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Riemann-Liouville Fractional Calculus and Multiindex Dzrbashjan-Gelfond-Leontiev Differentiation and Integration with Multiindex Mittag-Leffler Function

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Abstract—The multiindex Mittag-Leffler (M-L) function and the multiindex Dzrbashjan-Gelfond-Leontiev (D-G-L) differentiation and integration play a very pivotal role in the theory and applications of generalized fractional calculus. The object of this paper is to investigate the relations that exist between the Riemann-Liouville fractional calculus and multiindex Dzrbashjan-Gelfond-Leontiev differentiation and integration with multiindex Mittag-Leffler function.

Keywords—Multiindex Mittag-Leffler function, Multiindex Dzrbashjan-Gelfond-Leontiev differentiation and integration, Riemann-Liouville fractional integrals and derivatives.

I. Introduction

THE Mittag-Leffler (M-L) function, titled as "Queen"-function of fractional calculus (FC) due to mainly its applications in the solutions of fractional-order differential and integral equations arising problems of mathematical, physical, biological and engineering areas.

The Mittag-Leffler (M-L) functions E_α (Mittag-Leffler, 1902-1905) and $E_{\alpha,\beta}$ (Wiman 1905, Agarwal 1953) are defined by the power series

$$\left.\begin{array}{c} E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\alpha>0 \\ E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\alpha>0,\beta>0. \end{array}\right\} \quad (1)$$

An important identity of M-L function [4] is

$$E_{\alpha,\beta}(z)=\frac{1}{\Gamma(\beta)}+zE_{\alpha,\alpha+\beta}(z), \quad (2)$$

which will be required later on.

In the section 2 of the book by Samko, Kilbas and Marichev [9], the left and right sided operators of Riemann-Liouville fractional calculus are defined as follows:

$$\left(I_{0+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}dt,\alpha>0 \quad (3)$$

$$\left(I_{-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty\frac{f(t)}{(t-x)^{1-\alpha}}dt,\alpha>0 \quad (4)$$

$$\left(D_{0+}^{\alpha}f\right)(x)=\left(\tfrac{d}{dx}\right)^{[\alpha]+1}\left(I_{0+}^{1-\{\alpha\}}f\right)(x)$$

$$=\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_0^x\frac{f(t)}{(x-t)^{\{\alpha\}}},\alpha>0 \quad (5)$$

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$$\left(D_{-}^{\alpha}f\right)(x)=\left(-\tfrac{d}{dx}\right)^{[\alpha]+1}\left(I_{-}^{1-\{\alpha\}}f\right)(x)$$

$$=\frac{1}{\Gamma(1-\{\alpha\})}\left(-\frac{d}{dx}\right)^{[\alpha]+1}\int_x^\infty\frac{f(t)}{(t-x)^{\{\alpha\}}},\alpha>0, \quad (6)$$

where $[\alpha]$ means the maximal integer not exceeding α and $\{\alpha\}$ is the fractional part of α .

II. Multiindex M-L function and multiindex D-G-L differentiation and integration

The multiindex (multiple, m-tuple) M-L function is introduced by Kiryakova [6, 7] and the multiindex D-G-L differentiation and integration, generated by the multiindex M-L function are introduced and studied by Kiryakova [8].

Definition II.1 [6, 7] *Let $m>1$ be an integer, $\rho_1,\cdots,\rho_m>0$ and $\mu_1,\cdots,\mu_m$ be arbitrary real numbers, then the multiindex M-L function are defined by means of the power series*

$$E_{(\frac{1}{\rho_i}),(\mu_i)}(z)=\sum_{k=0}^{\infty}\phi_k z^k=\sum_{k=0}^{\infty}\frac{z^k}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}. \quad (7)$$

For $m=1$, this is the classical M-L function $E_{(\frac{1}{\rho}),(\mu)}(z)$, and for $\frac{1}{\rho}=\alpha>0,\mu=\beta>0$, it is the M-L function $E_{\alpha,\beta}(z)$ considered by Wiman 1905 and Agarwal 1953, which is given in (1).

For $m=2$, (7) reduces to the M-L function considered first by Dzrbashjan [3]. He denoted it by $\phi_{\rho_1,\rho_2}(z;\mu_1,\mu_2)$ and defined in the following form, see also [8, Appendix]:

$$\begin{aligned} E_{(\frac{1}{\rho_1},\frac{1}{\rho_2}),(\mu_1,\mu_2)}(z) &= \phi_{\rho_1,\rho_2}(z;\mu_1,\mu_2) \\ &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\Gamma(\mu_2+\frac{k}{\rho_2})} \end{aligned} \quad (8)$$

Definition II.2 [8, 6, 5] *Let $f(z)$ be analytic function in a disk $\Delta_R=\{|z|<R\}$ and $\rho_i>0,\mu_i\in\Re(i=1,\cdots,m)$ be arbitrary parameters, then the correspondences:* $f(z)=\sum_{k=0}^{\infty}a_kz^k\longmapsto\tilde{D}f(z)=D_{(\rho_i),(\mu_i)}f(z),\tilde{L}f(z)=L_{(\rho_i),(\mu_i)}f(z)$, *defined by*

$$\left.\begin{array}{c} \tilde{D}f(z)=\sum_{k=1}^{\infty}a_k\prod_{j=1}^{m}\left(\frac{\Gamma(\mu_j+\frac{k}{\rho_j})}{\Gamma(\mu_j+\frac{k-1}{\rho_j})}\right)z^{k-1} \\ \tilde{L}f(z)=\sum_{k=0}^{\infty}a_k\prod_{j=1}^{m}\left(\frac{\Gamma(\mu_j+\frac{k}{\rho_j})}{\Gamma(\mu_j+\frac{k+1}{\rho_j})}\right)z^{k+1} \end{array}\right\} \quad (9)$$

are called multiindex D-G-L differentiations and integrations, respectively.

For $m = 1$, the operators (9) are D-G-L operators of differentiation and integration, studied by Dimovski and Kiryakova [1, 2], Kiryakova [8]:

$$\left.\begin{aligned} D_{\rho,\mu}f(z) &= \sum_{k=1}^{\infty} a_k \frac{\Gamma(\mu+\frac{k}{\rho})}{\Gamma(\mu+\frac{k-1}{\rho})} z^{k-1} \\ L_{\rho,\mu}f(z) &= \sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu+\frac{k}{\rho})}{\Gamma(\mu+\frac{k+1}{\rho})} z^{k+1} \end{aligned}\right\} \quad (10)$$

Lemma II.3 [6, 7] *The multiple M-L function (7) satisfy the following relations* ($\lambda \neq 0$):

$$D_{(\rho_i),(\mu_i)} E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) = \lambda E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) \quad (11)$$

$$L_{(\rho_i),(\mu_i)} E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) = \frac{1}{\lambda} E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) - \frac{1}{\lambda \prod_{j=1}^{m} \Gamma(\mu_j)} \quad (12)$$

One can verify the above lemma easily by applying the definition (7) and (9).

III. Relations with Riemann-Liouville fractional calculus

In this section we derive certain relations that exist between the multiindex D-G-L operators of differentiation and integration connecting with multiindex M-L function and the left and right sided operators of Riemann-Liouville fractional integrals and derivatives.

Theorem III.1 *Let $\alpha > 0, \rho_i > 0, \mu_i \in \Re (i = 1, \cdots, m), \lambda \neq 0$, and let I_{0+}^{α} be the left sided operator of Riemann-Liouville fractional integral (3), then there holds the formula*

$$\left(I_{0+}^{\alpha}\left[t^{\mu_1-1}\tilde{D}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{\frac{1}{\rho_1}})\right]\right)(x)$$
$$= x^{\alpha+\mu_1-\frac{1}{\rho_1}-1}\Big[E_{(\frac{1}{\rho_i}),(\mu_1+\alpha-\frac{1}{\rho_1},\mu_2-\frac{1}{\rho_2},\cdots,\mu_m-\frac{1}{\rho_m})}(\lambda x^{\frac{1}{\rho_1}})$$
$$-\frac{1}{\Gamma(\mu_1+\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\Big] \quad (13)$$

Proof: By virtue of (3) and (11), we have

$$\begin{aligned} \Lambda &\equiv \left(I_{0+}^{\alpha}\left[t^{\mu_1-1}\tilde{D}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{\frac{1}{\rho_1}})\right]\right)(x) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\lambda\sum_{k=0}^{\infty}\frac{\lambda^k t^{\frac{k}{\rho_1}+\mu_1-1}}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}dt \end{aligned}$$

Interchanging the order of integration and summation and evaluating the inner integral by means of beta function formula, it gives

$$\begin{aligned} \Lambda &= x^{\alpha+\mu_1-1}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}x^{\frac{k}{\rho_1}}}{\Gamma(\mu_1+\alpha+\frac{k}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})} \\ &= x^{\alpha+\mu_1-1}\sum_{k=1}^{\infty}\frac{\lambda^{k}x^{\frac{k-1}{\rho_1}}}{\Gamma(\mu_1+\alpha+\frac{k-1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k-1}{\rho_j})} \\ &= x^{\alpha+\mu_1-\frac{1}{\rho_1}-1} \\ &\cdot\Big[\sum_{k=0}^{\infty}\frac{\lambda^k x^{\frac{k}{\rho_1}}}{\Gamma((\mu_1+\alpha-\frac{1}{\rho_1})+\frac{k}{\rho_1})\prod_{j=2}^{m}\Gamma((\mu_j-\frac{1}{\rho_j})+\frac{k}{\rho_j})} \\ &\quad -\frac{1}{\Gamma(\mu_1+\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\Big] \\ &= x^{\alpha+\mu_1-\frac{1}{\rho_1}-1}\Big[E_{(\frac{1}{\rho_i}),(\mu_1+\alpha-\frac{1}{\rho_1},\mu_2-\frac{1}{\rho_2},\cdots,\mu_m-\frac{1}{\rho_m})}(\lambda x^{\frac{1}{\rho_1}}) \\ &\quad -\frac{1}{\Gamma(\mu_1+\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\Big] \end{aligned}$$

Corollary III.2 *Let $\alpha > 0, \rho_i > 0, \mu_i \in \Re (i = 1, \cdots, m), \lambda \neq 0$ and $\frac{1}{\rho_1} = \alpha$, there holds the formula*

$$\left(I_{0+}^{\alpha}\left[t^{\mu_1-1}\tilde{D}E_{(\alpha,\frac{1}{\rho_2},\cdots,\frac{1}{\rho_m}),(\mu_i)}(\lambda t^{\alpha})\right]\right)(x)$$
$$= x^{\mu_1-1}\Big[E_{(\alpha,\frac{1}{\rho_2},\cdots,\frac{1}{\rho_m}),(\mu_1,\mu_2-\frac{1}{\rho_2},\cdots,\mu_m-\frac{1}{\rho_m})}(\lambda x^{\alpha})$$
$$-\frac{1}{\Gamma(\mu_1)\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\Big] \quad (14)$$

For $m = 1, \frac{1}{\rho_1} = \alpha$ and $\mu_1 = \mu$, (13) reduces to

Corollary III.3 *For $\alpha > 0, \rho > 0, \mu \in \Re$ and $\lambda \neq 0$, there holds the formula*

$$\left(I_{0+}^{\alpha}\left[t^{\mu-1}D_{\rho,\mu}E_{\alpha,\mu}(\lambda t^{\alpha})\right]\right)(x) = x^{\mu-1}\Big[E_{\alpha,\mu}(\lambda x^{\alpha}) - \frac{1}{\Gamma(\mu)}\Big] \quad (15)$$

Theorem III.4 *Let $\alpha > 0, \rho_i > 0, \mu_i \in \Re (i = 1, \cdots, m), \lambda \neq 0$, and let I_{0+}^{α} be the left sided operator of Riemann-Liouville fractional integral (3), then there holds the formula*

$$\left(I_{0+}^{\alpha}\left[t^{\mu_1-1}\tilde{L}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{\frac{1}{\rho_1}})\right]\right)(x) = \frac{1}{\lambda}x^{\alpha+\mu_1-1}$$
$$\cdot\Big[E_{(\frac{1}{\rho_i}),(\mu_1+\alpha,\mu_2,\cdots,\mu_m)}(\lambda x^{\frac{1}{\rho_1}}) - \frac{1}{\Gamma(\mu_1+\alpha)\prod_{j=2}^{m}\Gamma(\mu_j)}\Big] \quad (16)$$

Proof: By virtue of (3) and (12), we have

$$\begin{aligned} \Lambda &\equiv \left(I_{0+}^{\alpha}\left[t^{\mu_1-1}\tilde{L}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{\frac{1}{\rho_1}})\right]\right)(x) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\frac{1}{\lambda}\Big[\sum_{k=0}^{\infty}\frac{\lambda^k t^{\frac{k}{\rho_1}+\mu_1-1}}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})} \\ &\quad -\frac{t^{\mu_1-1}}{\prod_{j=1}^{m}\Gamma(\mu_j)}\Big]dt \end{aligned}$$

Now splitting the integral into two integrals and changing the order of integration and summation in the first integral and then evaluating the two integrals with the help of beta function formula, we obtain

$$\begin{aligned} \Lambda &= \frac{1}{\lambda}x^{\alpha+\mu_1-1}\Big[\sum_{k=0}^{\infty}\frac{\lambda^k x^{\frac{k}{\rho_1}}}{\Gamma((\mu_1+\alpha)+\frac{k}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})} \\ &\quad -\frac{1}{\Gamma(\mu_1+\alpha)\prod_{j=2}^{m}\Gamma(\mu_j)}\Big] \\ &= \frac{1}{\lambda}x^{\alpha+\mu_1-1}\Big[E_{(\frac{1}{\rho_i}),(\mu_1+\alpha,\mu_2,\cdots,\mu_m)}(\lambda x^{\frac{1}{\rho_1}}) \\ &\quad -\frac{1}{\Gamma(\mu_1+\alpha)\prod_{j=2}^{m}\Gamma(\mu_j)}\Big] \end{aligned}$$

This completes the proof of the theorem III.4

For $m = 1, \frac{1}{\rho_1} = \beta$ and $\mu_1 = \mu$, the above theorem reduces to

Corollary III.5 *For $\alpha > 0, \rho > 0, \mu \in \Re$ and $\lambda \neq 0$, there holds the formula*

$$\left(I_{0+}^{\alpha}\left[t^{\mu-1}L_{\rho,\mu}E_{\beta,\mu}(\lambda t^{\beta})\right]\right)(x) = \frac{1}{\lambda}x^{\alpha+\mu-1}\Big[E_{\beta,\mu+\alpha}(\lambda x^{\beta}) - \frac{1}{\Gamma(\mu+\alpha)}\Big] \quad (17)$$

On using the identity (2) on the right hand side of (17), it reduces to the result

$$\left(I_{0+}^{\alpha}\left[t^{\mu-1}L_{\rho,\mu}E_{\beta,\mu}(\lambda t^{\beta})\right]\right)(x) = x^{\alpha+\beta+\mu-1}E_{\beta,\alpha+\beta+\mu}(\lambda x^{\beta}) \quad (18)$$

Theorem III.6 *Let $\alpha > 0, \rho_i > 0, \mu_i \in \Re(i = 1, \cdots, m), \lambda \neq 0$, and let I_{-}^{α} be the right sided operator of Riemann-Liouville fractional integral (4), then there holds the formula*

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\mu_1}\tilde{D}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{-\frac{1}{\rho_1}})\right]\right)(x)$$
$$= x^{-\mu_1+\frac{1}{\rho_1}}\Bigg[E_{(\frac{1}{\rho_i}),(\mu_1+\alpha-\frac{1}{\rho_1},\mu_2-\frac{1}{\rho_2},\cdots,\mu_m-\frac{1}{\rho_m})}(\lambda x^{-\frac{1}{\rho_1}}) - \frac{1}{\Gamma(\mu_1+\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\Bigg] \quad (19)$$

Proof: By virtue of (4) and (11), we have

$$\Lambda \equiv \left(I_{-}^{\alpha}\left[t^{-\alpha-\mu_1}\tilde{D}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{-\frac{1}{\rho_1}})\right]\right)(x)$$
$$= \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\mu_1}\lambda\sum_{k=0}^{\infty}\frac{\lambda^k t^{-\frac{k}{\rho_1}}}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}dt$$

If we interchange the order of integration and summation and substitute $t = \frac{x}{u}$ to evaluate the inner integral, we obtain

$$\Lambda = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}x^{-\mu_1-\frac{k}{\rho_1}}}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}\int_0^1 u^{\frac{k}{\rho_1}+\mu_1-1}(1-u)^{\alpha-1}du$$
$$= x^{-\mu_1}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}x^{-\frac{k}{\rho_1}}}{\Gamma(\mu_1+\alpha+\frac{k}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}$$
$$= x^{-\mu_1}\sum_{k=1}^{\infty}\frac{\lambda^{k}x^{-\frac{(k-1)}{\rho_1}}}{\Gamma(\mu_1+\alpha+\frac{k-1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k-1}{\rho_j})}$$
$$= x^{-\mu_1+\frac{1}{\rho_1}}\cdot\Bigg[\sum_{k=0}^{\infty}\frac{\lambda^k x^{-\frac{k}{\rho_1}}}{\Gamma((\mu_1+\alpha-\frac{1}{\rho_1})+\frac{k}{\rho_1})\prod_{j=2}^{m}\Gamma((\mu_j-\frac{1}{\rho_j})+\frac{k}{\rho_j})} - \frac{1}{\Gamma(\mu_1+\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\Bigg]$$
$$= x^{-\mu_1+\frac{1}{\rho_1}}\Bigg[E_{(\frac{1}{\rho_i}),(\mu_1+\alpha-\frac{1}{\rho_1},\mu_2-\frac{1}{\rho_2},\cdots,\mu_m-\frac{1}{\rho_m})}(\lambda x^{-\frac{1}{\rho_1}}) - \frac{1}{\Gamma(\mu_1+\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\Bigg]$$

Corollary III.7 *Let $\alpha > 0, \rho_i > 0, \mu_i \in \Re(i = 1, \cdots, m), \lambda \neq 0$ and $\frac{1}{\rho_1} = \alpha$, there holds the formula*

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\mu_1}\tilde{D}E_{(\alpha,\frac{1}{\rho_2},\cdots,\frac{1}{\rho_m}),(\mu_i)}(\lambda t^{-\alpha})\right]\right)(x)$$
$$= x^{-\mu_1+\alpha}\Bigg[E_{(\alpha,\frac{1}{\rho_2},\cdots,\frac{1}{\rho_m}),(\mu_1,\mu_2-\frac{1}{\rho_2},\cdots,\mu_m-\frac{1}{\rho_m})}(\lambda x^{-\alpha}) - \frac{1}{\Gamma(\mu_1)\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\Bigg] \quad (20)$$

For $m = 1, \frac{1}{\rho_1} = \alpha$ and $\mu_1 = \mu$, (19) reduces to

Corollary III.8 *For $\alpha > 0, \rho > 0, \mu \in \Re$ and $\lambda \neq 0$, there holds the formula*

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\mu}D_{\rho,\mu}E_{\alpha,\mu}(\lambda t^{-\alpha})\right]\right)(x) = x^{-\mu+\alpha}\Big[E_{\alpha,\mu}(\lambda x^{-\alpha}) - \frac{1}{\Gamma(\mu)}\Big] \quad (21)$$

Theorem III.9 *Let $\alpha > 0, \rho_i > 0, \mu_i \in \Re(i = 1, \cdots, m), \lambda \neq 0$, and let I_{-}^{α} be the right sided operator of Riemann-Liouville fractional integral (4), then there holds the formula*

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\mu_1}\tilde{L}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{-\frac{1}{\rho_1}})\right]\right)(x)$$
$$= \tfrac{1}{\lambda}x^{-\mu_1}\Bigg[E_{(\frac{1}{\rho_i}),(\mu_1+\alpha,\mu_2,\cdots,\mu_m}(\lambda x^{-\frac{1}{\rho_1}}) - \frac{1}{\Gamma(\mu_1+\alpha)\prod_{j=2}^{m}\Gamma(\mu_j)}\Bigg] \quad (22)$$

Proof: By virtue of (4) and (12), we have

$$\Lambda \equiv \left(I_{-}^{\alpha}\left[t^{-\alpha-\mu_1}\tilde{L}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{-\frac{1}{\rho_1}})\right]\right)(x)$$
$$= \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\mu_1}\cdot\frac{1}{\lambda}\Bigg[\sum_{k=0}^{\infty}\frac{\lambda^k t^{-\frac{k}{\rho_1}}}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})} - \frac{1}{\prod_{j=1}^{m}\Gamma(\mu_j)}\Bigg]dt$$

Now splitting the integral into two integrals and interchanging the order of integration and summation in the first integral and substitute $t = \frac{x}{u}$ to evaluate the integrals, we obtain

$$\Lambda = \tfrac{1}{\lambda}x^{-\mu_1}\Bigg[\sum_{k=0}^{\infty}\frac{\lambda^k x^{-\frac{k}{\rho_1}}}{\Gamma((\mu_1+\alpha)+\frac{k}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})} - \frac{1}{\Gamma(\mu_1+\alpha)\prod_{j=2}^{m}\Gamma(\mu_j)}\Bigg]$$
$$= \tfrac{1}{\lambda}x^{-\mu_1}\Bigg[E_{(\frac{1}{\rho_i}),(\mu_1+\alpha,\mu_2,\cdots,\mu_m)}(\lambda x^{-\frac{1}{\rho_1}}) - \frac{1}{\Gamma(\mu_1+\alpha)\prod_{j=2}^{m}\Gamma(\mu_j)}\Bigg]$$

For $m = 1, \frac{1}{\rho_1} = \beta$ and $\mu_1 = \mu$, (22) reduces to

Corollary III.10 *For $\alpha > 0, \rho > 0, \mu \in \Re$ and $\lambda \neq 0$, there holds the formula*

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\mu}L_{\rho,\mu}E_{\beta,\mu}(\lambda t^{-\beta})\right]\right)(x) = \frac{1}{\lambda}x^{-\mu}\Big[E_{\beta,\mu+\alpha}(\lambda x^{-\beta}) - \frac{1}{\Gamma(\mu+\alpha)}\Big] \quad (23)$$

Making use of the identity (2), (23) reduces to the result

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\mu}L_{\rho,\mu}E_{\beta,\mu}(\lambda t^{-\beta})\right]\right)(x)=x^{-\mu-\beta}E_{\beta,\alpha+\beta+\mu}(\lambda x^{-\beta}) \tag{24}$$

We now proceed to derive certain other theorems of $\tilde{D}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z)$ and $\tilde{L}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z)$ associated with the fractional derivative operators D_{0+}^{α} and D_{-}^{α} defined by (5) and (6), respectively.

Theorem III.11 *Let $\alpha>0,\rho_i>0,\mu_i\in\Re(i=1,\cdots,m),\lambda\neq 0$, and let D_{0+}^{α} be the left sided operator of Riemann-Liouville fractional derivative (5), then there holds the formula*

$$\begin{aligned}&\left(D_{0+}^{\alpha}\left[t^{\mu_1-1}\tilde{D}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{\frac{1}{\rho_1}})\right]\right)(x)\\&=x^{\mu_1-\alpha-\frac{1}{\rho_1}-1}\left[E_{(\frac{1}{\rho_i}),(\mu_1-\alpha-\frac{1}{\rho_1},\mu_2-\frac{1}{\rho_2},\cdots,\mu_m-\frac{1}{\rho_m})}(\lambda x^{\frac{1}{\rho_1}})\right.\\&\left.-\frac{1}{\Gamma(\mu_1-\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\right]\end{aligned} \tag{25}$$

Proof: By virtue of (5) and (11), we have

$$\begin{aligned}\Lambda&\equiv\left(D_{0+}^{\alpha}\left[t^{\mu_1-1}\tilde{D}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{\frac{1}{\rho_1}})\right]\right)(x)\\&=\left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{0+}^{1-\{\alpha\}}\left[t^{\mu_1-1}\tilde{D}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{\frac{1}{\rho_1}})\right]\right)(x)\\&=\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\frac{1}{\Gamma(1-\{\alpha\})}\\&\quad\cdot\int_0^x(x-t)^{-\{\alpha\}}t^{\frac{k}{\rho_1}+\mu_1-1}dt\\&=\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{\Gamma(\frac{k}{\rho_1}+\mu_1+1-\{\alpha\})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}\\&\quad\cdot\left(\frac{d}{dx}\right)^{[\alpha]+1}x^{\frac{k}{\rho_1}+\mu_1-\{\alpha\}}\\&=\sum_{k=0}^{\infty}\frac{\lambda^{k+1}x^{\frac{k}{\rho_1}+\mu_1-\alpha-1}}{\Gamma(\frac{k}{\rho_1}+\mu_1-\alpha)\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}\\&=x^{\mu_1-\alpha-1}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}x^{\frac{k}{\rho_1}}}{\Gamma(\mu_1-\alpha+\frac{k}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}\\&=x^{\mu_1-\alpha-1}\sum_{k=1}^{\infty}\frac{\lambda^{k}x^{\frac{k-1}{\rho_1}}}{\Gamma(\mu_1-\alpha+\frac{k-1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k-1}{\rho_j})}\\&=x^{\mu_1-\alpha-\frac{1}{\rho_1}-1}\\&\quad\cdot\left[\sum_{k=0}^{\infty}\frac{\lambda^{k}x^{\frac{k}{\rho_1}}}{\Gamma((\mu_1-\alpha-\frac{1}{\rho_1})+\frac{k}{\rho_1})\prod_{j=2}^{m}\Gamma((\mu_j-\frac{1}{\rho_j})+\frac{k}{\rho_j})}\right.\\&\quad\left.-\frac{1}{\Gamma(\mu_1-\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\right]\\&=x^{\mu_1-\alpha-\frac{1}{\rho_1}-1}\left[E_{(\frac{1}{\rho_i}),(\mu_1-\alpha-\frac{1}{\rho_1},\mu_2-\frac{1}{\rho_2},\cdots,\mu_m-\frac{1}{\rho_m})}(\lambda x^{\frac{1}{\rho_1}})\right.\\&\quad\left.-\frac{1}{\Gamma(\mu_1-\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\right]\end{aligned}$$

For $m=1,\frac{1}{\rho_1}=\beta$ and $\mu_1=\mu$, (25) reduces to

Corollary III.12 *For $\alpha>0,\rho>0,\mu\in\Re$ and $\lambda\neq 0$, there holds the formula*

$$\begin{aligned}&\left(D_{0+}^{\alpha}\left[t^{\mu-1}D_{\rho,\mu}E_{\beta,\mu}(\lambda t^{\beta})\right]\right)(x)\\&=x^{\mu-\alpha-\beta-1}\left[E_{\beta,\mu-\alpha-\beta}(\lambda x^{\beta})-\frac{1}{\Gamma(\mu-\alpha-\beta)}\right]\end{aligned} \tag{26}$$

Theorem III.13 *Let $\alpha>0,\rho_i>0,\mu_i\in\Re(i=1,\cdots,m),\lambda\neq 0$, and let D_{0+}^{α} be the left sided operator of Riemann-Liouville fractional derivative (5), then there holds the formula*

$$\begin{aligned}&\left(D_{0+}^{\alpha}\left[t^{\mu_1-1}\tilde{L}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{\frac{1}{\rho_1}})\right]\right)(x)\\&=\tfrac{1}{\lambda}x^{\mu_1-\alpha-1}\left[E_{(\frac{1}{\rho_i}),(\mu_1-\alpha,\mu_2,\cdots,\mu_m)}(\lambda x^{\frac{1}{\rho_1}})\right.\\&\left.-\frac{1}{\Gamma(\mu_1-\alpha)\prod_{j=2}^{m}\Gamma(\mu_j)}\right]\end{aligned} \tag{27}$$

The proof of the above theorem can be developed by the similar lines to that of the theorem III.11

For $m=1,\frac{1}{\rho_1}=\beta$ and $\mu_1=\mu$, (27) reduces to

Corollary III.14 *For $\alpha>0,\rho>0,\mu\in\Re$ and $\lambda\neq 0$, there holds the formula*

$$\begin{aligned}&\left(D_{0+}^{\alpha}\left[t^{\mu-1}L_{\rho,\mu}E_{\beta,\mu}(\lambda t^{\beta})\right]\right)(x)\\&=\frac{1}{\lambda}x^{\mu-\alpha-1}\left[E_{\beta,\mu-\alpha}(\lambda x^{\beta})-\frac{1}{\Gamma(\mu-\alpha)}\right]\end{aligned} \tag{28}$$

On using the identity (2), (28) reduces to the result

$$\begin{aligned}&\left(D_{0+}^{\alpha}\left[t^{\mu-1}L_{\rho,\mu}E_{\beta,\mu}(\lambda t^{\beta})\right]\right)(x)\\&=x^{\mu-\alpha+\beta-1}E_{\beta,\mu+\beta-\alpha}(\lambda x^{\beta})\end{aligned} \tag{29}$$

Theorem III.15 *Let $\alpha>0,\rho_i>0,\mu_i\in\Re(i=1,\cdots,m),\mu_1-[\alpha]>1,\lambda\neq 0$, and let D_{-}^{α} be the right sided operator of Riemann-Liouville fractional derivative (6), then there holds the formula*

$$\begin{aligned}&\left(D_{-}^{\alpha}\left[t^{\alpha-\mu_1}\tilde{D}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{-\frac{1}{\rho_1}})\right]\right)(x)\\&=x^{-\mu_1+\frac{1}{\rho_1}}\left[E_{(\frac{1}{\rho_i}),(\mu_1-\alpha-\frac{1}{\rho_1},\mu_2-\frac{1}{\rho_2},\cdots,\mu_m-\frac{1}{\rho_m})}(\lambda x^{-\frac{1}{\rho_1}})\right.\\&\left.-\frac{1}{\Gamma(\mu_1-\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\right]\end{aligned} \tag{30}$$

Proof: By virtue of (6) and (11), we have

$$\begin{aligned}\Lambda&\equiv\left(D_{-}^{\alpha}\left[t^{\alpha-\mu_1}\tilde{D}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{-\frac{1}{\rho_1}})\right]\right)(x)\\&=\left(-\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{-}^{1-\{\alpha\}}\left[t^{\alpha-\mu_1}\tilde{D}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{-\frac{1}{\rho_1}})\right]\right)(x)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}\left(-\frac{d}{dx}\right)^{[\alpha]+1}\frac{1}{\Gamma(1-\{\alpha\})}\\
&\quad\cdot\int_x^{\infty}(t-x)^{-\{\alpha\}}t^{-\frac{k}{\rho_1}+\alpha-\mu_1}dt\\
&= \sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{\Gamma(\mu_1-\alpha+\frac{k}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}\\
&\quad\cdot\left(-\frac{d}{dx}\right)^{[\alpha]+1}x^{-\{\alpha\}-\frac{k}{\rho_1}+\alpha-\mu_1+1}\\
&= \sum_{k=0}^{\infty}\frac{\lambda^{k+1}x^{-\frac{k}{\rho_1}-\mu_1}}{\Gamma(\mu_1-\alpha+\frac{k}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}\\
&= x^{-\mu_1}\sum_{k=1}^{\infty}\frac{\lambda^k x^{-\frac{(k-1)}{\rho_1}}}{\Gamma(\mu_1-\alpha+\frac{k-1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j+\frac{k-1}{\rho_j})}\\
&= x^{-\mu_1+\frac{1}{\rho_1}}\\
&\quad\cdot\Bigg[\sum_{k=0}^{\infty}\frac{\lambda^k x^{-\frac{k}{\rho_1}}}{\Gamma((\mu_1-\alpha-\frac{1}{\rho_1})+\frac{k}{\rho_1})\prod_{j=2}^{m}\Gamma((\mu_j-\frac{1}{\rho_j})+\frac{k}{\rho_j})}\\
&\qquad-\frac{1}{\Gamma(\mu_1-\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\Bigg]\\
&= x^{-\mu_1+\frac{1}{\rho_1}}\Bigg[E_{(\frac{1}{\rho_i}),(\mu_1-\alpha-\frac{1}{\rho_1},\mu_2-\frac{1}{\rho_2},\cdots,\mu_m-\frac{1}{\rho_m})}(\lambda x^{-\frac{1}{\rho_1}})\\
&\qquad-\frac{1}{\Gamma(\mu_1-\alpha-\frac{1}{\rho_1})\prod_{j=2}^{m}\Gamma(\mu_j-\frac{1}{\rho_j})}\Bigg]
\end{aligned}
$$

If we set $m=1, \frac{1}{\rho_1}=\beta, \mu_1=\mu$, (30) reduces to

Corollary III.16 *If $\alpha>0, \rho>0, \mu\in\Re, \mu-[\alpha]>1$ and $\lambda\neq 0$, then there holds the formula*
$\left(D_-^{\alpha}\left[t^{\alpha-\mu}D_{\rho,\mu}E_{\beta,\mu}(\lambda t^{-\beta})\right]\right)(x)$

$$= x^{-\mu+\beta}\Big[E_{\beta,\mu-\alpha-\beta}(\lambda x^{-\beta})-\frac{1}{\Gamma(\mu-\alpha-\beta)}\Big] \tag{31}$$

Theorem III.17 *Let $\alpha>0, \rho_i>0, \mu_i\in\Re(i=1,\cdots,m), \mu_1-[\alpha]>1, \lambda\neq 0$, and let D_-^{α} be the right sided operator of Riemann-Liouville fractional derivative (6), then there holds the formula*
$\left(D_-^{\alpha}\left[t^{\alpha-\mu_1}\tilde{L}E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda t^{-\frac{1}{\rho_1}})\right]\right)(x)$
$= \frac{1}{\lambda}x^{-\mu_1}\Bigg[E_{(\frac{1}{\rho_i}),(\mu_1-\alpha,\mu_2,\cdots,\mu_m)}(\lambda x^{-\frac{1}{\rho_1}})$

$$-\frac{1}{\Gamma(\mu_1-\alpha)\prod_{j=2}^{m}\Gamma(\mu_j)}\Bigg] \tag{32}$$

The above theorem can be proved by the similar lines to that of the theorem III.15

$m=1, \frac{1}{\rho_1}=\beta, \mu_1=\mu$, (32) reduces to

Corollary III.18 *If $\alpha>0, \rho>0, \mu\in\Re, \mu-[\alpha]>1$ and $\lambda\neq 0$, there holds the formula*
$\left(D_-^{\alpha}\left[t^{\alpha-\mu}L_{\rho,\mu}E_{\beta,\mu}(\lambda t^{-\beta})\right]\right)(x)$

$$= \frac{1}{\lambda}x^{-\mu}\Big[E_{\beta,\mu-\alpha}(\lambda x^{-\beta})-\frac{1}{\Gamma(\mu-\alpha)}\Big] \tag{33}$$

On using the identity (2), (33) reduces to the result

$$\left(D_-^{\alpha}\left[t^{\alpha-\mu}L_{\rho,\mu}E_{\beta,\mu}(\lambda t^{-\beta})\right]\right)(x) = x^{-\mu-\beta}E_{\beta,\beta+\mu-\alpha}(\lambda x^{-\beta}) \tag{34}$$

IV. Conclusion

It is expected that some of the results derived in this survey may find applications in the solution of certain fractional order differential and integral equations arising problems of physical sciences and engineering areas, where the D-G-L differentiation and integration as well as multiindex M-L functions leading a pivotal role.

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