] \right)^{\frac{1}{2}},$

then there exists a sufficiently small $\hat{\varepsilon} > 0$ such that for $\varepsilon \in (0, \hat{\varepsilon}]$, the prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ is guaranteed. Furthermore, a suitable filter is of the form (6) with

$$\begin{aligned} \hat{A}_{ij}(\varepsilon) &= \left[Y_\varepsilon^{-1} - X_\varepsilon \right]^{-1} \mathcal{M}_{0_{ij}}(\varepsilon) Y_\varepsilon^{-1} \\ \hat{B}_i &= \left[Y_0^{-1} - X_0 \right]^{-1} \mathcal{B}_{0_i} \\ \hat{C}_i &= \mathcal{C}_{0_i} Y_0^{-1} \end{aligned} \qquad (22)$$

where

$$\mathcal{M}_{0_{ij}}(\varepsilon) = -A_i^T - X_\varepsilon A_i Y_\varepsilon - \left[Y_\varepsilon^{-1} - X_\varepsilon \right] \hat{B}_i C_{2_j} Y_\varepsilon$$
$$- \tilde{C}_{1_i}^T \left[\tilde{C}_{1_j} Y_\varepsilon + \tilde{D}_{12} \hat{C}_j Y_\varepsilon \right]$$
$$- \gamma^{-2} \left\{ X_\varepsilon \tilde{B}_{1_i} + \left[Y_\varepsilon^{-1} - X_\varepsilon \right] \hat{B}_i \tilde{D}_{21_i} \right\} \tilde{B}_{1_j}^T$$
$$X_\varepsilon = \left\{ X_0 + \varepsilon \tilde{X} \right\} E_\varepsilon \; and \; Y_\varepsilon^{-1} = \left\{ Y_0^{-1} + \varepsilon N_\varepsilon \right\} E_\varepsilon$$

with $\tilde{X} = D \left(X_0^T - X_0 \right)$ *and* $N_\varepsilon = D \left((Y_0^{-1})^T - Y_0^{-1} \right).$

Proof: The proof has been omitted for brevity. ∎

B. Case II–$\nu(t)$ is unavailable for feedback

Now, the premise variable of the fuzzy model $\nu(t)$ is unavailable for feedback which implies μ_i is unavailable for feedback. Hence, we cannot select our filter which depends on μ_i. Thus, we select our filter as follows:

$$\begin{aligned} E_\varepsilon \dot{\hat{x}}(t) &= \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j \left[\hat{A}_{ij}(\varepsilon) \hat{x}(t) + \hat{B}_i y(t) \right] \\ \hat{z}(t) &= \sum_{i=1}^r \hat{\mu}_i \hat{C}_i \hat{x}(t) \end{aligned} \qquad (23)$$

where $\hat{\mu}_i$ depends on the premise variable of the filter which is different from μ_i.

By applying the same technique used in Case I, we have the following theorem.

Theorem 2: Consider the system (4). Given a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ and a positive constant δ, if there exist matrices $X_0, Y_0, \mathcal{B}_{0_i}$ and $\mathcal{C}_{0_i}$, $i = 1, 2, \cdots, r$, satisfying the following ε-independent linear matrix inequalities:

$$\begin{bmatrix} X_0 E + D X_0 & I \\ I & Y_0 E + D Y_0 \end{bmatrix} > 0 \quad (24)$$

$$EX_0^T = X_0 E, \quad X_0^T D = D X_0, \quad X_0 E + D X_0 > 0 \quad (25)$$

$$EY_0^T = Y_0 E, \quad Y_0^T D = D Y_0, \quad Y_0 E + D Y_0 > 0 \quad (26)$$

$$\Psi_{11_{ii}} < 0, \quad i = 1, 2, \cdots, r \quad (27)$$

$$\Psi_{22_{ii}} < 0, \quad i = 1, 2, \cdots, r \quad (28)$$

$$\Psi_{11_{ij}} + \Psi_{11_{ji}} < 0, \quad i < j \leq r \quad (29)$$

$$\Psi_{22_{ij}} + \Psi_{22_{ji}} < 0, \quad i < j \leq r \quad (30)$$

where $E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$,

$$\Psi_{11_{ij}} = \begin{pmatrix} A_i Y_0^T + Y_0 A_i^T + \gamma^{-2} \tilde{\bar{B}}_{1_i} \tilde{\bar{B}}_{1_j}^T & (*)^T \\ \left[Y_0 \tilde{\bar{C}}_{1_i}^T + \mathcal{C}_{0_i}^T \tilde{\bar{D}}_{12}^T \right]^T & -I \end{pmatrix}$$

$$\Psi_{22_{ij}} = \begin{pmatrix} \begin{pmatrix} A_i^T X_0 + X_0 A_i + \mathcal{B}_{0_i} C_{2_j} \\ + C_{2_i}^T \mathcal{B}_{0_j}^T + \tilde{\bar{C}}_{1_i}^T \tilde{\bar{C}}_{1_j} \end{pmatrix} & (*)^T \\ \left[X_0 \tilde{\bar{B}}_{1_i} + \mathcal{B}_{0_i} \tilde{\bar{D}}_{21_j} \right]^T & -\gamma^2 I \end{pmatrix}$$

with

$$\tilde{\bar{B}}_{1_i} = \begin{bmatrix} \delta I & I & 0 & B_{1_i} & 0 \end{bmatrix},$$

$$\tilde{\bar{C}}_{1_i} = \begin{bmatrix} \frac{\gamma \bar{\rho}}{\delta} \bar{H}_{1_i}^T & \frac{\gamma \bar{\rho}}{\delta} \bar{H}_{5_i}^T & \sqrt{2}\bar{\lambda}\bar{\rho} \bar{H}_{4_i}^T & \sqrt{2}\bar{\lambda} C_{1_i}^T \end{bmatrix}^T,$$

$$\tilde{\bar{D}}_{12} = \begin{bmatrix} 0 & 0 & 0 & -\sqrt{2}\bar{\lambda} I \end{bmatrix}^T,$$

$$\tilde{\bar{D}}_{21_i} = \begin{bmatrix} 0 & 0 & \delta I & D_{21_i} & I \end{bmatrix}$$

and $\bar{\lambda} = \left(1 + \bar{\rho}^2 \sum_{i=1}^{r} \sum_{j=1}^{r} \left[\|\bar{H}_{2_i}^T \bar{H}_{2_j}\| + \|\bar{H}_{7_i}^T \bar{H}_{7_j}\| \right] \right)^{\frac{1}{2}},$

then there exists a sufficiently small $\hat{\varepsilon} > 0$ such that for $\varepsilon \in (0, \hat{\varepsilon}]$, the prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ is guaranteed. Furthermore, a suitable filter is of the form (23) with

$$\hat{A}_{ij}(\varepsilon) = \left[Y_\varepsilon^{-1} - X_\varepsilon \right]^{-1} \mathcal{M}_{0_{ij}}(\varepsilon) Y_\varepsilon^{-1}$$
$$\hat{B}_i = \left[Y_0^{-1} - X_0 \right]^{-1} \mathcal{B}_{0_i} \quad (31)$$
$$\hat{C}_i = \mathcal{C}_{0_i} Y_0^{-1}$$

where

$$\mathcal{M}_{0_{ij}}(\varepsilon) = -A_i^T - X_\varepsilon A_i Y_\varepsilon - \left[Y_\varepsilon^{-1} - X_\varepsilon \right] \hat{B}_i C_{2_j} Y_\varepsilon$$
$$- \tilde{\bar{C}}_{1_i}^T \left[\tilde{\bar{C}}_{1_j} Y_\varepsilon + \tilde{\bar{D}}_{12} \hat{C}_j Y_\varepsilon \right]$$
$$- \gamma^{-2} \left\{ X_\varepsilon \tilde{\bar{B}}_{1_i} + \left[Y_\varepsilon^{-1} - X_\varepsilon \right] \hat{B}_i \tilde{\bar{D}}_{21_i} \right\} \tilde{\bar{B}}_{1_j}^T$$
$$X_\varepsilon = \left\{ X_0 + \varepsilon \tilde{X} \right\} E_\varepsilon \text{ and } Y_\varepsilon^{-1} = \left\{ Y_0^{-1} + \varepsilon N_\varepsilon \right\} E_\varepsilon$$

with $\tilde{X} = D \left(X_0^T - X_0 \right)$ *and* $N_\varepsilon = D \left((Y_0^{-1})^T - Y_0^{-1} \right)$.

Proof: It can be shown by employing the same technique used in the proof for Theorem 1. ∎

IV. EXAMPLE

Consider the tunnel diode circuit shown in Figure 1 where the tunnel diode is characterized by

$$i_D(t) = 0.01 v_D(t) + 0.05 v_D^3(t).$$

Assuming that the inductance, L, is the parasitic parameter
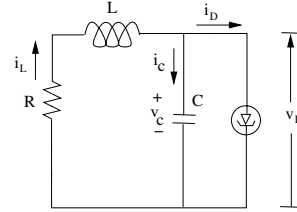

Fig. 1. Tunnel diode circuit.

and letting $x_1(t) = v_C(t)$ and $x_2(t) = i_L(t)$ as the state variables, we have

$$\begin{aligned} C\dot{x}_1(t) &= -0.01 x_1(t) - 0.05 x_1^3(t) + x_2(t) \\ L\dot{x}_2(t) &= -x_1(t) - R x_2(t) + 0.1 w_2(t) \\ y(t) &= Jx(t) + 0.1 w_1(t) \\ z(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned} \quad (32)$$

where $w(t)$ is the disturbance noise input, $y(t)$ is the measurement output, $z(t)$ is the state to be estimated and J is the sensor matrix. Note that the variables $x_1(t)$ and $x_2(t)$ are treated as the deviation variables (variables deviate from the desired trajectories). The parameters of the circuit are $C = 100 \ mF$, $R = 10 \pm 10\% \ \Omega$ and $L = \varepsilon \ H$. With these parameters (32) can be rewritten as

$$\begin{aligned} \dot{x}_1(t) &= -0.1 x_1(t) + 0.5 x_1^3(t) + 10 x_2(t) \\ \varepsilon \dot{x}_2(t) &= -x_1(t) - (10 + \Delta R) x_2(t) + 0.1 w_2(t) \\ y(t) &= Jx(t) + 0.1 w_1(t) \\ z(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned} \quad (33)$$

For the sake of simplicity, we will use as few rules as possible. Assuming that $|x_1(t)| \leq 3$, the nonlinear network system (33) can be approximated by the following TS fuzzy model:

Plant Rule 1: IF $x_1(t)$ is $M_1(x_1(t))$ THEN

$$\begin{aligned} E_\varepsilon \dot{x}(t) &= [A_1 + \Delta A_1] x(t) + B_{1_1} w(t), \quad x(0) = 0, \\ z(t) &= C_{1_1} x(t), \\ y(t) &= C_{2_1} x(t) + D_{21_1} w(t). \end{aligned}$$

Plant Rule 2: IF $x_1(t)$ is $M_2(x_1(t))$ THEN

$$\begin{aligned} E_\varepsilon \dot{x}(t) &= [A_2 + \Delta A_2] x(t) + B_{1_2} w(t), \quad x(0) = 0, \\ z(t) &= C_{1_2} x(t), \\ y(t) &= C_{2_2} x(t) + D_{21_2} w(t) \end{aligned}$$

where $x(t) = [x_1^T(t) \; x_2^T(t)]^T$, $w(t) = [w_1^T(t) \; w_2^T(t)]^T$,

$$A_1 = \begin{bmatrix} -0.1 & 10 \\ -1 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -4.6 & 10 \\ -1 & -1 \end{bmatrix},$$

$$B_{1_1} = B_{1_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C_{2_1} = C_{2_2} = J, \; D_{21} = \begin{bmatrix} 0.1 & 0 \end{bmatrix},$$

$$\Delta A_1 = F(x(t),t)H_{1_1}, \; \Delta A_2 = F(x(t),t)H_{1_2}$$

$$\text{and} \; E_\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}.$$

Now, by assuming that $\|F(x(t),t)\| \le \rho = 1$ and since the values of R are uncertain but bounded within 10% of their nominal values given in (32), we have

$$H_{1_1} = H_{1_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the plot of the membership function Rules 1 and 2 is the same as in Figure 2. By employing the results given in Lemma 1 and the Matlab LMI solver, it is easy to realize that $\varepsilon < 0.006$ for the fuzzy filter design in Case I and $\varepsilon < 0.008$ for the fuzzy filter design in Case II, the LMIs become ill-conditioned and the Matlab LMI solver yields the error message, "Rank Deficient".
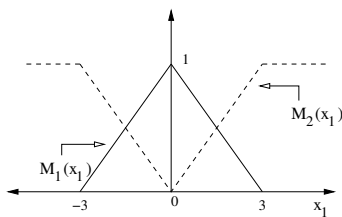

Fig. 2. Membership functions for the two fuzzy set.

Case I-$\nu(t)$ are available for feedback

In this case, $x_1(t) = \nu(t)$ is assumed to be available for feedback; for instance, $J = [1 \; 0]$. This implies that μ_i is available for feedback. Using the LMI optimization algorithm and Theorem 1 with $\varepsilon = 100 \; \mu H$, $\gamma = 0.6$ and $\delta = 1$, we obtain the following results:

$$\hat{A}_{11}(\varepsilon) = \begin{bmatrix} -0.0674 & -0.3532 \\ -30.7181 & -4.3834 \end{bmatrix},$$
$$\hat{A}_{12}(\varepsilon) = \begin{bmatrix} -0.0674 & -0.3532 \\ -30.7181 & -4.3834 \end{bmatrix},$$
$$\hat{A}_{21}(\varepsilon) = \begin{bmatrix} -0.0928 & -0.3138 \\ -34.7355 & -3.8964 \end{bmatrix},$$
$$\hat{A}_{22}(\varepsilon) = \begin{bmatrix} -0.0928 & -0.3138 \\ -34.7355 & -3.8964 \end{bmatrix},$$
$$\hat{B}_1 = \begin{bmatrix} 1.5835 \\ 3.2008 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 1.2567 \\ 3.8766 \end{bmatrix},$$
$$\hat{C}_1 = \begin{bmatrix} -1.7640 & -0.8190 \end{bmatrix},$$

$$\hat{C}_2 = \begin{bmatrix} 4.5977 & -0.8190 \end{bmatrix}.$$

Hence, the resulting fuzzy filter is

$$E_\varepsilon \dot{\hat{x}}(t) = \sum_{i=1}^{2} \sum_{j=1}^{2} \mu_i \mu_j \hat{A}_{ij}(\varepsilon)\hat{x}(t) + \sum_{i=1}^{2} \mu_i \hat{B}_i y(t)$$

$$\hat{z}(t) = \sum_{i=1}^{2} \mu_i \hat{C}_i \hat{x}(t)$$

where

$$\mu_1 = M_1(x_1(t)) \; \text{and} \; \mu_2 = M_2(x_1(t)).$$

Case II: $\nu(t)$ are unavailable for feedback

In this case, $x_1(t) = \nu(t)$ is assumed to be unavailable for feedback; for instance, $J = [0 \; 1]$. This implies that μ_i is unavailable for feedback. Using the LMI optimization algorithm and Theorem 2 with $\varepsilon = 100 \; \mu H$, $\gamma = 0.6$ and $\delta = 1$, we obtain the following results:

$$\hat{A}_{11}(\varepsilon) = \begin{bmatrix} -2.3050 & -0.4186 \\ -32.3990 & -4.4443 \end{bmatrix},$$
$$\hat{A}_{12}(\varepsilon) = \begin{bmatrix} -2.3050 & -0.4186 \\ -32.3990 & -4.4443 \end{bmatrix},$$
$$\hat{A}_{21}(\varepsilon) = \begin{bmatrix} -2.3549 & -0.3748 \\ -32.4539 & -3.9044 \end{bmatrix},$$
$$\hat{A}_{22}(\varepsilon) = \begin{bmatrix} -2.3549 & -0.3748 \\ -32.4539 & -3.9044 \end{bmatrix},$$
$$\hat{B}_1 = \begin{bmatrix} -0.3053 \\ 3.9938 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} -0.3734 \\ 5.1443 \end{bmatrix},$$
$$\hat{C}_1 = \begin{bmatrix} 4.3913 & -0.1406 \end{bmatrix},$$
$$\hat{C}_2 = \begin{bmatrix} 1.9832 & -0.1406 \end{bmatrix}.$$

The resulting fuzzy filter is

$$E_\varepsilon \dot{\hat{x}}(t) = \sum_{i=1}^{2} \sum_{j=1}^{2} \hat{\mu}_i \hat{\mu}_j \hat{A}_{ij}(\varepsilon)\hat{x}(t) + \sum_{i=1}^{2} \hat{\mu}_i \hat{B}_i y(t)$$

$$\hat{z}(t) = \sum_{i=1}^{2} \hat{\mu}_i \hat{C}_i \hat{x}(t)$$

where

$$\hat{\mu}_1 = M_1(\hat{x}_1(t)) \; \text{and} \; \hat{\mu}_2 = M_2(\hat{x}_1(t)).$$

Remark 2: The ratios of the filter error energy to the disturbance input noise energy are depicted in Figure 3 when $\varepsilon = 100 \; \mu H$. The disturbance input signal, $w(t)$, which was used during the simulation is the rectangular signal (magnitude 0.9 and frequency 0.5 Hz). Figures 4(a) - 4(b), respectively, show the responses of $x_1(t)$ and $x_2(t)$ in Cases I and II. Table I shows the performance index γ with different values of ε in Cases I and II. After 50 seconds, the ratio of the filter error energy to the disturbance input noise energy tends to a constant value which is about 0.02 in Case I and 0.08 in Case II. Thus, in Case I where $\gamma = \sqrt{0.02} = 0.141$ and in Case II where $\gamma = \sqrt{0.08} = 0.283$, both are less than the prescribed value 0.6. From Table 9.1, the maximum value of ε that guarantees the $\mathcal{L}_2$-gain of the

mapping from the exogenous input noise to the filter error energy being less than 0.6 *is* 0.30 *H, i.e.,* $\varepsilon \in (0, 0.30]$ *H in Case I, and* 0.25 *H, i.e.,* $\varepsilon \in (0, 0.25]$ *H in Case II.* □

V. CONCLUSION

The problem of designing a robust $\mathcal{H}_\infty$ fuzzy ε-independent filter for a TS fuzzy descriptor system with parametric uncertainties has been considered. Sufficient conditions for the existence of the robust $\mathcal{H}_\infty$ fuzzy filter have been derived in terms of a family of ε-independent LMIs. A numerical simulation example has been also presented to illustrate the theory development.
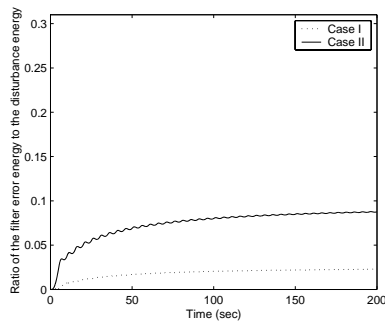

Fig. 3.   The ratio of the filter error energy to the disturbance noise energy:
$$\left(\frac{\int_0^{T_f} (z(t) - \hat{z}(t))^T (z(t) - \hat{z}(t)) dt}{\int_0^{T_f} w^T(t) w(t) dt} \right).$$

TABLE I

THE PERFORMANCE INDEX γ OF THE SYSTEM WITH DIFFERENT

VALUES OF ε.

ε	The performance index γ	
	Output-feedback in Case I	Output-feedback in Case II
0.0001	0.141	0.283
0.1	0.316	0.509
0.25	0.479	0.596
0.26	0.500	> 0.6
0.30	0.591	> 0.6
0.31	> 0.6	> 0.6

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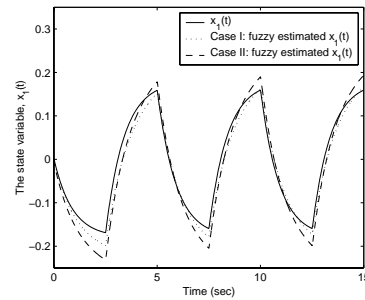
(a) The histories of $x_1(t)$
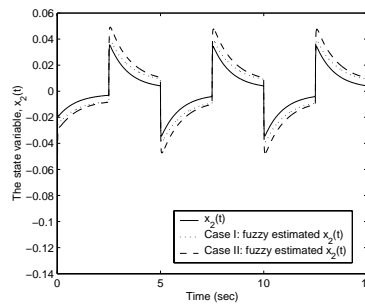
(b) The histories of $x_2(t)$

Fig. 4.   The histories of the state variables, $x_1(t)$ and $x_2(t)$.

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