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Research of a Multistep Method Applied to Numerical Solution of Volterra Integro-Differential Equation

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***Abstract*—**Solution of some practical problems is reduced to the solution of the integro-differential equations. But for the numerical solution of such equations basically quadrature methods or its combination with multistep or one-step methods are used. The quadrature methods basically is applied to calculation of the integral participating in right hand side of integro-differential equations. As this integral is of Volterra type, it is obvious that at replacement with its integrated sum the upper limit of the sum depends on a current point in which values of the integral are defined. Thus we receive the integrated sum with variable boundary, to work with is hardly. Therefore multistep method with the constant coefficients, which is free from noted lack and gives the way for finding it's coefficients is present.

***Keywords*—**Volterra integro-differential equations, multistep methods, finite-difference methods, initial value problem

I. Introduction

As is known, from 1908 to 1913 Volterra was persistently engaged in the theory of residual action and proved that differential equations of mechanics and electrodynamics of continuum are only approximations of more exact integro-differential equations. Heat he applied to investigation of some problems of geophysics considering heat elastic properties of the earth's crust preserve residual effect responding to very large time intervals [1-2]. If we generalize these integro-differential equations we can get the following nonlinear Volterra integro-differential equations.

$$y'(x) = f(x,y) + \int_{x_0}^{x} K(x,s,y(s))ds. \qquad (1)$$

Let's admit that continuous functions $f(x,y)$ and $K(x,s,y)$ are defined on their respective domain

$$G = \{x_0 \le x \le X,\ |y| \le a\}$$

and $\overline{G} = \{x_0 \le s \le x+\varepsilon \le X+\varepsilon,\ |y| \le a\}$ (ε tend to zero as $h \to 0$) and satisfy a Lipchitz condition in the variable y, and the solution of the equation (1) at the point x_0 satisfies following condition:

$$y(x_0) = y_0. \qquad (2)$$

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Suppose that the initial-value problem (1) - (2) has a unique solution defined on some segment $[x_0, X]$. The main of this report is to obtain a numerical solution of the initial-value problem (1) - (2). Therefore the segment $[x_0, X]$ is divided in to N equal parts by the positive and constant step size h. The mesh points are defines as the $x_m = x_0 + mh \ \ (m = 0,1,2,...,N)$. Denote by the y_m, y'_m approximate and $y(x_n)$, $y'(x_n)$ exact values of function $y(x)$ and it's derivative on the point x_m $(m = 0,1,2,..)$. It is known that the initial value problem (1)-(2) can be reduced to an equivalent problem which has the following form [3]:

$$y'(x) = f(x,y) + v(x), \quad y(x_0) = y_0, \qquad (3)$$

$$v(x) = \int_{x_0}^{x} K(x,s,y(s))ds. \qquad (4)$$

After application of the multistep method to the solution of problem (3) and equation (4) (see [3]), we receive the following system of difference equations:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f_{n+i} + h\sum_{i=0}^{k} \beta_i v_{n+i}, \qquad (5)$$

$$\sum_{i=0}^{k} \alpha'_i v_{n+i} = h\sum_{j=0}^{k}\sum_{i=0}^{k} \gamma_{i,j} K(x_{n+j}, x_{n+i}, y_{n+i}), \qquad (6)$$

where coefficients $\alpha_i, \alpha'_i, \beta_i, \gamma_{i,j}$ $(i,j = 0,1,....,k)$ are some real numbers, and $\alpha_k \ne 0, \alpha'_k \ne 0$. If we apply the quadrature method for finding solution of the initial value problem (1) and (2) then we receive the following [4], [5]:

$$y'(x_n) = f(x_n, y_n) + \sum_{i=0}^{n} a_i K(x_n, x_i, y(x_i)), \qquad (7)$$
$$y(x_0) = y_0.$$

The received correlation is the approximation of problem (1) - (2) whose error is defined by the remainder term of quadratures methods. We will notice that correlation (7) is initial value problem for the ordinary differential equations of the first order. Difficulty at the numerical solution of a problem (7) is that with increase in values of n, the quantity of members in the integrated sum increases, and also there appears of necessity

calculations of the values of $K(x_n, s_i, y_i)$ $(i = 0,1,2,...,n)$.

There are many methods for the numerical solution of the initial value problem (7). Among them we can note classical methods of Runge-Kutta and Adams. Now semi implicit and implicit methods of Runge-Kutta, and also generalisation of Adams methods which is called a multistep method with constant coefficients or in the more general form – multistep method of Obreshkov type investigated in [6] are used. Unlike the methods, the method here is applied to the numerical solution of a problem (1) - (2).

One of the important questions at researching the method (5) – (6) is how much closer the found values of the solution of a problem (1) - (2) by the scheme (5) - (6) to exact values of the solution of a problem (1) - (2). Similar researches for the equation (4) were carried out in [7]-[8].

For solving initial value problem (1) and (2) we suggested a multistep method with the constant coefficients to construction of which the next paragraph is devoted

II. Definition of coefficients in formulas (6)

As it is known, accuracy and some properties of methods depend on the values of their coefficients. Here we will consider definitions of coefficients in formulas (6). Concerning definition of coefficients α_i, β_i $(i = 0,1,2,...,k)$ in the formula (5) notice that their values coincide with the values coefficients of the finite-difference methods which were well investigated. Therefore we will consider definition of values of coefficients $\alpha_i', \gamma_{i,j}$ $(i, j = 0,1,2,...,k)$.

Considering that function $y(x)$ depends from $v(x)$, the correlation (5) can be considered as the integral equation from which we will receive the following [9]:

$$v(x_{n+k}) - v(x_{n+k-1}) = \int_{x_0}^{x_{n+k-1}} (K(x_{n+k}, s, y(s)) - K(x_{n+k-1}, s, y(s))ds + \int_{x_{n+k-1}}^{x_{n+k}} (K(x_{n+k}, s, y(s))ds. \quad (8)$$

After using the Lagrange theorem, we will have:

$$v(x_{n+k}) - v(x_{n+k-1}) = h \int_{x_0}^{x_{n+k-1}} K_x'(\xi_{n+k}, s, y(s))ds + \int_{x_{n+k-1}}^{x_{n+k}} K(x_{n+k}, s, y(s))ds, \quad (9)$$

where $x_{n+k-1} < \xi_{n+k} < x_{n+k}$.

It is easy to show that from a correlation (4) we can write the following:

$$v'(x) = K(x, x, y(x)) + \int_{x_0}^{x} K_x'(x, s, y(s))ds. \quad (10)$$

Here we will put $x = \xi_{n+k}$. Then we have:

$$\int_{x_0}^{\xi_{n+k}} K_x'(\xi_{n+k}, s, y(s))ds = v'(\xi_{n+k}) - K(\xi_{n+k}, \xi_{n+k}, y(\xi_{n+k})). \quad (11)$$

After the account received in equality (1.2), we have:

$$v(x_{n+k}) - v(x_{n+k-1}) = hv'(\xi_{n+k}) - hK(\xi_{n+k}, \xi_{n+k}, y(\xi_{n+k})) - h\int_{x_{n+k-1}}^{\xi_{n+k}} K_x'(\xi_{n+k}, s, y(s))ds + \int_{x_{n+k-1}}^{x_{n+k}} K(x_{n+k}, s, y(s))ds. \quad (12)$$

Using replacement of values of derivatives of function through its values we have:

$$hv'(\xi_{n+k}) = \sum_{i=0}^{k} \overline{\alpha}_i v(x_{n+i}) + O(h^{p+1}). \quad (13)$$

Considering (13) in equality (12) we can write:

$$\sum_{i=0}^{k} \alpha_i v(x_{n+i}) = \int_{x_{n+k-1}}^{x_{n+k}} K(x_{n+k}, s, y(s))ds - h\int_{x_{n+k-1}}^{\xi_{n+k}} K_x'(\xi_{n+k}, s, y(s))ds - hK(\xi_{n+k}, \xi_{n+k}, y(\xi_{n+k})) + O(h^{p+1}). \quad (14)$$

Applying the formula (13) to value $K_x'(\xi_{n+k}, s, y(s))$ and replacement of integrals corresponding quadrature formulas and rejecting remainder terms in (14) we will receive a multistep method after which generalization it is had:

$$\sum_{i=0}^{k} \alpha_i v(x_{n+i}) = h\sum_{j=0}^{k}\sum_{i=0}^{k} \beta_{i,j} K(x_{n+j}, x_{n+i}, y_{n+i}). \quad (15)$$

Coefficients $\alpha_i, \beta_{i,j}$ $(i, j = 0,1,2,...,k)$ in a method (15) can be defined with the help above the described schemes. However, with that end in view, one scheme in which result for definition of coefficients the known system of the linearly-algebraic equations turns out here is suggested.

Let's consider a special case when function $K(x, s, y)$ is independent of x and we will put $F(s, y) = K(x, s, y)$. Then from (4) we have:

$$v' = F(x, y), \quad v(x_0) = 0. \quad (16)$$

Let's apply to the numerical solving of a problem (16) the following finite difference method:

$$\sum_{i=0}^{k} \alpha_i v(x_{n+i}) = h\sum_{j=0}^{k} \gamma_i v'_{n+i}, \quad (17)$$

Then we have:

$$\sum_{i=0}^{k} \alpha_i v_{n+i} = h\sum_{i=0}^{k} \gamma_i F(x_{n+i}, y_{n+i}). \quad (18)$$

It is easy to be convinced that if ratio (1.8) we will put $F(s,y)=K(x,s,y)$ and designate by $\gamma_i=\sum_{j=0}^{k}\beta_{i,j}$, then from (15) follows (18).

As it is known, coefficients $\alpha_i,\gamma_i\ (i=0,1,...,k)$ finite-difference method (17) can be defined from the following system of the linearly-algebraic equations:

$$\sum_{i=0}^{k}\alpha_i=0,\sum_{i=0}^{k}\left((i^l/l!)\alpha_i-(i^{l-1}/(l-1)!)\gamma_i\right)=0 \quad (l=1,2,...,p;\ 0!=1). \tag{19}$$

In this system quantity of the equations equally $p+1$, and quantity of unknown equally $2k+2$. It is easy to show that systems (19) to have not trivial solution there should be $p+1<2k+2$. Hence $p\le 2k$.

Let's consider a case $k=2$. As it is known, in this case by, solving system (18), we will find coefficients of method with the maximum degree of accuracy $p_{\max}=4$, which have the following appearance:

$$\alpha_2=-\alpha_0=1,\ \ \alpha_1=0,\ \ \gamma_0=\gamma_2=1/3,\ \ \gamma_1=4/3.$$

The corresponding method registers in a kind:

$$\nu_{n+2}=\nu_n+h(\nu'_{n+2}+4\nu'_{n+1}+\nu'_n)/3. \tag{20}$$

This method is well-known Simpson's method which is unique at $k=2$ and has the degree $p=4$, and there is no method with the degree $p>4$.

On the basis of Simpson's method we will construct method of type (15). With that end in view we will solve the following system of the linearly-algebraic equations:

$$\sum_{i=0}^{k}\beta_{i,j}=\gamma_i,\quad (i=0,1,2,...,k). \tag{21}$$

Let's notice that in this system quantity of unknown persons more than quantity of the equations; hence it has a solution more than one. Therefore the method of type (15) with degree $p=4$ at $k=2$ is not unique. One them looks like:

$$\begin{aligned}\nu_{n+2}=\nu_n+h(&2K(x_n,x_n,y_n)+K(x_n,x_{n+1},y_{n+1})+\\&+K(x_n,x_{n+2},y_{n+2})+K(x_{n+1},x_n,y_n)+\\&+4K(x_{n+1},x_{n+1},y_{n+1})+K(x_{n+1},x_{n+2},y_{n+2})-\\&+2K(x_{n+2},x_n,y_n)-K(x_{n+2},x_{n+1},y_{n+1})-\\&-K(x_{n+2},x_{n+2},y_{n+2}))/3.\end{aligned} \tag{22}$$

However, by means of selection of coefficients $\beta_{i,j}\ (i,j=0,1,2,...,k)$ it is possible to expand area of stability or to reduce absolute value of coefficients at a body in asymptotic decomposition of an error of a method.

For using the method (22) we must to find y_{n+2} the value of $y(x)$ at the point x_{n+2}. For this aim we here suggested the next sequence of multistep method:

$$\begin{aligned}\bar{y}_{n+2}=y_{n+1}+h(3f_{n+1}-f_n)/2+\\+h(3\nu_{n+1}-\nu_n)/2\end{aligned} \tag{23}$$

$$\begin{aligned}\hat{y}_{n+2}=y_{n+1}+h(5f_{n+2}+8f_{n+1}-f_n)/12+\\+h(5\hat{\nu}_{n+2}+8\nu_{n+1}+\nu_n)/12\end{aligned} \tag{24}$$

$$\begin{aligned}y_{n+2}=y_n+h(\hat{f}_{n+2}+4f_{n+1}+f_n)/3+\\+h(\nu_{n+2}+4\nu_{n+1}+\nu_n)/3\end{aligned} \tag{25}$$

Here

$$f_m=f(x_m,y_m),\ \nu_m=\nu(x_m),$$

$$\bar{f}_m=f(x_m,\bar{y}_m),\ \hat{f}_m=f(x_m,y_m),\ (m=0,1,2,...).$$

But for the calculation ν_{n+2} the value of $\nu(x)$ at the point x_{n+2}, we present the next predictor-corrector method:

$$\begin{aligned}\hat{\nu}_{n+2}=\nu_{n+1}+h(&2K(x_{n+1},x_n,y_n)+\\&+K(x_{n+1},x_{n+1},y_{n+1})+K(x_n,x_{n+1},y_{n+1})-\\&-2K(x_n,x_n,y_n))/2\end{aligned} \tag{26}$$

$$\begin{aligned}\nu_{n+2}=\nu_{n+1}+h(&2K(x_{n+2},x_{n+2},y_{n+2})+\\&+2K(x_{n+2},x_{n+1},y_{n+1})+K(x_{n+2},x_n,y_n)+\\&+K(x_{n+1},x_{n+2},y_{n+2})+6K(x_{n+1},x_{n+1},y_{n+1})+\\&+K(x_{n+1},x_n,y_n)-K(x_n,x_n,y_{n+1})-\\&+K(x_n,x_{n+1},y_{n+1})-K(x_n,x_{n+2},y_{n+2}))/12.\end{aligned} \tag{27}$$

Let's note that above used constructed methods there is a necessity of calculations a kernel of integral in points being above diagonal $s=x$. It is obvious, that at $h\to 0$ these points come nearer to points being on a diagonal $s=x$. That is what a domain of definition of a kernel of integral we'll expand to ε. With such situations one faces in application of some approximated methods. However, from above offered multistep methods with constant coefficients it is possible to receive methods which answer to the requirement of the classical theory of Volterra integral equations. For this purpose it is enough to put $\beta_{ji}=0$ at $j<i$ ($i,j=0,1,...,k$). Below some concrete methods are present:

$$\begin{aligned}\hat{\nu}_{n+2}=\nu_n+h(&2K(x_{n+2},x_{n+1},y_{n+1})+\\&+2K(x_{n+1},x_{n+1},y_{n+1})-K(x_{n+1},x_n,y_n)+\\&+K(x_n,x_n,y_n))/2\end{aligned} \tag{28}$$

$$\begin{aligned}\nu_{n+2}=\nu_n+h(&K(x_{n+2},x_{n+2},\hat{y}_{n+2})+\\&+2K(x_{n+2},x_{n+1},y_{n+1})+\\&+K(x_{n+2},x_n,y_n)+2K(x_{n+1},x_{n+1},y_{n+1})+\\&+K(x_{n+1},x_n,y_n)-K(x_n,x_n,y_n))/3\end{aligned} \tag{29}$$

To approximate the solution of the initial-value problem

$$y'(x) = f(x,y) + \int_{x_0}^{x} K(x,s,y(s))ds,$$

$$x_0 \le x \le X, y(x_0) = y_0$$

At N equally spaced numbers in the interval $[x_0, X]$:

INPUT end point x_0, X ; integer N, initial conditions $y_0; y_1; \nu_0; \nu_1$.

OUTPUT approximation y_n to $y(x_n)$ at the N values of x.

Step 1 Set $h = (X - x_0)/N$;

$x_0 = a$;

$y_0 = b$;

OUTPUT (x_0, y_0).

Step 2 For i=2,3,…N do Steps 3-8.

Step 3 Set $x_i + a + ih.(comput\ x_i)$

Step 4 $\bar{y}_i; \hat{\nu}_i$ $(comput\ \bar{y}_i, \hat{\nu}_i$ by the predictor methods (1.16) and (1.19) or (1.21))

Step 5 $\bar{y}_i; \nu_i$ $(comput\ \bar{y}_i, \nu_i$ by the predictor methods (1.17) and (1.20) or (1.22))

Step 6 y_i (comput values of the initial-value problem by the corrector method (1.18))

Step 7 OUTPUTS (x_i, y_i)

Step 8 STOP.

III. Conclusion

In an inference we'll note, that methods constructed in view of property of Volterra integral are simpler (see for example methods (28) and (29)), however they in some cases can be less exact than methods of type (27). We investigated approximately 10 Volterra integral equations which have the solution of different properties - increase, decrease, oscillation, etc. In many cases outcomes of evaluations on methods (22) and (29) almost coincided, but in some cases the method (22) has appeared more exact. It shows that in the given method (22) we use more amount of the information about solutions of the considered equation, than in the method (29).

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