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Robust Fuzzy Observer Design for Nonlinear Systems

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***Abstract*—**This paper shows a new method for design of fuzzy observers for Takagi-Sugeno systems. The method is based on Linear matrix inequalities (LMIs) and it allows to insert $\mathbf{H}_\infty$ constraint into the design procedure. The speed of estimation can tuned be specification of a decay rate of the observer closed loop system. We discuss here also the influence of parametric uncertainties at the output control system stability.

***Keywords*—** $\mathbf{H}_\infty$ norm, Linear Matrix Inequalities, Observers, Takagi-Sugeno Systems, Parallel Distributed Compensation

I. Introduction

In the past some design techniques were developed for modeling and control of nonlinear uncertain systems. Very interesting approach were done in the fuzzy modeling and control, especially with Takagi-Sugeno (T-S) fuzzy modeling [4] and related Parallel Distributed Compensation (PDC) control algorithm [1][2]. This method uses local linear models in the consequent of fuzzy rules. Decision variables are designed to divide the state space of the system into areas, where the linear local models describe precisely the nonlinear system. In the overlapped parts of these areas are local models interpolated according to fuzzy membership functions. Takagi-Sugeno model based control of nonlinear systems is quite popular now for its simple and effective design based usually on Linear Matrix Inequalities (LMIs)[5].

The design of an observer in Takagi-Sugeno fuzzy systems is very important part of a control system design. Estimation of unmeasured states must be fast and should depend on the noise and system uncertainties as less as possible. Since we can look at the parametric uncertainties as at an input disturbance signal, which should be attenuated, $\mathbf{H}_\infty$ norm of the closed loop system should be minimized. It could be also useful, if we can implement the parametric uncertainties into the design method. This problem is discussed here, but it is quite difficult and exceeds the scope of this paper.

Although many design methods for a fuzzy controller (Parallel Distributed Compensator – PDC) are developed [1][5][6], there can be found only a few methods for a T-S fuzzy observer design in a literature.

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The observer and controller has some similar properties from a mathematical point of view [2][3][7], that can simplify the development of observer design methods. We can say, that they are dual problems. Presented research is inspired by the paper published in [1], which deals with a robust $\mathbf{H}_\infty$ PDC controller design. On its base is derived a method for an observer design.

II. Takagi-Sugeno System with Observer and Controller

A. T-S model

The standard Takagi-Sugeno fuzzy model consists of the set of fuzzy rules with linear consequent, that describe system in a local areas ***i***. For our purpose we suppose following form:

Rule i:
IF $z_1(t)$ is M_1^i and $z_2(t)$ is M_2^i and … and $z_n(t)$ is M_n^i
THEN

$$\dot{\mathbf{x}}(t)=(\mathbf{A}_i+\Delta\mathbf{A}_i)\mathbf{x}(t)+(\mathbf{B}_{1i}+\Delta\mathbf{B}_{1i})\mathbf{w}(t)+(\mathbf{B}_{2i}+\Delta\mathbf{B}_{2i})\mathbf{u}(t),$$
$$\mathbf{y}(t)=\mathbf{C}_i\mathbf{x}(t) \qquad (1)$$

where $\mathbf{z}^T(t)=[z_1(t),...,z_n(t)]$ are some premise variables,
$\mathbf{y}^T(t)=[y_1(t),...,y_l(t)]$ is an output vector,
$\mathbf{x}^T(t)=[x_1(t),...,x_n(t)]$ is a state vector,
$\mathbf{u}^T(t)=[u_1(t),...,u_m(t)]$ is a control input vector and
$\mathbf{w}^T(t)=[w_1(t),...,w_p(t)]$ is a disturbance vector.

i = 1,2,…,r denotes the area's number. r is the number of areas and thus also of fuzzy rules. M_j^i is a fuzzy set ($M_j^i(\mathbf{z}(t))$ is the grade of membership of premise variable $z_j(t)$ in the area number i). m is the number of inputs and l is the number of outputs of T-S fuzzy system. Matrixes $\mathbf{A}_i\in R^{n\times n}$, $\mathbf{B}_{1i}\in R^{n\times m}$, $\mathbf{B}_{2i}\in R^{n\times m}$, $\mathbf{C}_i\in R^{l\times n}$ describe the system in the area number i.

Matrixes $\Delta\mathbf{A}_i$, $\Delta\mathbf{B}_{1i}$ a $\Delta\mathbf{B}_{2i}$ represent uncertainties in the system. We will suppose, that uncertainties in the output matrix $\mathbf{C}_i$ can be transformed into matrices $\Delta\mathbf{B}_{1i}$ a $\Delta\mathbf{B}_{2i}$. To incorporate the uncertainties into a PDC controller design methods is commonly used the following assumption:

$$(\Delta\mathbf{A}_i,\Delta\mathbf{B}_{1i},\Delta\mathbf{B}_{2i})=\mathbf{D}\mathbf{F}_i(t)[\mathbf{E}_{1i},\mathbf{E}_{2i},\mathbf{E}_{3i}]$$
$$\mathbf{F}_i^T(t)\mathbf{F}_i(t)\le\mathbf{I} \tag{2}$$

where $\mathbf{D},\mathbf{E}_{1i},\mathbf{E}_{2i},\mathbf{E}_{3i}$ are known, real, constant matrixes with appropriate dimensions. $\mathbf{I}$ is identity matrix and $\mathbf{F}_i(t)$ are unknown matrix functions with Lebesgue-measurable elements.

The defuzzified output can be represented as follows:

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\{[\mathbf{A}_i+\Delta\mathbf{A}_i]\mathbf{x}(t)+ +[\mathbf{B}_{1i}+\Delta\mathbf{B}_{1i}]\mathbf{w}(t)+[\mathbf{B}_{2i}+\Delta\mathbf{B}_{2i}]\mathbf{u}(t)\} \tag{3}$$
$$\mathbf{y}(t)=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\mathbf{C}_i\mathbf{x}(t)$$

where $$h_i(\mathbf{z}(t))=\frac{\prod_{j=1}^{n}M_j^i(z_j(t))}{\sum_{i=1}^{r}\prod_{j=1}^{n}M_j^i(z_j(t))} \tag{4}$$

If $\mathbf{z}(t)$ is in the specified range, then $\sum_{i=1}^{r}h_i(\mathbf{z}(t))=1$

This representation can be easily implemented into a Matlab model.

B. Fuzzy State Observer

Here we can suppose, that the disturbance signal is not measured. Also the observer don't have information about parametric uncertainties. Fuzzy state observer is described by the following fuzzy rules:

Observer rule i:

IF $z_1(t)$ is M_1^i and $z_2(t)$ is M_2^i and … and $z_n(t)$ is M_n^i

THEN $\dot{\hat{\mathbf{x}}}(t)=\mathbf{A}_i\hat{\mathbf{x}}(t)+\mathbf{B}_{2i}\mathbf{u}(t)+\mathbf{G}_i[\mathbf{y}(t)-\hat{\mathbf{y}}(t)]$,

$$\hat{\mathbf{y}}(t)=\mathbf{C}_i\hat{\mathbf{x}}(t), \tag{5}$$

where $\mathbf{G}_i$ is an observer gain in the area number *i*.

$\hat{\mathbf{x}}^T(t)=[\hat{x}_1(t),...,\hat{x}_n(t)]$ is estimated state vector and

$\hat{\mathbf{y}}^T(t)=[\hat{y}_1(t),...,\hat{y}_l(t)]$ is a vector of estimated outputs.

After defuzzyfication we get the following equations:

$$\dot{\hat{\mathbf{x}}}(t)=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\{\mathbf{A}_i\hat{\mathbf{x}}(t)+\mathbf{B}_{2i}\mathbf{u}(t)+\mathbf{G}_i[\mathbf{y}(t)-\hat{\mathbf{y}}(t)]\},$$
$$\hat{\mathbf{y}}(t)=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\mathbf{C}_i\hat{\mathbf{x}}(t). \tag{6}$$

The aim of the observer is to eliminate estimation error as fast as possible. Estimation error is defined as

$$\mathbf{e}(t)=\mathbf{x}(t)-\hat{\mathbf{x}}(t)$$

It will be useful to convert the equation (6) in the form with estimation error $\mathbf{e}(t)$ instead of $\hat{\mathbf{x}}(t)$. The computation of time derivative of estimation error is then:

$$\dot{\mathbf{e}}(t)=\dot{\mathbf{x}}(t)-\dot{\hat{\mathbf{x}}}(t)=$$
$$=\sum_{i=1}^{r}h_i(\mathbf{z}(t)).\left\{\begin{matrix}\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_{1i}\mathbf{w}(t)+\mathbf{B}_{2i}\mathbf{u}(t)+\\+\Delta\mathbf{A}_i\mathbf{x}(t)+\Delta\mathbf{B}_{1i}\mathbf{w}(t)+\Delta\mathbf{B}_{2i}\mathbf{u}(t)\end{matrix}\right\}-$$
$$-\sum_{i=1}^{r}h_i(\mathbf{z}(t))\{\mathbf{A}_i\hat{\mathbf{x}}(t)+\mathbf{B}_{2i}\mathbf{u}(t)+\mathbf{G}_i[\mathbf{y}(t)-\hat{\mathbf{y}}(t)]\}=$$
$$=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\left\{\begin{matrix}\mathbf{A}_i[\mathbf{x}(t)-\hat{\mathbf{x}}(t)]-\mathbf{G}_i[\mathbf{y}(t)-\hat{\mathbf{y}}(t)]+\mathbf{B}_{1i}\mathbf{w}(t)+\\+\Delta\mathbf{A}_i\mathbf{x}(t)+\Delta\mathbf{B}_{1i}\mathbf{w}(t)+\Delta\mathbf{B}_{2i}\mathbf{u}(t)\end{matrix}\right\}=$$
$$=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\left\{\begin{matrix}\mathbf{A}_i[\mathbf{x}(t)-\hat{\mathbf{x}}(t)]-\\-\mathbf{G}_i\sum_{j=1}^{r}h_j(\mathbf{z}(t))\mathbf{C}_j[\mathbf{x}(t)-\hat{\mathbf{x}}(t)]+\mathbf{B}_{1i}\mathbf{w}(t)+\\+\Delta\mathbf{A}_i\mathbf{x}(t)+\Delta\mathbf{B}_{1i}\mathbf{w}(t)+\Delta\mathbf{B}_{2i}\mathbf{u}(t)\end{matrix}\right\}= \tag{7}$$
$$=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\sum_{j=1}^{r}h_j(\mathbf{z}(t))\left[\begin{matrix}(\mathbf{A}_i-\mathbf{G}_i\mathbf{C}_j)\mathbf{e}(t)+\mathbf{B}_{1i}\mathbf{w}(t)+\\+\Delta\mathbf{A}_i\mathbf{x}(t)+\Delta\mathbf{B}_{1i}\mathbf{w}(t)+\Delta\mathbf{B}_{2i}\mathbf{u}(t)\end{matrix}\right]$$

In the case that the matrixes $\Delta\mathbf{A}_i=0,\ \Delta\mathbf{B}_{1i}=0,\ \Delta\mathbf{B}_{2i}=0$ we get

$$\dot{\mathbf{e}}(t)=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\sum_{j=1}^{r}h_j(\mathbf{z}(t))[(\mathbf{A}_i-\mathbf{G}_i\mathbf{C}_j)\mathbf{e}(t)+\mathbf{B}_{1i}\mathbf{w}(t)]$$
$$\vartheta(t)=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\mathbf{C}_i\mathbf{e}(t) \tag{8}$$

where $\vartheta(t)=\mathbf{y}(t)-\hat{\mathbf{y}}(t)$ is an output estimation error.

If we define $\mathbf{L}_{ij}=\mathbf{A}_i-\mathbf{G}_i\mathbf{C}_j$, then we get

$$\dot{\mathbf{e}}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\mathbf{z}(t))h_j(\mathbf{z}(t))[\mathbf{L}_{ij}\mathbf{e}(t)+\mathbf{B}_{1i}\mathbf{w}(t)]. \tag{9}$$

C. PDC Fuzzy Control Algorithm

The control signal of PDC controller is computed as

$$\mathbf{u}(t)=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\mathbf{K}_i\mathbf{x}(t) \tag{10}$$

where $\mathbf{K}_i\in R^{m\times n}$ is a constant feedback gain.

If some state variables have to be estimated by a fuzzy observer, then the control output is computed in the following way:

$$\mathbf{u}(t)=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\mathbf{K}_i\hat{\mathbf{x}}(t) \tag{11}$$

The closed loop system we get by combination of (10) and (3). Then

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\mathbf{z}(t))h_j(\mathbf{z}(t))\cdot \\ \cdot[(\mathbf{A}_i+\Delta\mathbf{A}_i+(\mathbf{B}_{2i}+\Delta\mathbf{B}_{2i})\mathbf{K}_j)\mathbf{x}(t)+\mathbf{B}_{1i}\mathbf{w}(t)+\Delta\mathbf{B}_{1i}\mathbf{w}(t)] \quad (12)$$

In the case that the matrixes $\Delta\mathbf{A}_i=0$, $\Delta\mathbf{B}_{1i}=0$, $\Delta\mathbf{B}_{2i}=0$ we get

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\mathbf{z}(t))h_j(\mathbf{z}(t))[(\mathbf{A}_i+\mathbf{B}_i\mathbf{K}_j)\mathbf{x}(t)+\mathbf{B}_{1i}\mathbf{w}(t)] \quad (13)$$

If we define $\mathbf{F}_{ij}=\mathbf{A}_i+\mathbf{B}_i\mathbf{K}_j$, then we get the system equation

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\mathbf{z}(t))h_j(\mathbf{z}(t))[\mathbf{F}_{ij}\mathbf{x}(t)+\mathbf{B}_{1i}\mathbf{w}(t)], \quad (14)$$

which is formally identical to (9). The important result is that fuzzy observer and PDC controller are dual problems and we may transform the PDC controller design methods to be used for design of an observer.

III. Robust Controller Design

The method have been published by Lee, Jeung and Park in [1]. The objective is to minimize $\boldsymbol{H}_\infty$ norm, that determine a disturbance attenuation and is frequently used also as a robustness measure. $\boldsymbol{H}_\infty$ norm is defined as

$$\|\mathbf{T}_{\mathbf{yw}}\|_\infty=\sup_{\|\mathbf{w}\|_2\neq 0}\frac{\|\mathbf{y}(t)\|_2}{\|\mathbf{w}(t)\|_2}. \quad (15)$$

where $\mathbf{T}_{\mathbf{yw}}$ is transfer matrix of a system (13) and supremum is taken from all trajectories from the initial conditions $\mathbf{x}(0)=0$.

We suppose, that

$$\|\mathbf{y}(t)\|_2\leq\lambda\|\mathbf{w}(t)\|_2 \quad (16)$$

The method minimizes λ, that is an upper bound of $\|\mathbf{T}_{\mathbf{yw}}\|_\infty$.

Decay rate α is specified by a condition

$$\dot{V}(\mathbf{x}(t))\leq-2\alpha V(\mathbf{x}(t)), \quad (17)$$

where $V(\mathbf{x}(t))$ is a Lyapunov function, that must be always positive with a negative time derivative.

The controller design method is summarised in Theorem 1.

Theorem 1: Equilibrium of a T-S fuzzy system with a PDC control (10) is asymptotically stable in large with a decay rate $\alpha>0$ and $\|\mathbf{T}_{yw}\|_\infty<\lambda$, if there exists a common positive definite matrix $\mathbf{Z}>0$, $\lambda>0$ and matrices $\mathbf{M}_i$, $i=1,2,...,r$ such that the following LMIs conditions hold.

$$\begin{bmatrix}\boldsymbol{\Delta}_{ii} & \mathbf{B}_{1i} & \mathbf{Z}\mathbf{C}_i^T\\ \mathbf{B}_{1i}^T & -\lambda^2\mathbf{I} & 0\\ \mathbf{C}_i\mathbf{Z} & 0 & -\mathbf{I}\end{bmatrix}\leq 0,\ i=1,2,...,r, \quad (18)$$

$$\begin{bmatrix}\boldsymbol{\Delta}_{ij}+\boldsymbol{\Delta}_{ji} & \mathbf{B}_{1i}+\mathbf{B}_{1j} & \mathbf{Z}\mathbf{C}_i^T+\mathbf{Z}\mathbf{C}_j^T\\ \mathbf{B}_{1i}^T+\mathbf{B}_{1j}^T & -2\lambda^2\mathbf{I} & 0\\ \mathbf{C}_i\mathbf{Z}+\mathbf{C}_j\mathbf{Z} & 0 & -2\mathbf{I}\end{bmatrix}\leq 0,\ 0\leq i<j\leq r, \quad (19)$$

where $\boldsymbol{\Delta}_{ij}=\mathbf{A}_i\mathbf{Z}+\mathbf{Z}\mathbf{A}_i^T+\mathbf{B}_{2i}\mathbf{M}_j+\mathbf{M}_j^T\mathbf{B}_{2i}^T+2\alpha\mathbf{Z}$,

The gain of the robust controller the will be:

$$\mathbf{K}_i=\mathbf{M}_i\mathbf{Z}^{-1} \quad (20)$$

The optimization objective is to minimize λ.

Proof: We choose the following substitution:

$$\overline{\mathbf{A}}=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\mathbf{z}(t))h_j(\mathbf{z}(t))[\mathbf{A}_i+\mathbf{B}_{2i}\mathbf{K}_j],$$

$$\overline{\mathbf{B}}_w=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\mathbf{B}_{1i},$$

$$\overline{\mathbf{C}}_y=\sum_{i=1}^{r}h_i(\mathbf{z}(t))\mathbf{C}_i,$$

Then the closed loop system (13) will be transformed to

$$\dot{\mathbf{x}}(t)=\overline{\mathbf{A}}\mathbf{x}(t)+\overline{\mathbf{B}}_w\mathbf{w}(t) \\ \mathbf{y}(t)=\overline{\mathbf{C}}_y\mathbf{x}(t) \quad (21)$$

Lyapunov function we choose $V(\mathbf{x}(t))=\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$, where $\mathbf{P}>0$. If we suppose zero initial conditions, then from (17) we get

$$\mathbf{x}^T(t)(\overline{\mathbf{A}}^T\mathbf{P}+\mathbf{P}\overline{\mathbf{A}}+2\alpha\mathbf{P})\mathbf{x}(t)\leq 0 \quad (22)$$

The system then will be stable with a decay rate $\alpha>0$. It holds if there exists matrix $\mathbf{P}>0$ that fulfill the matrix inequality

$$\overline{\mathbf{A}}^T\mathbf{P}+\mathbf{P}\overline{\mathbf{A}}+2\alpha\mathbf{P}\leq 0 \quad (23)$$

To implement the $\boldsymbol{H}_\infty$ norm we will now suppose that

$$\dot{V}(\mathbf{x}(t))+\mathbf{y}^T(t)\mathbf{y}(t)-\lambda^2\mathbf{w}^T(t)\mathbf{w}(t)\leq 0. \quad (24)$$

Then $\boldsymbol{H}_\infty$ norm of the system (21) is less than λ [5]. The condition (24) is equivalent to

$$\mathbf{x}^T(t)(\overline{\mathbf{A}}^T\mathbf{P}+\mathbf{P}\overline{\mathbf{A}}+\overline{\mathbf{C}}_y^T\overline{\mathbf{C}}_y)\mathbf{x}(t)+\mathbf{x}^T(t)\mathbf{P}\overline{\mathbf{B}}_w\mathbf{w}(t)+ \\ +\mathbf{w}^T(t)\overline{\mathbf{B}}_w^T\mathbf{P}\mathbf{x}(t)-\lambda^2\mathbf{w}^T(t)\mathbf{w}(t)\leq 0 \quad (25)$$

This is equivalent to

$$\begin{bmatrix} \overline{\mathbf{A}}^T\mathbf{P}+\mathbf{P}\overline{\mathbf{A}}+\overline{\mathbf{C}}_y^T\overline{\mathbf{C}}_y & \mathbf{P}\overline{\mathbf{B}}_w \\ \overline{\mathbf{B}}_w^T\mathbf{P} & -\lambda^2\mathbf{I} \end{bmatrix} \leq 0 \qquad (26)$$

To fulfill both conditions (22) and (25) must hold:

$$\begin{bmatrix} \overline{\mathbf{A}}^T\mathbf{P}+\mathbf{P}\overline{\mathbf{A}}+\overline{\mathbf{C}}_y^T\overline{\mathbf{C}}_y+2\alpha\mathbf{P} & \mathbf{P}\overline{\mathbf{B}}_w \\ \overline{\mathbf{B}}_w^T\mathbf{P} & -\lambda^2\mathbf{I} \end{bmatrix} \leq 0 \qquad (27)$$

This inequality is not linear, but we can transform it into a LMI. At first we introduce a matrix $\mathbf{Z}=\mathbf{P}^{-1}$ and we will multiply (27) by $\begin{bmatrix} \mathbf{Z} & 0 \\ 0 & \mathbf{I} \end{bmatrix}$. Using S-procedure we obtain

$$\begin{bmatrix} \mathbf{Z}\overline{\mathbf{A}}^T+\overline{\mathbf{A}}\mathbf{Z}+2\alpha\mathbf{Z} & \overline{\mathbf{B}}_w & \mathbf{Z}\overline{\mathbf{C}}_y^T \\ \overline{\mathbf{B}}_w^T & -\lambda^2\mathbf{I} & 0 \\ \overline{\mathbf{C}}_y\mathbf{Z} & 0 & -\mathbf{I} \end{bmatrix} \leq 0, \qquad (28)$$

that is equivalent to

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\mathbf{z}(t))h_j(\mathbf{z}(t)) \begin{bmatrix} \mathbf{Z}\mathbf{F}_{ij}^T+\mathbf{F}_{ij}\mathbf{Z}+2\alpha\mathbf{Z} & \mathbf{B}_{1i} & \mathbf{Z}\mathbf{C}_i^T \\ \mathbf{B}_{1i}^T & -\lambda^2\mathbf{I} & 0 \\ \mathbf{C}_i\mathbf{Z} & 0 & -\mathbf{I} \end{bmatrix} \leq 0 \qquad (29)$$

where $\mathbf{F}_{ij}=\mathbf{A}_i-\mathbf{B}_{2i}\mathbf{K}_j$.

This LMI can be also written as

$$\sum_{i=1}^{r} h_i^2(\mathbf{z}(t)) \begin{bmatrix} \mathbf{\Delta}_{ii} & \mathbf{B}_{1i} & \mathbf{Z}\mathbf{C}_i^T \\ \mathbf{B}_{1i}^T & -\lambda^2\mathbf{I} & 0 \\ \mathbf{C}_i\mathbf{Z} & 0 & -\mathbf{I} \end{bmatrix} + \sum_{i=1}^{r}\sum_{i<j}^{r} h_i(\mathbf{z}(t))h_j(\mathbf{z}(t)) \cdot$$
$$\cdot \begin{bmatrix} \mathbf{\Delta}_{ij}+\mathbf{\Delta}_{ji} & \mathbf{B}_{1i}+\mathbf{B}_{1j} & \mathbf{Z}\mathbf{C}_i^T+\mathbf{Z}\mathbf{C}_j^T \\ \mathbf{B}_{1i}^T+\mathbf{B}_{1j}^T & -2\lambda^2\mathbf{I} & 0 \\ \mathbf{C}_i\mathbf{Z}+\mathbf{C}_j\mathbf{Z} & 0 & -2\mathbf{I} \end{bmatrix} \leq 0 \qquad (30)$$

where $\mathbf{\Delta}_{ij}=\mathbf{A}_i\mathbf{Z}+\mathbf{Z}\mathbf{A}_i^T+\mathbf{B}_{2i}\mathbf{M}_j+\mathbf{M}_j^T\mathbf{B}_{2i}^T+2\alpha\mathbf{Z}$
and $\mathbf{M}_i=\mathbf{K}_i\mathbf{Z}$.

This is equivalent to the LMI conditions in the Theorem 1. □

IV. Fuzzy Observer Design

Now we will derive the LMI method for an observer design. The objective is again to minimize $\boldsymbol{H}_\infty$ norm, that determine now a disturbance attenuation on the output estimation error. $\boldsymbol{H}_\infty$ norm $\|\mathbf{T}_{\vartheta w}\|_\infty<\lambda_e$ is then defined as

$$\|\mathbf{T}_{\vartheta \mathbf{w}}\|_\infty = \sup_{\|\mathbf{w}\|_2\neq 0} \frac{\|\vartheta(t)\|_2}{\|\mathbf{w}(t)\|_2}. \qquad (31)$$

where $\mathbf{T}_{\vartheta \mathbf{w}}$ is transfer matrix of a system (8).

We suppose, that

$$\|\vartheta(t)\|_2 \leq \lambda_e \|\mathbf{w}(t)\|_2 \qquad (32)$$

The method minimizes λ_e, that is an upper bound of $\|\mathbf{T}_{\vartheta \mathbf{w}}\|_\infty$.

The decay rate α_e is specified by a condition

$$\dot{V}_e(\mathbf{x}(t)) \leq -2\alpha_e V_e(\mathbf{x}(t)), \qquad (33)$$

where $V_e(\mathbf{x}(t))$ is a Lyapunov function, that must be always positive with a negative time derivative.

We will start from the equation (27), where remain matrices $\overline{\mathbf{B}}_w=\sum_{i=1}^{r} h_i(\mathbf{z}(t))\mathbf{B}_{1i}$ and $\overline{\mathbf{C}}_y=\sum_{i=1}^{r} h_i(\mathbf{z}(t))\mathbf{C}_i$ and newly defined is $\overline{\mathbf{A}}_e=\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\mathbf{z}(t))h_j(\mathbf{z}(t))[\mathbf{A}_i-\mathbf{G}_i\mathbf{C}_j]$ that replaces matrix $\overline{\mathbf{A}}$. Matrix $\mathbf{P}$ is replaced by $\mathbf{Y}$. The closed loop system (8) is then transformed to

$$\begin{aligned} \dot{\mathbf{e}}(t) &= \overline{\mathbf{A}}_e\mathbf{e}(t)+\overline{\mathbf{B}}_w\mathbf{w}(t) \\ \vartheta(t) &= \overline{\mathbf{C}}_y\mathbf{e}(t) \end{aligned} \qquad (34)$$

Lyapunov function we choose

$$V_e(\mathbf{e}(t))=\mathbf{e}^T(t)\mathbf{Y}\mathbf{e}(t), \qquad (35)$$

where $\mathbf{Y}>0$.

For $\boldsymbol{H}_\infty$ norm must hold that

$$\dot{V}_e(\mathbf{e}(t))+\vartheta^T(t)\vartheta(t)-\lambda_e^2\mathbf{w}^T(t)\mathbf{w}(t) \leq 0. \qquad (36)$$

Then the $\boldsymbol{H}_\infty$ norm of a system with observer (31) is less than λ_e [5]. The condition (27) is transformed to

$$\begin{bmatrix} \overline{\mathbf{A}}^T\mathbf{Y}+\mathbf{Y}\overline{\mathbf{A}}+\overline{\mathbf{C}}_y^T\overline{\mathbf{C}}_y+2\alpha\mathbf{Y} & \mathbf{Y}\overline{\mathbf{B}}_w \\ \overline{\mathbf{B}}_w^T\mathbf{Y} & -\lambda_e^2\mathbf{I} \end{bmatrix} \leq 0 \qquad (37)$$

Inequality (37) consists all requirements, whose are the stability, the $\boldsymbol{H}_\infty$ norm determined by its upper bound λ_e and the decay rate α. This inequality we can write also as

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\mathbf{z}(t))h_j(\mathbf{z}(t)) \begin{bmatrix} \mathbf{L}_{ij}^T\mathbf{Y}+\mathbf{Y}\mathbf{L}_{ij}+\mathbf{C}_i^T\mathbf{C}_i+2\alpha\mathbf{Y} & \mathbf{Y}\mathbf{B}_{1i} \\ \mathbf{B}_{1i}^T\mathbf{Y} & -\lambda_e^2\mathbf{I} \end{bmatrix} \leq 0 \qquad (38)$$

or

$$\sum_{i=1}^{r} h_i^2(\mathbf{z}(t)) \begin{bmatrix} \Phi_{ii} & \mathbf{Y}\mathbf{B}_{1i} \\ \mathbf{B}_{1i}^T\mathbf{Y} & -\lambda_e^2\mathbf{I} \end{bmatrix} +$$
$$+\sum_{i=1}^{r}\sum_{i<j}^{r} h_i(\mathbf{z}(t))h_j(\mathbf{z}(t)) \begin{bmatrix} \Phi_{ij}+\Phi_{ji} & \mathbf{Y}\mathbf{B}_{1i}+\mathbf{Y}\mathbf{B}_{1j} \\ \mathbf{B}_{1i}^T\mathbf{Y}+\mathbf{B}_{1j}^T\mathbf{Y} & -2\lambda_e^2\mathbf{I} \end{bmatrix} \leq 0 \qquad (39)$$

where $\Phi_{ij}=\mathbf{A}_i^T\mathbf{Y}-\mathbf{C}_j^T\mathbf{J}_i^T+\mathbf{Y}\mathbf{A}_i-\mathbf{J}_i\mathbf{C}_j+\mathbf{C}_i^T\mathbf{C}_i+2\alpha\mathbf{Y}$.

We are now able to formulate the LMI conditions for the observer design method.

Theorem 2: State fuzzy observer (13) maintains asymptotical stability of an estimation error $\mathbf{e}(t)\rightarrow 0$, $\boldsymbol{H}_\infty$ norm $\|\mathbf{T}_{\vartheta w}\|_\infty < \lambda_e$ and decay rate α, if there will exist matrix $\mathbf{J}_i$, positive definite matrix $\mathbf{P}>0$ and scalars $\lambda_e>0$ and $\alpha>0$, that the following LMI conditions hold

$$\begin{bmatrix} \Phi_{ii} & \mathbf{YB}_{1i} \\ \mathbf{B}_{1i}^T\mathbf{Y} & -\lambda_e^2\mathbf{I} \end{bmatrix} < 0 \quad i=1,2,...,r, \tag{40}$$

$$\begin{bmatrix} \Phi_{ij}+\Phi_{ji} & \mathbf{YB}_{1i}+\mathbf{YB}_{1j} \\ \mathbf{B}_{1i}^T\mathbf{Y}+\mathbf{B}_{1j}^T\mathbf{Y} & -2\lambda_e^2\mathbf{I} \end{bmatrix} \le 0 \quad i,j=1,2,...,r, \tag{41}$$

where $\Phi_{ij}=\mathbf{A}_i^T\mathbf{Y}-\mathbf{C}_j^T\mathbf{J}_i^T+\mathbf{YA}_i-\mathbf{J}_i\mathbf{C}_j+\mathbf{C}_i^T\mathbf{C}_i+2\alpha\mathbf{Y}$,
and $\mathbf{J}_i=-\mathbf{YG}_i$.

The observer gain is then computed as $\mathbf{G}_i=-\mathbf{Y}^{-1}\mathbf{J}_i$.

Proof: It implies from the above results. □

V. Observer Based Fuzzy Control Design

Observer is a very important part of the control system. The quality of the observer may dramatically change the speed and parameters of a controller, e.g. the $\boldsymbol{H}_\infty$ norm. The observer should attenuate disturbances, be as fast, as possible and should not depend much on parametric uncertainties. Hopefully Ma et Sun in [3] found so called separation property that if both controller and observer are stable, then the overall observer based control system is stable too. Only the control quality parameters will change. We are thus allowed to design these parts separately.

By combination of the T-S fuzzy system (3) without parametric uncertainties, observer (6) and PDC control (11) we get the overall state equation, from which is the separation property well observable:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_i+\mathbf{B}_i\mathbf{K}_j & -\mathbf{B}_i\mathbf{K}_j \\ 0 & \mathbf{A}_i-\mathbf{G}_i\mathbf{C}_j \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{1i} \\ \mathbf{B}_{1i} \end{bmatrix}\mathbf{w}(t) \tag{42}$$

If we incorporate parametric uncertainties into the system, then the design problem is more complicated.

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\mathbf{z}(t))h_j(\mathbf{z}(t))\cdot \\ &\cdot\left\{ \begin{bmatrix} \mathbf{A}_i+\Delta\mathbf{A}_i+(\mathbf{B}_{2i}+\Delta\mathbf{B}_{2i})\mathbf{K}_j & -(\mathbf{B}_{2i}+\Delta\mathbf{B}_{2i})\mathbf{K}_j \\ \Delta\mathbf{A}_i+\Delta\mathbf{B}_{2i}\mathbf{K}_j & \mathbf{A}_i-\mathbf{G}_i\mathbf{C}_j \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \right. \\ &\left. + \begin{bmatrix} \mathbf{B}_{1i}+\Delta\mathbf{B}_{1i} \\ \mathbf{B}_{1i}+\Delta\mathbf{B}_{1i} \end{bmatrix}\mathbf{w}(t) \right\} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\mathbf{z}(t))h_j(\mathbf{z}(t))\cdot\left\{ \widetilde{\mathbf{A}}_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \widetilde{\mathbf{B}}_i\mathbf{w}(t) \right\} \end{aligned} \tag{43}$$

We can see, that the stability can be changed since the zero element disappeared. The separation property will not hold. The problem is difficult to solve since the design method must assure the Lyapunov stability of the overall system. In the sense of quadratic Lyapunov stability, that is commonly used for T-S fuzzy systems it means that the there must exist a common matrix $\widetilde{\mathbf{P}}$ that

$$\widetilde{\mathbf{A}}_i^T\widetilde{\mathbf{P}}+\widetilde{\mathbf{P}}\widetilde{\mathbf{A}}_i<0 \tag{43}$$

holds for $i=1,2,...,r$.

This condition have to be incorporated into a design method. The parametric uncertainties can be included in the form (2). In further research we will concern on this problem.

V. Conclusions

The paper deals with a fuzzy observer design problem. The method is based on a convex numerical optimization of a set of LMIs. The LMI conditions are derived here and allow us to increase robustness against input disturbances and parametric uncertainties by minimizing of the upper bound of the $\boldsymbol{H}_\infty$ norm. The fast state estimation is ensured by specifying of minimal decay rate parameter.

The duality of the controller and observer is shown here. The observer design method is then derived on a base of results in [1], where a method for a PDC controller design is published.

The paper also discuss the problem of design of the observer based control fuzzy system, where can be both parts designed separately due to a separation principle. The different situation comes if we would like to incorporate the parametric uncertainties into the design process. It was shown, that there are some differences between the observer and PDC controller in this case and that it is possible to find a method, that could cope with this problem, only if we will work with an overall system consisting both state variables an estimation error variables.

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