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Reconstruction of Binary Matrices Satisfying Neighborhood Constraints by Simulated Annealing

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***Abstract*—**This paper considers the NP-hard problem of reconstructing binary matrices satisfying exactly-1-4-adjacency constraint from its row and column projections. This problem is formulated into a maximization problem. The objective function gives a measure of adjacency constraint for the binary matrices. The maximization problem is solved by the simulated annealing algorithm and experimental results are presented.

***Keywords*—**Discrete Tomography, exactly-1-4-adjacency, simulated annealing.

I. Introduction

DISCRETE Tomography (DT) is the reconstruction of discrete objects from their projections. The number of projections is generally two to four. Books by Herman and Kuba [1], [2] are excellent references for the foundations, algorithms and applications of Discrete Tomography.

One of the classical problems in DT is the reconstruction of a binary matrix from its row and column projections. This problem is highly underdetermined which makes the size of solution space very large. In order to reduce the size of solution space, additional constraints are imposed on the binary matrices such as convexity [3], connectivity [4], etc. Some constraints make the reconstruction problem NP-hard. We consider the local neighborhood constraint which also makes the problem NP-hard [5]. In 2008, Frosini, Picoleau, and Rinaldi [5] showed that the problem of reconstructing a binary matrix satisfying exactly-1-4-adjacency constraint from the row and column projections is NP-hard.

Meta-heuristics are useful methods for solving NP-hard problems. Simulated Annealing (SA) is a meta-heuristic based on the principle of physical annealing of solids. SA has been used to solve many hard combinatorial optimization problems since its introduction in 1983 [6]. A good survey on simulated annealing can be found in [7]. In DT, Jarray and Tlig [8] applied SA for reconstructing hv-convex binary matrices. We have applied the simulated annealing method for the NP-hard problem of reconstructing a binary matrix satisfying exactly-1-4-adjacency constraint from the row and column projections. This paper is structured as follows.

In Section II we give the necessary definitions and preliminaries. We convert our problem into a maximization problem in Section III. Then in Section IV we present the simulated annealing algorithm for our problem. Section V gives the experimental results and we conclude in Section VI.

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II. Definitions and Preliminaries

A binary matrix is represented in two ways, viz., matrix representation and pixel representation, as shown in Fig. 1. In the pixel representation the white pixels correspond to value 1 and the black pixels correspond to value 0.

0	1	0	0
1	1	0	1
1	1	1	0
0	0	0	1

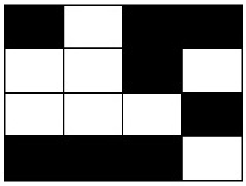

Fig. 1 Two representations of the same binary matrix

Let $A = [a_{ij}]$ be a binary matrix of size $m \times n$. The row and column projections of A are given by $R = (r_1, r_2, \dots, r_m)$ & $C = (c_1, c_2, \dots, c_n)$ respectively, where

$$r_i = \sum_{j=1}^{n} a_{ij} \qquad i = 1,2,\dots,m \tag{1}$$

and

$$c_j = \sum_{i=1}^{m} a_{ij} \qquad j = 1,2,\dots,n. \tag{2}$$

For given projections (R, C), we denote by $\mathfrak{U}(R, C)$ the set of binary matrices having R and C as row and column projections respectively. Fig. 2 shows a binary matrix and its row and column projections.

A switching component of a binary matrix is a sub-matrix of the form $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ or $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$. A switching operation is the transformation of the matrix $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ into the matrix $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ or vice versa. Note that by performing a switching operation on a binary matrix, its row and column projections do not change. The following is a fundamental theorem given by Ryser [9].

Theorem [9]: Let $A, B \in \mathfrak{U}(R, C)$. Then A can be transformed into B or vice versa by a finite number of switching operations.

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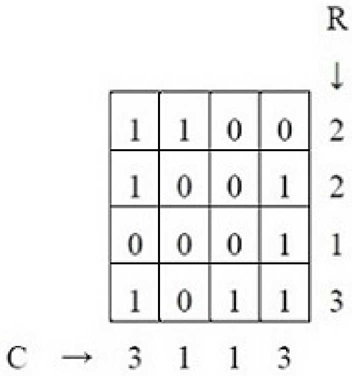

Fig. 2 A 4×4 binary matrix and its row projection $R = (2,2,1,3)$ and column projection $C = (3,1,1,3)$.

Now we define the adjacency constraint used in this paper. Let a_{ij} be an element of a binary matrix A of order $m \times n$. Then the 4-adjacent neighbors of a_{ij} are the elements (if exists) $a_{i,j-1}$, $a_{i-1,j}$, $a_{i,j+1}$, and $a_{i+1,j}$. Next, a binary matrix $A = [a_{ij}]_{m\times n}$ is said to be exactly-1-4-adjacent if $a_{ij} = 1$ implies that exactly one among its 4-adjacent neighbors has value 1. Fig. 3 illustrates an exactly-1-4-adjacent matrix. We now define our reconstruction problem.

Reconstruction Problem (RP)

Input: Non-negative integer vectors (R, C).

Output: A binary matrix $X \in \mathfrak{U}(R, C)$ and satisfying the exactly-1-4-adjacency constraint.

0	0	0	1	1	0	0	1
0	1	1	0	0	1	0	0
1	0	0	0	0	1	0	0
1	0	1	1	0	0	1	1
0	1	0	0	1	0	0	0
0	1	0	0	1	0	1	1
0	0	0	1	0	1	0	0
1	1	0	1	0	1	0	1

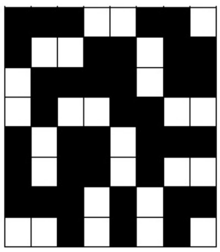

Fig. 3 An exactly-1-4-adjacent binary matrix

III. Formulation of Maximization Problem

In this section we define two maximization problems for reconstruction problem RP. Each problem contains an objective function which gives a measure of exactly-1-4-adjacency constraint on a binary matrix.

A. Problem P1

Let $X = [x_{ij}]$ be a binary matrix of size $m \times n$. In order to define a measure for exactly-1-4-adjacency we introduce the following function-

For each $x_{ij} \in X$, define $\alpha(i,j)$ as-

$\alpha(i,j) = 0 \; if \; x_{ij} = 0$

$\alpha(i,j) = 0 \; if \; x_{ij} = 1$ and all its 4-adjacent neighbors has value 0

$\alpha(i,j) = 1 \;\; if \; x_{ij} = 1$ and exactly one of its 4-adjacent neighbors has value 1

$\alpha(i,j) = -1 \;\; if \; x_{ij} = 1$ and exactly two of its 4-adjacent neighbors has value 1

$\alpha(i,j) = -2 \;\; if \; x_{ij} = 1$ and exactly three of its 4-adjacent neighbors has value 1

$\alpha(i,j) = -3 \;\; if \; x_{ij} = 1$ and exactly four of its 4-adjacent neighbors has value 1

For example for $X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ the values for α are

$\alpha(1,1) = 1, \alpha(1,2) = 0, \alpha(1,3) = 0$

$\alpha(2,1) = -2, \alpha(2,2) = -2, \alpha(2,3) = 1$

$\alpha(3,1) = -1, \alpha(3,2) = -1, \alpha(3,3) = 0$

Based on the function $\alpha(i,j)$ we define the objective function $f_1(X)$ as

$$f_1(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha(i,j) \tag{3}$$

Clearly a binary matrix satisfying the exactly-1-4-adjacent constraint will maximize the function $f_1(X)$. So we define our optimization problem as

Maximize

$$f_1(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha(i,j) \tag{4}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = r_i \qquad for \; i = 1,2, \dots, m \tag{5}$$

$$\sum_{i=1}^{m} x_{ij} = c_j \qquad for \; j = 1,2, \dots, n \tag{6}$$

B. Problem P2

Let $X = [x_{ij}]_{m\times n}$ be a binary matrix. Two horizontally consecutive positions $(i,j), (i,j+1)$ is said to be a pair if $x_{ij} = x_{i,j+1} = 1$ and $x_{i,j-1} = x_{i-1,j} = x_{i-1,j+1} = x_{i,j+2} = x_{i+1,j-1} = x_{i+1,j+1} = 0$ whenever they exists. Similarly two vertically consecutive positions $(i,j), (i+1,j)$ is said to be a pair if $x_{ij} = x_{i+1,j} = 1$ and $x_{i-1,j} = x_{i,j+1} = x_{i+1,j+1} = x_{i+2,j} = x_{i+1,j-1} = x_{i,j-1} = 0$ whenever they exists. For example let $X = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ then the positions $\{(2,2),(2,3)\}, \{(1,5),(2,5)\}, \{(3,4),(4,4)\}, \{(5,1),(5,2)\}$ forms pairs. We see that an exactly-1-4-adjacent binary matrix has maximum number of pairs in the set $\mathfrak{U}(R, C)$. So we define problem P2 as

Maximize

$$f_2(X) = number \; of \; pairs \; in \; X \tag{7}$$

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subject to

$$\sum_{j=1}^{n} x_{ij} = r_i \qquad for\ i = 1,2,\dots,m \tag{8}$$

$$\sum_{i=1}^{m} x_{ij} = c_j \qquad for\ j = 1,2,\dots,n \tag{9}$$

C. Proposition 1

Let $X = [x_{ij}]$ be a binary matrix of size $m \times n$. Let (R, C) be the row and column projections of X. Then X is exactly-1-4-adjacent if and only if

$$a)\ f_1(X) = \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j \tag{10}$$

$$b)\ f_2(X) = \frac{1}{2}\sum_{i=1}^{m} r_i = \frac{1}{2}\sum_{j=1}^{n} c_j \tag{11}$$

Proof:

a) $X = [x_{ij}]$ is exactly-1-4-adjacent iff each $x_{ij} = 1$ has exactly one 4-adjacent neighbor with value 1

iff $x_{ij} = 1 \Rightarrow \alpha(i,j) = 1$

iff $\sum_{i=1}^{m}\sum_{j=1}^{n} \alpha(i,j) = \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$

iff $f_1(X) = \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$.

b) $X = [x_{ij}]$ is exactly-1-4-adjacent iff each $x_{ij} = 1$ has exactly one 4-adjacent neighbor with value 1

iff number of pairs in $X = \frac{1}{2}\sum_{i=1}^{m} r_i = \frac{1}{2}\sum_{j=1}^{n} c_j$

iff $f_2(X) = \frac{1}{2}\sum_{i=1}^{m} r_i = \frac{1}{2}\sum_{j=1}^{n} c_j$.

IV. Simulated Annealing Algorithm

In this section we present a simulated annealing algorithm for solving optimization problems P1 and P2. The algorithm starts with an initial solution and at an initial temperature. The initial solution is obtained by Ryser's algorithm. The algorithm proceeds from one solution to its neighbor solution in search of the optimal solution. Two binary matrices are said to be neighbors of each other if they differ by a single switching component, i.e. one matrix can be obtained from the other by a single switching operation. The neighbor solution is accepted if it gives a better objective function value otherwise it is accepted with some probability. The temperature is decreased gradually and at each temperature a fixed number of iterations are performed to attain equilibrium at that temperature. The algorithm is terminated when the temperature reaches a predefined final value. The best solution obtained is given as the output. The algorithm is as follows-

1) *Find initial solution $X = X_0$. Set initial temperature $T = 100$. Set number of iterations $itrs = iterations$.*
2) *Repeat (3) to (8) while $T > 0.01$*
3) *Set iteration count $itr = 0$.*
4) *Repeat (5), (6), (7) while $itr < itrs$*
5) *Generate a random neighbor of X, say Y. Calculate $D = f(Y) - f(X)$.*
6) *If $D \geq 0$ then $X = Y$ and $itr = itr + 1$.*
7) *If $D < 0$ then generate a random number $r(0 < r < 1)$. If $r < e^{(D/T)}$ then $X = Y$ and $itr = itr + 1$.*
8) *Decrease temperature $T = T \times 0.9$.*

The flowchart of the algorithm is shown in Fig. 4.

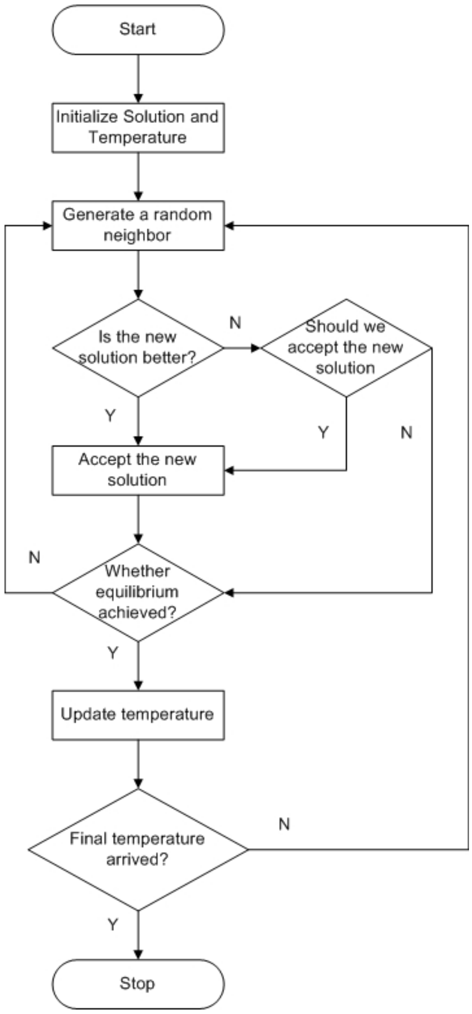


Fig. 4 Flowchart of the simulated annealing algorithm

V. Experimental Results

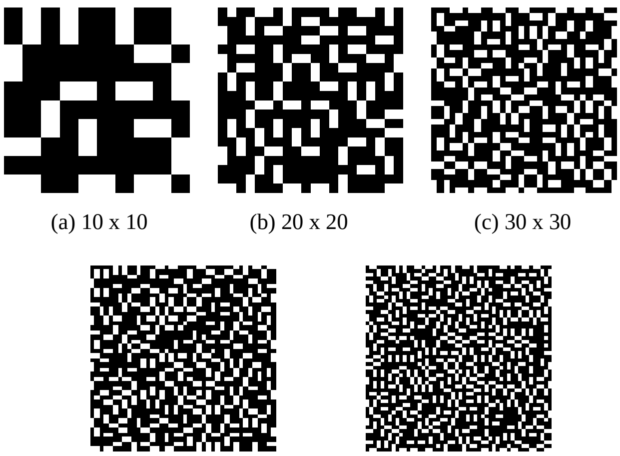
The experiments were carried on a laptop with Intel core2duo processor, 2GB RAM and 2GHz speed. The programs were written in MATLAB R2010a. The images were of sizes 10x10, 20x20, 30x30, 40x40, and 50x50. The results are shown in Tables I and II. The first column in the tables gives the size of the images. The "Iterations" column gives the number of iterations performed at each temperature value to attain equilibrium at that temperature. The column of the objective functions $f_1(X), f_2(X)$ value contains two sub-columns titled "Maximum" and "Obtained". The "Maximum" column consists of the maximum value of the objective function as calculated using Proposition 1 and the "Obtained" column consists of the objective function value of the reconstructed images obtained from the simulated annealing algorithm. Figs. 5 and 6 show the reconstructed images obtained from the simulated annealing algorithm of problems P1 and P2 respectively.

TABLE I
RECONSTRUCTION RESULTS FOR PROBLEM P1

Image size	Iterations	$f_1(X)$	
		Maximum	Obtained
10 x 10	20	34	34
20 x 20	40	126	124
30 x 30	60	302	296
40 x 40	80	512	500
50 x 50	100	828	798

TABLE II
RECONSTRUCTION RESULTS FOR PROBLEM P2

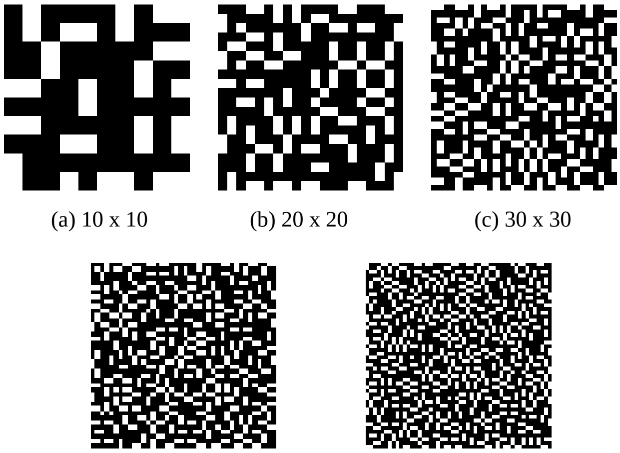
Image size	Iterations	$f_2(X)$	
		Maximum	Obtained
10 x 10	20	17	16
20 x 20	40	63	60
30 x 30	60	151	147
40 x 40	80	256	245
50 x 50	100	414	393



(a) 10 x 10 (b) 20 x 20 (c) 30 x 30

(d) 40 x 40 (e) 50 x 50

Fig. 5 Reconstructed images of various sizes of Problem P1 by simulated annealing algorithm



(a) 10 x 10 (b) 20 x 20 (c) 30 x 30

(d) 40 x 40 (e) 50 x 50

Fig. 6 Reconstructed images of various sizes of Problem P2 by simulated annealing algorithm

VI. CONCLUSION

In this paper we considered the reconstruction problem in the class of exactly-1-4-adjacent binary matrices. Since the problem is NP-hard we used a meta-heuristic algorithm to solve the problem. We defined two maximization problems for reconstructing exactly-1-4-adjacent matrices. Then we presented the simulated annealing algorithm for solving the maximization problems. We obtained good and comparable approximations of matrices in the class of exactly-1-4-adjacent matrices. Further work can be done on the analysis of the algorithm and also simulated annealing algorithm can be applied on other problems of Discrete Tomography.

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